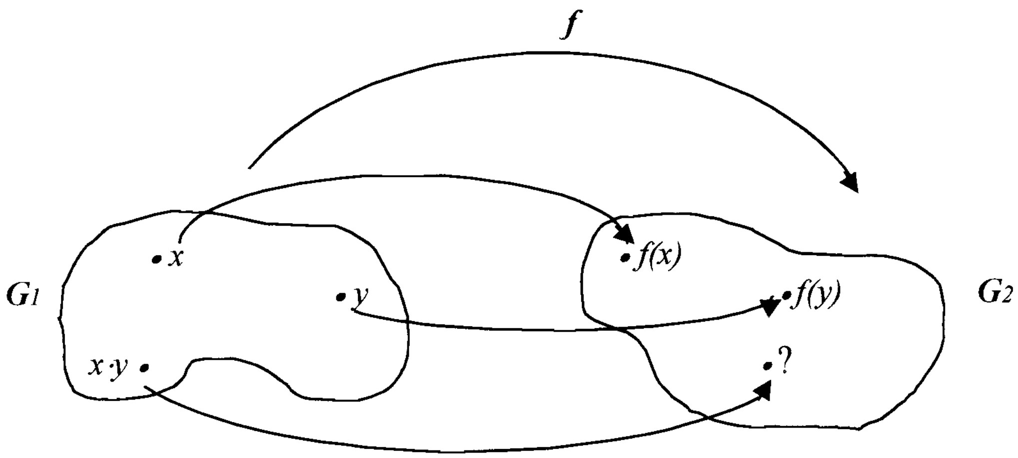
# Homomorphisms

Let $G_1$ and $G_2$ be groups. A surjective function $f : G_1 \to G_2$ is said to be a homomorphism iff $f(x \cdot y) = f(x) \cdot f(y)$ for all $x$ and $y$ in $G_1$. This last equation will be called the Homomorphism Test Equation and abbreviated HTE. The HTE is usually misunderstood by beginners. The group operator on the left of the equation is not the same as the group operator on the right. On the left we have $f(x \cdot y)$. Since $x$ and $y$ are in $G_1$, this operator must be the operator in $G_1$. On the right we have $f(x) \cdot f(y)$. Since both $f(x)$ and $f(y)$ are in $G_2$, this operator must be the $G_2$ operator. A homomorphism not only converts the elements of $G_1$ into elements of $G_2$ but must also convert the $G_1$ operator into the $G_2$ operator. A graphic might be instructive.




The question mark in the diagram has to be $f(x) \cdot f(y)$ if $f$ is a homomorphism. We say that group homomorphisms "preserve multiplication". As we'll see, that's not all they "preserve".

**Theorem:** If $f : G_1 \to G_2$ is a group homomorphism then $f(e_1) = e_2$ where $e_1$ is the identity of $G_1$ and $e_2$ is the identity of $G_2$.

**Proof:** Let $x$ be an element of $G_1$. $f(x)$ is an element of $G_2$. Consider:

$$f(x \cdot e_1) = f(x) \cdot f(e_1)$$

However, $f(x \cdot e_1)$ also must equal $f(x)$ since $e_1$ is the identity of $G_1$.

$$\therefore f(x) = f(x) \cdot f(e_1)$$

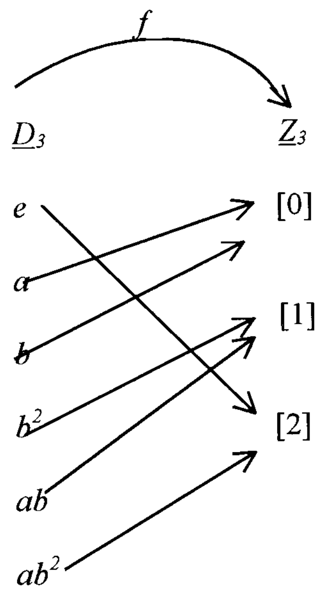
Since $e_2$ is the identity of $G_2$ and $f(x)$ is in $G_2$ :

$$f(x) \cdot e_2 = f(x) \cdot f(e_1)$$

$\therefore$ (by RHC) $e_2 = f(e_1)$

**QED**

Now we know that homomorphisms "preserve" identities. We also know that a chart like:




can't be a homomorphism since $f(e)$ does not equal 0. In other words, group homomorphisms "link" identities.

