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Chapter One

Introduction to *Mathematica*

1.1 *Mathematica's* Conventions

1.1.1 Naming

Built-in *Mathematica* commands, functions, constants, and other expressions begin with capital letters, and are - for the most part- one or more full-length English words. *Mathematica* is case sensitive and so a common cause of error is the failure to capitalize command names. For example, **Plot**, **Integrate**, **FindRoot**, **ParametricPlot3D** are valid function names. **Sin**, **Exp**, **Det**, **GCD**, **Max** are some of the standard mathematical abbreviations that are exceptions to the full-length-English-word(s) rule. User defined functions and variables can be any mixture of upper and lower case letters and numbers. However, a name cannot begin with a number. By convention, user defined functions begin with a lower-case letter, but this is not required. For example **f**, **g1**, **myPlot**, **r12**, **sOLution**, **Method1**, are permissible function names.

Example 1: Evaluate $\cos(\pi)$

Solution: We will use this example to show two possible errors before giving the correct answer.

```
In[1]:= cos [Pi]
```

```
Out[1]= cos [π]
```

This is a typical capitalization error one makes when using *Mathematica*. The comment before the output $\text{Out}[1] = \cos[\pi]$ indicates a possible error.

```
In[2]:= Cos [pi]
```

```
Out[2]= Cos [π]
```

Here *Mathematica* did not recognize **pi** and hence returned the command unevaluated. The correct command to evaluate $\cos(\pi)$ is

```
In[3]:= Cos [Pi]
```

```
Out[3]= -1
```

NOTE: It is worth noting that *Mathematica* keeps track of the calculations it performs in a given session. The inputs are denoted by **In[k]** and the outputs are denoted by **Out[k]**. Thus, **In[1]** is the first input and **Out[1]** is the output of the first input.

1. 1. 2 Parentheses, Square Brackets and Braces

Mathematica interprets various types of brackets differently. Using an incorrect type of bracket is another common source of error. *Mathematica*'s bracketing conventions are:

1) **Parentheses**, (), are used only for grouping. For example $(x-y)^2$, $1/(a+b)$, $(x^3-y)/(x+3y^2)$. It is worth noting that *Mathematica* understands **f(2)** as **f** times **2**, and not as the **value of the function f at 2**.

2) **Square Brackets**, [], are used for function arguments. Thus **Sin[Pi]**, **Simplify[(x^3-y^3)/(x-y)]**, **Sqrt[346]** are proper use of square brackets. If **f** is a user-defined function, then **f[2]** is **f** evaluated at **2**.

3) **Braces or curly brackets**, { }, are used for lists, ranges, and iterators. In all cases, elements are separated by commas. Here are the typical use of braces:

{1, 4, 9, 16, 25, 36} the list of the square of the first six positive integers.

Plot[f[x], {x, -5, 5}] specifies the range of values for x.

Table[m^3, {m, 1, 100}] specifies the values of the iterator m, i.e., the values of m; m goes from 1 to 100.

NOTE: We can write comments with an input. The comments should be enclosed by (* and *). For example, **Table[n^2, {n, 1, 6}]** (*returns a list of the square of the first six positive integers*)

Example 2: Below we give examples of the correct and incorrect usage of brackets.

In[4]:= $(2^3 - 4) / (4 + 3 * 9^2)$ (* a correct use of (...) *)

Out[4]= $\frac{4}{247}$

In[5]:= $[2^3 - 4] / [4 + 3 * 9^2]$ (* an incorrect use of [...] *)

In[5]:= **Sqrt[343]** (* a correct use of [...] *)

Out[5]= $7 \sqrt{7}$

In[6]:= **Sqrt(343)** (* an incorrect use of (...) *)

Out[6]= 343 Sqrt

1. 1. 3 Equal Signs

The following are *Mathematica* rules of using the equal sign symbol.

1) A **single equal sign** (=) assigns a value to a variable. Thus **q=3** means that **q** will be assigned the value **3**. If we enter **10+q^3**, *Mathematica* will return **37**. As another example take **y=x^3-x+1**. Now, if we assign a value for x, say **x=2**, then *Mathematica* evaluates y using that value of **x** and uses the result, **7** in our case, in any input containing y afterwards.