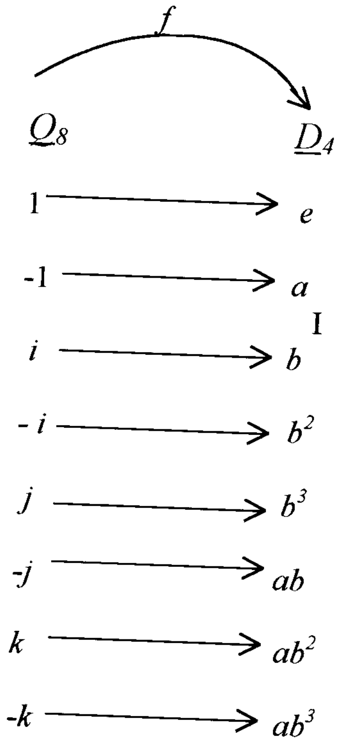
**Theorem:** If $f : G_1 \rightarrow G_2$ is a group homomorphism then for every $x$ in $G_1$,
$f(x^{-1}) = (f(x))^{-1}$

**Proof:** Consider: $f(x \cdot x^{-1}) = f(e_1) = e_2$
However, $f(x \cdot x^{-1})$ also equals $f(x) \cdot f(x^{-1})$
Therefore, $f(x) \cdot f(x^{-1}) = e_2$
$\therefore f(x^{-1}) = (f(x))^{-1}$

**QED**

This theorem can also be quite misunderstood. Consider the equation $f(x^{-1}) = (f(x))^{-1}$. The inverse on the left is the $G_1$ inverse of the element $x$ in $G_1$. The inverse on the right is the $G_2$ inverse of the element $f(x)$ in $G_2$. We now know that homomorphisms "preserve" inverses.

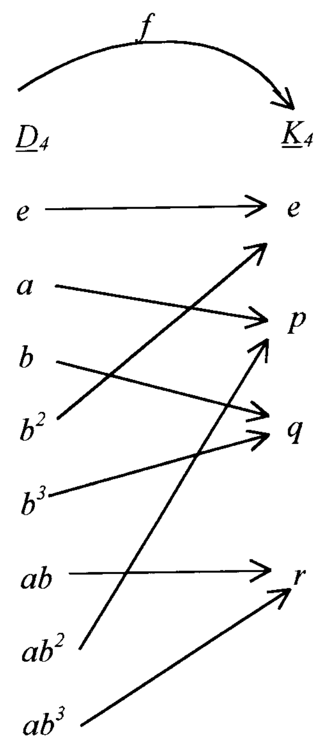
Consider this diagram:




This function cannot be a homomorphism. Let's consider one reason it can't because of our new theorem. $f(i) = b$. Therefore $f(i^{-1})$ should equal $\left(f(i)\right)^{-1} = b^{-1}$. Since $i^{-1} = -i$ and since $b^{-1} = b^3$, we now know that if $i$ and $b$ are linked then $-i$ and $b^3$ would be linked if $f$ were a homomorphism. Thus $f$ is not a homomorphism.

Suppose $f : G_1 \to G_2$ is a group homomorphism. Suppose an element $x$ in $G$ is self-invertible. Suppose $f(x) = y$ in $G_2$. Is it possible that $y$ is not self-invertible? Suppose $y^{-1} \neq y$. We know $f(x) = y \therefore f(x^{-1}) = y^{-1}$. However, $x^{-1} = x$. $\therefore f(x)$ equals both $y$ and $y^{-1}$. This is impossible if $f$ is a function. $\therefore y$ must be self-invertible. $\therefore$ the image of a self-invertible element under a homomorphism is also a self-invertible element.

A warning, however, is in order at this point. Suppose an element $y$ in $G_2$ is self-invertible. If $x$ is a pre-image of $y$, $x$ need not be self-invertible. The equations $f(x^{-1}) = \left(f(x)\right)^{-1} = y^{-1} = y$ do imply that the inverse of $x$ in $G_1$ has to also be a pre-image of $y$. The following is a homomorphism:




Note that:

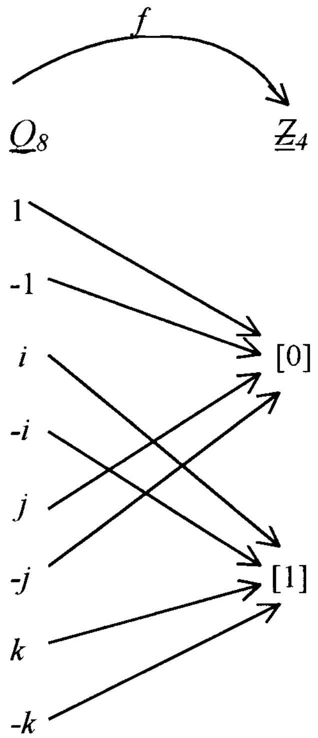
1. $f(e) = e$
2. Since $p, q$ and $r$ are self-invertible, whenever an element is a pre-image of one of them its inverse is also a pre-image of the same one.

**Definition:** Let $f : G_1 \rightarrow G_2$ be a group homomorphism. The set of all elements in $G_1$ that are pre-images of $e_2$ (the identity of $G_2$) is called the kernel of the homomorphism and denoted $K_f$.

$$\text{i.e. } K_f = \{x \mid x \in G_1 \text{ and } f(x) = e_2\}$$

In our example of a homomorphism that linked $D_4$ and $K_4$, $Ker_f = \{e, b^2\}$.