2) A **colon-equal sign** (:=) creates a delayed statement for an expression, used in defining a function. For example **f[x_]:=x^3-x+1** will reevaluate the function f for each new value of x. We will say more about defining functions in Section 1.3 below.

3) A **double equal sign** (==) is a test for the equality of two expressions. If we set **x = -5**, then evaluating **x == -5** returns **True**, and **x == 5** returns **False**. Another common usage of == is as in **Solve[x^3-x+1==0,x]** (See Section 1.5 below.)

1.1.4 Referring to Previous Results

Mathematica remembers all of the inputs and outputs in a session, and we can refer to a previous output by using the **percent sign (%)** notation. A single % refers to *Mathematica's* last output, %% refers to the next-to-last output, and so on. %k refers to output line numbered k. For example %45 refers to output number 45.

NOTE: **CTRL +L** reproduces the last input and **CTRL+SHIFT+L** reproduces the last output.

1.2 Basic Calculator Operations

Mathematica uses +, -, *, /, and ^ for **addition, subtraction, multiplication, division** and **power**, respectively. Multiplication can also be performed by leaving a blank space between factors.

Mathematica can convert numerical values to decimal form. The command for this is **N[expr]** or **N[expr,n]**. In most cases, **N[expr]** returns six digits of **expr** or gives **expr** in the form *n.abcd* * 10^m. **N[expr,n]** attempts to give n digits of **expr**. *Mathematica* can perform calculations to arbitrary precision and can handle numbers that are arbitrarily large or small.

Example 3: Here are some typical usages of **N[.]**

In[7]:= **Pi**

Out[7]= π

In[8]:= **N[%]**

Out[8]= 3.14159

Note that the command **N[%]** gives a decimalized version of the last output. Similarly **N[%k]** can be used to get the decimal equivalent of **Output[k]**.

In[9]:= **N[Pi, 200]**

Out[9]= 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482 \
534211706798214808651328230664709384460955058223172535940812848111745028410270193852110555 \
96446229489549303820

Exercise 1.1. Evaluate the following expressions:

a) $103.41+20*76$ b) $\frac{5^2+\pi}{1+\pi}$ c) $\frac{1}{1+\frac{1}{1+\frac{1}{\pi}}}$ d) $\frac{2.06*10^9}{0.99*10^{-8}}$

e) What is the remainder of 1998 divided by 13?

1.3 Functions

We can define our own function in *Mathematica*. The command for defining a function such as $f(x) = \frac{x^2+x+2}{x+1}$ in *Mathematica* is **f[x_]:= $\frac{x^2+x+2}{x+1}$** . Once we evaluate this, *Mathematica* will understand **f[x]** as $\frac{x^2+x+2}{x+1}$. Thus, evaluating **f[a]** returns the value of **f** at **a**, while evaluating **f[{a, b, c, ...}]** returns the values of **f** at **a, b, c, ...** respectively. Here is an example. But first an important **Reminder:** *Mathematica* reads **f(a)** as **f** times **a**.

- Example 4:** (a) Define the function $f(x) = \frac{x^2+x+2}{x+1}$ in *Mathematica*
 (b) Evaluate $f(10)$
 (c) Evaluate $f(x)$, for $x = -3, -2, -1, 0, 1, 2, 3$.

Solution:

```
In[10]:= f[x_] := (x^2 + x + 2) / (x + 1)
```

Observe that the output is not displayed. To check that we have defined our function correctly, we evaluate

```
In[11]:= f[10]
```

```
Out[11]=  $\frac{2 + x + x^2}{1 + x}$ 
```

Since addition is commutative, this is the given function $f(x)$. We now proceed to solve (b) and (c).

```
In[12]:= f[10]
```

```
Out[12]=  $\frac{112}{11}$ 
```

```
In[13]:= f[{-3, -2, -1, 0, 1, 2, 3}]
```

```
Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>
```

```
Out[13]= {-4, -4, ComplexInfinity, 2, 2,  $\frac{8}{3}$ ,  $\frac{7}{2}}$ 
```

Thus $f(10)=112/11$, and the last output shows that $f(-3)=-4$, $f(-2)=-4$, $f(-1)=\text{ComplexInfinity}$, $f(0)=2$, $f(1)=2$, $f(2)=8/3$, and $f(3)=7/2$. Note that *Mathematica* is warning us that it has encountered the undefined expression $1/0$ and gave us ComplexInfinity for $f(-1)$.