The following is a homomorphism:




For this homomorphism, $Ker_f = \{1, \text{-}1, j, \text{-}j\}$

**Theorem:** Let $f : G_1 \rightarrow G_2$ be a group homomorphism. $Ker_f$ is a normal subgroup of $G_1$.

**Proof:** Let's first establish that $Ker_f$ is a subgroup of $G_1$.

Closure

Let $x \in Ker_f$
Let $y \in Ker_f$
$f(x \cdot y) = f(x) \cdot f(y) = e_2 \cdot e_2 = e_2$
$\therefore x \cdot y \in Ker_f$

Inverse Closure

Let $x \in Ker_f$
$f(x^{-1}) = \left(f(x)\right)^{-1} = e_2^{-1} = e_2$
$\therefore x^{-1} \in Ker_f$

Now we know that $Ker_f$ is a subgroup of $G_1$. Let's establish normalicy.

NTC

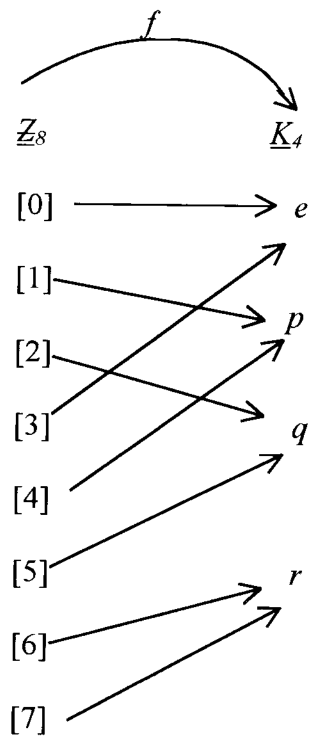
Let $x \in Ker_f$. Let $q$ be an arbitrary element of $G_1$.

$$f(qxq^{-1}) = f(q) \cdot f(x) \cdot f(q^{-1}) = f(q) \cdot e_2 \cdot f(q^{-1})$$
$$= f(q) \cdot f(q^{-1}) = f(q) \cdot (f(q))^{-1} = e_2$$
$$\therefore qxq^{-1} \in Ker_f$$
$$\therefore Ker_f \triangleleft G_1$$

**QED**

Notice that in our proof we used the fact that $f(q^{-1}xq) = f(q^{-1})f(x)f(q)$. It is easy to establish that the HTE can be extended to an arbitrary number of elements in a product.

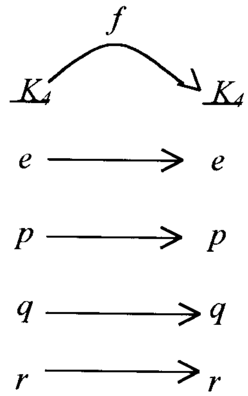
The following cannot be a homomorphism:




If it were, $Ker_f = \{[0], [3]\}$. However, $\{[0[, [3]\}$ is not a normal subgroup of $\mathbb{Z}_8$. It isn't even a subgroup. Can you find other reasons that the above function could not be a homomorphism?

Our theorem limits the number of different homomorphisms for which a given group $G$ can be the domain. The number is limited by the number of normal subgroups contained in $G$. For now, we do not know whether or not each normal subgroup of $G$ can be a kernel for a homomorphism with $G$ as the domain. We also don't know whether or not a given normal subgroup can generate more than one homomorphism.

Two rather trivial homomorphisms exist for any group $G$. The first is usually called the identity function and is defined by $f(x) = x$ for every $x$ in $G$. For example:



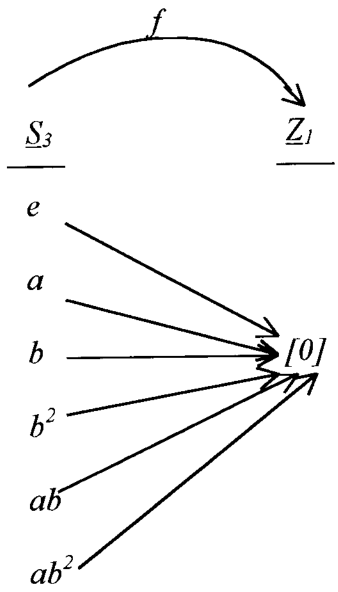

Note that the kernel for the identity function for any group is $\{e\}$. To prove that the identity function is a homomorphism, consider:

$$f(xy) = xy = f(x)f(y) \text{ for any } x,\ y \in G$$
$$\therefore\ f(xy) = f(x)f(y)$$

A second trivial homomorphism that can be defined on any group $G$ uses the group $\mathbb{Z}_1$ as the range group. $\mathbb{Z}_1$ is the group with only one element [0] whose operating table is:

| $\mathbb{Z}_1$ | [0] |
|---|---|
| [0] | [0] |

For any group $G$, define the function $f(x) = [0]$ for every $x$ in $G$. For example:




To confirm that $F$ is a homomorphism, consider:

$$f(x \cdot y) = [0] = [0] \cdot [0] = f(x) \cdot f(y)$$
$$\therefore f(xy) = f(x)f(y)$$

Note that the kernel of this homomorphism is $G$ itself.

Suppose $G_1$ is a finite group and $o(G_1) = n$. Let $f : G_1 \to G_2$ be a function where $G_2$ is also a finite group. In order to determine whether or not $f$ is a homomorphism, we need to establish that $f(xy) = f(x)f(y)$ for every $x$ and $y$ in $G_1$. How many equations would we need to consider to establish that $f$ is a homomorphism by exhausting every possibility for $x$ and $y$? We know $x$ can be any of $n$ elements. Similarly, $y$ can be any of $n$ elements. Therefore, $n^2$ equations would have to be satisfied where each is of the type $f(xy) = f(x)f(y)$.

**Theorem:** Let $f : G_1 \to G_2$ be a group homomorphism. Let $Ker_f = K$.

$$f(x) = f(y) \text{ iff } Kx = Ky.$$

**Proof:** Suppose that $Kx = Ky$. Since $x = e_1 x$ and $e_1 \in K$, we know that $x \in Kx$. Since $Kx = Ky$, we can deduce that $x \in Ky$. $\therefore$ $x = k_1 y$ for some $k_1 \in K$.

$$\therefore f(x) = f(k_1 y) = f(k_1)f(y) = e_2 f(y) = f(y)$$
$$\therefore f(x) = f(y).$$

Now suppose that $Kx \neq Ky$. We know that if two cosets are not equal, then they are disjoint. Could $f(x) = f(y)$? Suppose so.

Consider:

$$f(xy^{-1}) = f(x)\cdot f(y^{-1}) = f(x)[f(y)]^{-1} = f(x)[f(x)]^{-1} = e_2$$

$$\therefore xy^{-1} \in K$$

$$\text{Let } xy^{-1} = k_2$$

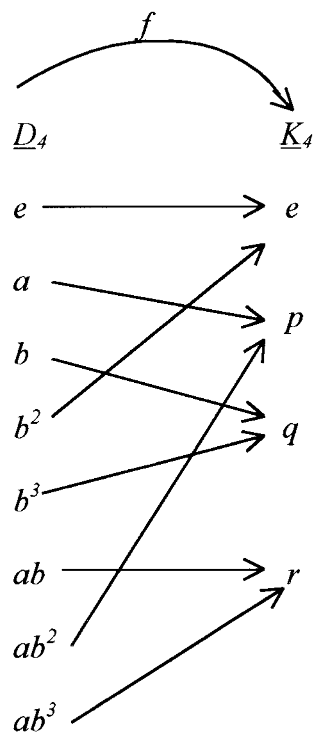
$$\therefore x = k_2 y$$

$$\therefore x \in Ky$$

We know that $x \in Kx$. $\therefore Kx$ and $Ky$ are not disjoint.

**QED**

This theorem gives us a great amount of insight concerning the structure of a homomorphism. The set of all pre-images of a specific range element is precisely one of the cosets generated by the kernel of the homomorphism. Let's consider again the function taking $D_4$ to $K_4$ that we itemized earlier in this chapter:




We claimed that this function was a homomorphism. In order to verify this we could consider all 64 equations of the form $f(x\cdot y) = f(x)\cdot f(y)$ where $x$ and $y$ are arbitrary elements of $D_4$. It is unnecessary for you to do this at this time, but all 64 equations are true. We should take a closer look at this function. The kernel of $f$ is $\{e, b^2\}$ which indeed is a normal subgroup of $D_4$ which illustrates one of our important theorems. Consider the cosets generated by the kernel $\{e, b^2\}$ which we will denote $K$.

$$Ke = \{e, b^2\}$$
$$Ka = \{a, ab^2\}$$
$$Kb = \{b, b^3\}$$
$$Kb^2 = \{b^2, e\}$$
$$Kb^3 = \{b^3, b\}$$
$$Kab = \{ab, ab^3\}$$
$$Kab^2 = \{ab^2, a\}$$
$$Kab^3 = \{ab^3, ab\}$$

The distinct cosets are $\{e, b^2\}$, $\{a, ab^2\}$, $\{b, b^3\}$ and $\{ab, ab^3\}$. The pre-image of each of the elements of $K_4$ is precisely one of these cosets. This is an illustration of another of our important theorems. It becomes clear that homomorphisms are highly structured. We will be interested in finding a way to resolve one key issue: What groups can a given group be linked to with a homomorphism where the given group is the domain? As we will see, the answer lies within the structure of the given group. Before we attack this issue, let's solve some example problems.