Sometimes we may want to deal with an expression such as $\frac{x^2+x+2}{x+1}$ temporarily. To evaluate such expressions at some given value(s) of x , say, $x = 10$ or $x = -3, -2, -1, 0, 1, 2, 3$, we do not need to define the expression as a function. We can use the **substitution** command */.* Here is how it works.

Example 5: Consider the expression $\frac{x^2+x+2}{x+1}$.

- (a) Find the value of this expression when $x = 10$.
 (b) Find the value of this expression for $x = -3, -2, -1, 0, 1, 2, 3$.

Solution:

```
In[14]:= (x^2 + x + 2) / (x + 1) /. x -> 10
```

```
Out[14]=  $\frac{112}{11}$ 
```

```
In[15]:=  $\frac{x^2 + x + 2}{x + 1}$  /. x -> {-3, -2, -1, 0, 1, 2, 3}
Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>
Out[15]:= {-4, -4, ComplexInfinity, 2, 2,  $\frac{8}{3}$ ,  $\frac{7}{2}$ }
```

Exercise 1.2. Enter the following functions into *Mathematica*:

a) $f(x) = 2x^3 - 6x^2 + x - 5$ b) $g(x) = \frac{x^2-1}{x^2+1}$ c) $h(x) = |\sqrt{x} - 3|$

Exercise 1.3. Evaluate the following functions using *Mathematica* and evaluate each function at the given point.

a) $f(x) = 1001 + x^4$ at $x = 25$ b) $1 + \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}$ at $x = \pi$

1.4 Basic Inputs

Mathematica 3.0 allows us to enter mathematical expressions and some of the commonly used commands from 6 different palettes. If we click on the **File** menu and go to **Palettes** we will find 7 options. The first six options contain most of the symbols we need. **BasicInput** is a palette which can be used to enter mathematical expressions involving exponents, rational expressions, radicals, integrals, derivatives, summation, product, matrices, some of the commonly used constants (π , e , i , ∞) and some of the Greek letters.

Example 6: Enter $\int_0^1 \frac{x^3}{\sqrt{2x-1}} dx$ in a notebook.

Solution: One way to enter this expression into a *Mathematica* notebook is as follows: Go to the **BasicInput** palette and click on the definite integration symbol $\int_{\square}^{\square} \square d\square$. Type 0 in the lower box and using the TAB key move to the top box and type 1. Use the TAB to go to the box where the integrand should be typed. While in this box go to the palette and click on $\frac{\square}{\square}$. Go back to the palette and click on \square^{\square} and type x^3 in this box. Go to the box in the denominator and then go to the palette to click on $\sqrt{\square}$ and type $2x-1$. Finally, using the TAB once again, go to the last box and type x .

Exercise 1.4: Enter the following expressions in a notebook.

(a) $\sqrt[3]{80}$ (b) $\frac{\sqrt[5]{1024}}{2^{-3}}$ (c) $\sqrt[3]{\sqrt{125}}$

(d) $\sqrt{\sqrt[3]{10a^7b}}$ (e) $\left(\frac{x^{-3}y^4}{5}\right)^{-3}$ (f) $\left(\frac{3m^{\frac{1}{6}}n^{\frac{1}{3}}}{4n^{\frac{-2}{3}}}\right)^2$

1.5 Basic Calculations

The **BasicCalculations** palette contains several options. These are **Arithmetic and Numbers**, **Algebra**, **Lisits and Matrices**, **Trigonometric and Exponential Functions**, **Calculus**, **Other Functions**, and **Graphics**. Each of these is preceded by a triangle. Click on the shape to see the contents of a palette in a category. In this section, we will discuss the options in **Algebra** and **Calculus** triangle.

1.5.1 Algebra

Algebra is subdivided into three parts: **Solving Equations**, **Polynomial Manipulation** and **Complex Numbers**.

Polynomial Manipulation contains the commands **Expand[]**, **Factor[]**, **Together[]**, **Apart[]**, **Cancel[]**, **Simplify[]**, **FullSimplify[]**.