**Problem 1** Let $G_1$ be the group of all non-singular $2 \times 2$ matrices under matrix multiplication. Let $G_2$ be the group of non-zero real numbers under the usual real number multiplication. Let $f : G_1 \to G_2$ be defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ [Note that $f$ is essentially the act of computing a determinant]. Is $f$ a homomorphism? If so, what is the kernel of $f$?

Solution: Consider first

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) =$$
$$f\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} =$$
$$(ae + bg)(cf + dh) - (af + bh)(ce + dg) =$$
$$aecf + aedh + bgcf + bgdh - afce - afdg - bhce - bhdg =$$
$$aedh + bgcf - afdg - bhce \quad \text{(Result 1)}$$

Now Consider $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\begin{pmatrix} e & f \\ g & h \end{pmatrix} =$

$$(ad - bc) \cdot (eh - fg) =$$
$$adeh - adfg - bceh + bcfg \quad \text{(Result 2)}$$

A comparison of results 1 and 2 reveal them to be equal. Therefore $f$ is a homomorphism.

Now we can consider the kernel of $f$. The identity of the range group is 1. In order for a $2 \times 2$ matrix to be in the kernel, its image under $f$ must be 1. The kernel is therefore the set of all $2 \times 2$ matrices whose determinants are 1. We now know that this set is a normal subgroup of the group of all $2 \times 2$ non-singular matrices. This result can be established directly by considering closure, inverse closure and the NTC.

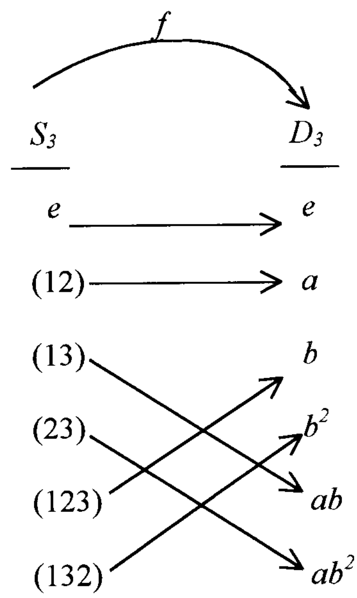
**Problem 2** Let $G_1$ be the group of all positive real numbers under usual multiplication. Let $G_2$ be the group of all reals under usual addition. Let $f : G_1 \rightarrow G_2$ be defined by $f(x) = \ell nx$. Is $f$ a homomorphism? If so, what is the kernel of $f$?

Solution: Consider
$$\begin{aligned} f(x \cdot y) &= \ell n(xy) \\ &= \ell nx + \ell ny \\ &= f(x) + f(y) \end{aligned}$$

Since multiplication is the operator for $G_1$ and addition is the operator for $G_2$, we have shown that the group theoretic equation $f(x \bigstar y) = f(x) \bigstar f(y)$ has been satisfied. Therefore $f$ is a homomorphism. The kernel of $f$ is the set of pre-images of the identity of $G_2$ which is 0.

$$\begin{aligned} \text{Therefore if } x \in Ker_f \text{ then } f(x) &= 0 \\ &\Rightarrow \ell nx = 0 \\ &\Rightarrow e^{\ell nx} = e^0 \\ &\Rightarrow x = 1 \\ &\therefore Ker_f = \{1\} \end{aligned}$$

An injective (one-to-one) homomorphism is called an isomorphism. The domain group and the range group of such a function are said to be isomorphic. Not only is there a one-to-one correspondence of their elements, but there is a smooth transition from the domain operator to the range operator. As such, the groups are considered to be structurally the same but with (possibly) different labels on the elements. Two of the groups you have in your library are isomorphic. Consider:



Clearly $f$ is injective. To establish that $f$ is a homomorphism by exhaustion requires confirming 36 equations. We must note that the fact that $S_3$ and $D_3$ are isomorphic does not imply that other members of the Symmetric family are isomorphic to members of the Dihedral family. Indeed, $S_4$ is not isomorphic to $D_{12}$. We could have gotten clues along the way that $S_3$ and $D_3$ were very similar. If you check your previous work, you will find that they both have the same number of subgroups. The orders of the subgroups match. The number of subgroups that are normal match. The number of order 2 and order 3 elements match.

A final word about isomorphisms is in order. Given the set of all groups, the relation "is isomorphic to" is an equivalence relation.

**Theorem:** Let $f : G_1 \to G_2$ be a group homomorphism. Let $K$ be the kernel of $f$. $G_1/K$ is isomorphic to $G_2$.

**Proof:** Let $h : G_1/K \to G_2$ be defined by $h(Kx) = f(x)$. Is $h$ well defined? Suppose $Kx = Ky$. Then $x \in Ky$. $\therefore \exists\ k_1$ such that $x = k_1 y$.

$\therefore f(x) = f(k_1 y) = f(k_1)f(y) = e_2 f(y) = f(y)$

Since $f(x) = f(y)$, $h(Kx) = h(Ky)$.

$\therefore h$ is well defined

Is $h$ a homomorphism?

Recall that in the last chapter we proved that $Kx \cdot Ky = Kxy$. Using this, consider:

$$\begin{aligned} h(Kx \cdot Ky) &= h(Kxy) \\ &= f(xy) \\ &= f(x)f(y) \\ &= h(Kx)h(Ky) \end{aligned}$$

$\therefore h$ is a homomorphism.

Is $h$ injective? Suppose $h(Kx) = h(Ky)$. Then we know $f(x) = f(y)$.

Consider:

$$\begin{aligned} f(xy^{-1}) &= f(x)f(y^{-1}) \\ &= f(x)[f(y)]^{-1} \\ &= f(x)[f(x)]^{-1} \\ &= e_2 \end{aligned}$$

$\therefore xy^{-1} \in K$

$\therefore x \in Ky$

$\therefore Kx = Ky$

Therefore, if $h(Kx) = h(Ky)$ it follows that $Kx = Ky$. This implies that $h$ is injective.

**QED**

Now we can find all groups to which a given group can be homomorphically linked. To achieve our goal we must:

1. Find all normal subgroups of the given group
2. Factor the given group by each normal subgroup
3. Identify the factor groups

4. Create the homomorphisms.

**Example 1:** Find all groups to which $Q_8$ can be homomorphically linked.

Solution: The normal subgroups of $Q_8$ are :
$\{1\}$, $Q_8$, $\{1,-1\}$, $\{1,-1,j,-j\}$ and $\{1,-1,k,-k\}$.

$\{1\}$

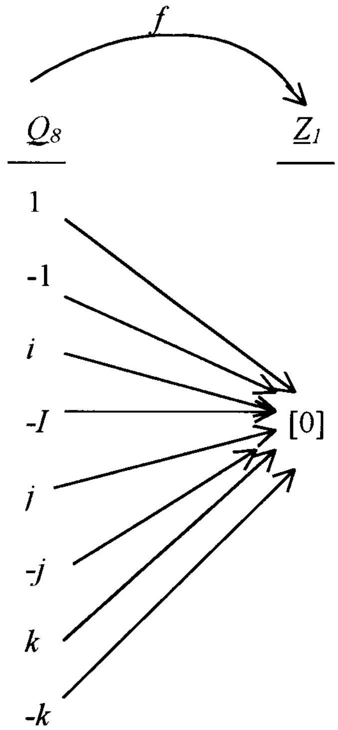
$Q_8/\{1\}$ is an isomorphic copy of $Q_8$. $\therefore$ $Q_8$ can be homomorphically linked to $Q_8$. The identity function will be the homomorphism (which is actually an isomorphism):

$f$

| $Q_8$ | | $Q_8$ |
|---|---|---|
| $1$ | $\longrightarrow$ | $1$ |
| $-1$ | $\longrightarrow$ | $-1$ |
| $i$ | $\longrightarrow$ | $i$ |
| $-i$ | $\longrightarrow$ | $-i$ |
| $j$ | $\longrightarrow$ | $j$ |
| $-j$ | $\longrightarrow$ | $-j$ |
| $k$ | $\longrightarrow$ | $k$ |
| $-k$ | $\longrightarrow$ | $-k$ |

$Q_8$

$Q_8/Q_8$ has only one distinct coset. $\therefore$ $Q_8$ can be homomorphically linked to $Z_1$. The function needed is:

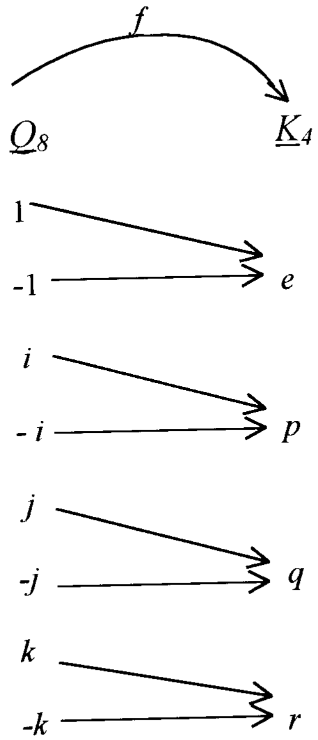



## $\{1, \text{-}1\}$

$Q_8/\{1, \text{-}1\}$ has four distinct cosets: $\{1, \text{-}1\}$, $\{i, \text{-}i\}$, $\{j, \text{-}j\}$ and {k, -$k$}. Letting $\{1, \text{-}1\}$ equal $K$, these correspond to $K$, $Ki$, $Kj$, and $Kk$. A table for $Q_8/K$ is:

| $Q_8/K$ | $K$ | $Ki$ | $Kj$ | $Kk$ |
|---|---|---|---|---|
| $K$ | $K$ | $Ki$ | $Kj$ | $Kk$ |
| $Ki$ | $Ki$ | $K$ | $Kk$ | $Kj$ |
| $Kj$ | $Kj$ | $Kk$ | $K$ | $Ki$ |
| $Kk$ | $Kk$ | $Kj$ | $Ki$ | $K$ |