NOTE: The commands **Expand**, **Factor**, **Together**, **Apart**, **Cancel**, **Simplify**, **FullSimplify**, **TrigExpand**, **TrigFactor**, **TrigReduce**, **ExpToTrig**, **PowerExpand**, and **ComplexExpand** can also be entered from **AlgebraicManipulation** palette of the **Palettes** submenu. However, the commands in **AlgebraicManipulation** palette are applied after entering and highlighting the expressions to which we apply the command and click on the button of the command we want from the palette. The result of a command will replace the input we have entered. In other words, the commands of **AlgebraicManipulation** palette are in-place simplifications.

Solving Equations contains different forms of the **Solve** commands. **Solve[lhs == rhs, var]** is the command to solve the equation **lhs = rhs** for the variable **var**. Thus, **Solve[x² - 1 == 0, x]** is the command to solve $x^2 - 1 = 0$ for x . In such cases where there is only one variable in the equation, we may not need to specify the variable. Thus, **Solve[x² - 1 == 0, x]** and **Solve[x² - 1 == 0]** give the same solutions. **Solve[{ lhs₁ == rhs₁, lhs₂ == rhs₂, ... }, {x₁, x₂, ...}]** solves a system of equations for the variables x_1, x_2, \dots

Numerical approximations of solutions may be obtained by the command **NSolve[lhs==rhs,var]**.

Example 7: Expand each of the following:

(a) $(x+1)(x-1)$ (b) $(x+y-2)(2x-3)$

Solution: The command we need is **Expand[...]**.

In[16]:= **a = Expand [(x + 1) (x - 1)]**

Out[16]= $-1 + x^2$

In[17]:= **b = Expand [(x + y - 2) (2 x - 3)]**

Out[17]= $6 - 7 x + 2 x^2 - 3 y + 2 x y$

Example 8: Factor each of the following:

(a) $x^3 - 2x^2 - 3x$ (b) $4x^{\frac{2}{3}} + 8x^{\frac{1}{3}} + 4$

Solution: We use the `Factor[...]` command.

In[18]:= `a = Factor [x3 - 2 x2 - 3 x]`

Out[18]= $(-3 + x) x (1 + x)$

In[19]:= `b = Factor [4 x2/3 + 8 x1/3 + 4]`

Out[19]= $4 (1 + x^{1/3})^2$

Example 9: Simplify the given expressions.

$$(a) \frac{x^2+4x-12}{3x-6} \quad (b) \frac{\left(\frac{2}{x}-3\right)}{1-\frac{1}{x-1}} \quad (c) x(1-2x)^{\frac{-3}{2}} + (1-2x)^{\frac{-1}{2}}$$

Solution: We will first use `Simplify` or `FullSimplify` command and then apply `Together`.

In[20]:= `a = Simplify [` $\frac{x^2 + 4 x - 12}{3 x - 6}$ `]`

Out[20]= $\frac{6 + x}{3}$

In[21]:= `b = FullSimplify [` $\frac{\frac{2}{x} - 3}{1 - \frac{1}{x-1}}$ `]`

Out[21]= $-3 - \frac{2}{-2 + x} + \frac{1}{x}$

In[22]:= `Together [%]`

Out[22]= $\frac{-2 + 5 x - 3 x^2}{(-2 + x) x}$

In[23]:= `c = FullSimplify [` $x(1 - 2 x)^{\frac{-3}{2}} + (1 - 2 x)^{\frac{-1}{2}}$ `]`

Out[23]= $\frac{1}{\sqrt{1 - 2 x}} + \frac{1}{x [1 - 2 x]^{3/2}}$

In[24]:= `Together [%]`

Out[24]= $\frac{\sqrt{1 - 2 x} + x [1 - 2 x]^{3/2}}{\sqrt{1 - 2 x} x [1 - 2 x]^{3/2}}$

Note that the last result of (c) is not in its simplest form $(1 + x - 2x^2) / (1 - 2x)^{\frac{3}{2}}$. Thus, we should keep in mind that the commands `Simplify` and `FullSimplify` are efficient for polynomials and rational function but may **not** be efficient for expressions involving radicals, trigonometric functions, and exponentials.