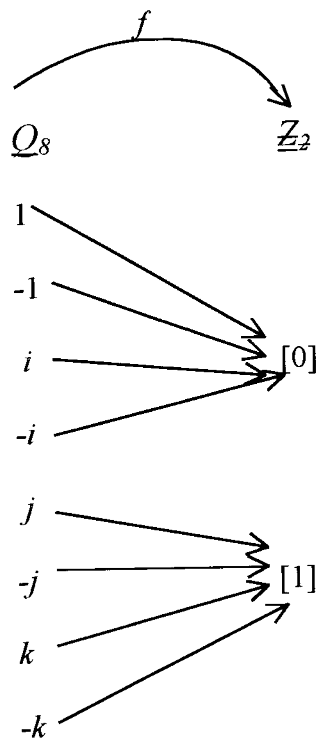
This group is an isomorphic copy of $K_4$. To homomorphically link $Q_8$ and $K_4$, we could use this function:




It is important to observe that each of the pre-images of elements in $K_4$ is a coset induced by factoring $Q_8$ by $\{1, -1\}$. Is the function above the only one possible? We could have made $i$ and $-i$ the pre-images of any one of the three elements $p$, $q$, $r$. Once one of these was selected, $j$ and $-j$ could have had either of the two remaining elements of $K_4$ as their image. Using combinatorics, there are 6 different possible homomorphic linkages of $Q_8$ and $K_4$.

<u>$\{1, -1, i, -i\}$</u>

$Q_8/\{1, -1, i, -i\}$ is a 2 element group. Since 2 is a prime number, the only group of order 2 is $\mathbb{Z}_2$. The function we use to create the homomorphic link is:



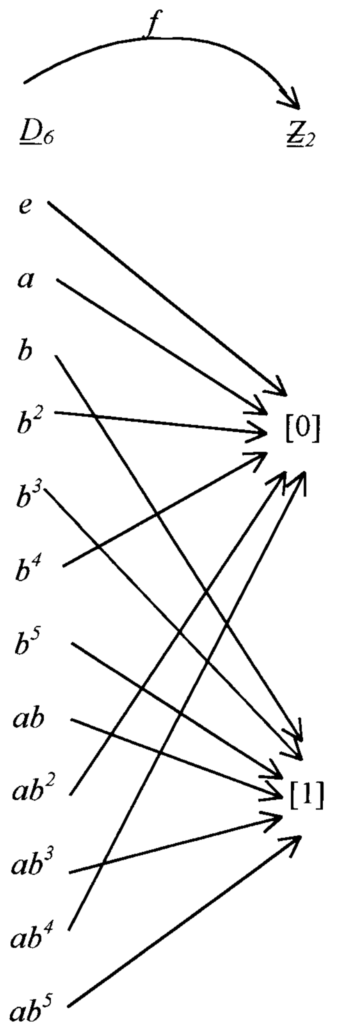

<u>$\{1, -1, j, -j\}$ and $\{1, -1, k, -k\}$</u>

These two normal subgroups can also be used as kernels for homomorphisms between $Q_8$ and $\mathbb{Z}_2$. Therefore, $Q_8$ can be linked homomorphically to $\mathbb{Z}_2$ in three different ways. To sum up, we know that $Q_8$ has only $\mathbb{Z}_1$, $\mathbb{Z}_2$, $K_4$ and $Q_8$ as homomorphic images.

**<u>Example 2</u>** Find all groups that are homomorphic images of $D_6$.

<u>Solution:</u>

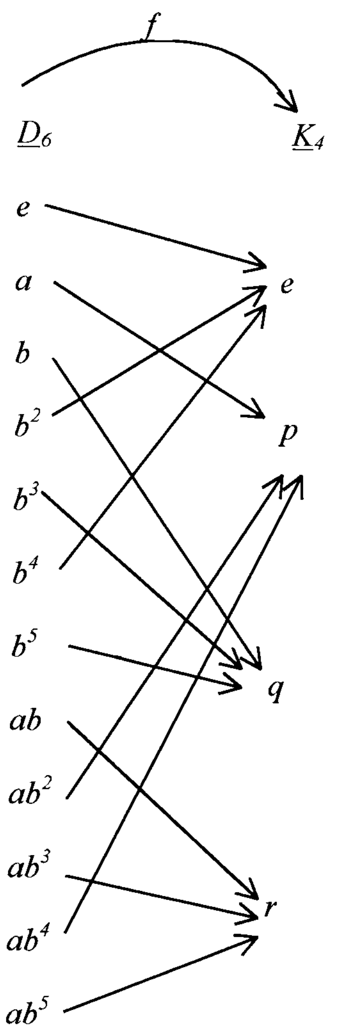
Using the trivial subgroups $\{e\}$ and $D_8$, we can link $D_6$ to $D_6$ and $\mathbb{Z}_1$ respectively. Using any of the 6 element subgroups (they all have to be normal), we can link $D_6$ to $\mathbb{Z}_2$. For example, using the 6 element subgroup $\{e, a, b^2, b^4, ab^2, ab^4\}$ we can create:




The only subgroup of order 3 in $D_6$ is $\{e, b^2, b^4\}$. Therefore this subgroup has to be normal. The cosets induced by this subgroup are : $\{e, b^2, b^4\}$, $\{a, ab^2, ab^4\}$, $\{b, b^3, b^5\}$ and $\{ab, ab^3, ab^5\}$. If we call these $K$, $Ka$, $Kb$ and $Kab$ respectively, we can obtain the operating table:

| $Q_6/K$ | $K$ | $Ka$ | $Kb$ | $Kab$ |
|---|---|---|---|---|
| $K$ | $K$ | $Ka$ | $Kb$ | $Kab$ |
| $Ka$ | $Ka$ | $K$ | $Kab$ | $Kb$ |
| $Kb$ | $Kb$ | $Kab$ | $K$ | $Ka$ |
| $Kab$ | $Kab$ | $Kb$ | $Ka$ | $K$ |

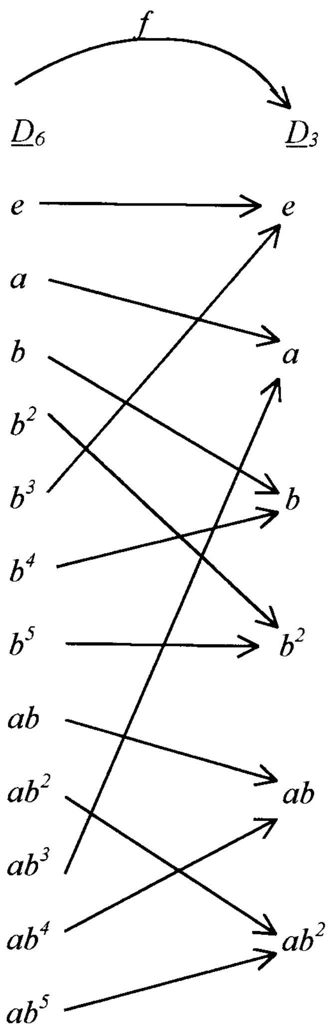
This is an isomorphic copy of $K_4$. One possible homomorphism is:




None of the 4 element subgroups of $D_6$ is normal. Therefore, $D_6$ can not be homomorphically linked to $\mathbb{Z}_3$. However, $\{e, b^3\}$ is a 2 element normal subgroup. It is the center of $D_6$. The cosets it generates are $\{e, b^2\}$, $\{a, ab^2\}$, $\{b, b^4\}$, $\{b^2, b^5\}$, $\{ab, ab^4\}$ and $\{ab^2, ab^5\}$. We'll call these respectively $K$, $Ka$, $Kb$, $Kb^2$, $Kab$ and $Kab^2$. An operating table is:

| | $K$ | $Ka$ | $Kb$ | $Kb^2$ | $Kab$ | $Kab^2$ |
|---|---|---|---|---|---|---|
| $K$ | $K$ | $Ka$ | $Kb$ | $Kb^2$ | $Kab$ | $Kab^2$ |
| $Ka$ | $Ka$ | $K$ | $Kab$ | $Kab^2$ | $Kb$ | $Kb^2$ |
| $Kb$ | $Kb$ | $Kab^2$ | $Kb^2$ | $K$ | $Ka$ | $Kab$ |
| $Kb^2$ | $Kb^2$ | $Kab$ | $K$ | $Kb$ | $Kab^2$ | $Ka$ |
| $Kab$ | $Kab$ | $Kb^2$ | $Kab^2$ | $Ka$ | $K$ | $Kb$ |
| $Kab^2$ | $Kab^2$ | $Kb$ | $Ka$ | $Kab$ | $Kb^2$ | $K$ |

This is an isomorphic copy of $D_3$. Therefore, $D_6$ can be linked to $D_3$. The pertinent homomorphism is:




Therfore, $D_6$ can have the following homomorphic images: $\mathbb{Z}_1$, $D_6$, $\mathbb{Z}_2$, $K_4$ and $D_3$.