Example 10: Perform the indicated operations:

$$(a) -\frac{1}{x} + \frac{2}{x^2+1} + \frac{1}{x^3+x} \quad (b) \left(\frac{5}{y} - \frac{6}{2y+1}\right) \div \left(\frac{5}{y} + 4\right)$$

Solution:

$$\text{In[25]:= a = Together} \left[-\frac{1}{x} + \frac{2}{x^2 + 1} + \frac{1}{x^3 + x} \right]$$

$$\text{Out[25]= } \frac{2 - x}{1 + x^2}$$

$$\text{In[26]:= b = Simplify} \left[\left(\frac{5}{y} - \frac{6}{2y + 1} \right) \div \left(\frac{5}{y} + 4 \right) \right]$$

$$\text{Out[26]= } \frac{1}{1 + 2y}$$

Example 11: Solve the following equations:

$$\text{a) } x^2 - x - 1 = 0 \qquad \text{b) } \begin{cases} 2x + 13y = 4 \\ 3x - 3y = -5 \end{cases}$$

Solution:

$$\text{In[27]:= a = Solve}[x^2 - x - 1 == 0]$$

$$\text{Out[27]= } \left\{ \left\{ x \rightarrow \frac{1}{2} (1 - \sqrt{5}) \right\}, \left\{ x \rightarrow \frac{1}{2} (1 + \sqrt{5}) \right\} \right\}$$

$$\text{In[28]:= b = Solve} [\{ 2x + 13y == 4, 3x - 3y == -5 \}]$$

$$\text{Out[28]= } \left\{ \left\{ x \rightarrow -\frac{53}{45}, y \rightarrow \frac{22}{45} \right\} \right\}$$

Exercise 1.5. Expand each of the following expressions:

$$\text{a) } (x + 1)(x - 1) \qquad \text{b) } (x + y - 2)(2x - 3)$$

Exercise 1.6. Factor each of the following expressions:

$$\text{a) } x^3 - 2x^2 - 3x \qquad \text{b) } 4x^{2/3} + 8x^{1/3} + 3.6 \qquad \text{c) } 6 + 2x - 3x^3 - x^4$$

Exercise 1.7. Simplify the following expressions using both of the commands **Simplify** and **FullSimplify**:

$$\text{a) } \frac{x^2 + 4x - 12}{3x - 6} \qquad \text{b) } \frac{\left(\frac{2}{x} - 3\right)}{1 - \frac{1}{x-1}} \qquad \text{c) } (x(1 - 2x))^{-3/2} + (1 - 2x)^{-1/2}$$

Exercise 1.8. Perform the indicated operations:

$$\text{a) } -\frac{1}{x} + \frac{2}{x^2 + 1} + \frac{1}{x^3 + x} \qquad \text{b) } \left(\frac{5}{y} - \frac{6}{2y + 1} \right) \div \left(\frac{5}{y} + 4 \right)$$

Exercise 1.9. Solve the following equations for x :

$$\text{a) } x^2 - x + 1 = 0 \qquad \text{b) } x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} = 0$$

Exercise 1.10. Approximate the solution of the equation $\tan x + e^x = 0$ near $x = 0$.

1.5.2 Calculus

The **Calculus** palette is subdivided into three parts: **Common Operations**, **Differential Equations** and **Numerical Root Finding**. **Common Operations** contains a palette for integration, derivative, limit, summation, product, and series. The symbols for all of these can also be entered from the **BasicInput** palette, which was discussed in Section 1.4 above.

Differential Equations contains a palette for solving differential equations symbolically as well as numerically. The last part, **Numeric Root Finding**, contains commands for finding numerical solutions for equations by specifying an initial approximation of a root. **FindRoot** [**lhs** == **rhs**, {**x**, **x₀**}] finds a root of the equation **lhs=rhs** in the variable **x** with a first approximation **x₀**. **FindRoot**[**lhs** == **rhs**, {**x**, **x₀**, **a**, **b**}] searches for a root between **a** and **b**. For transcendental equations or for search of a zero of a function near a given point **FindRoot** is recommended. We give an example below.

Example 12: Solve the equation: $\sin 2x + \cos x = 0$

Solution: We will use **Solve** command first and then **FindRoot**.

```
In[29]:= a = Solve[Sin[2 x] + Cos[x] == 0]
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

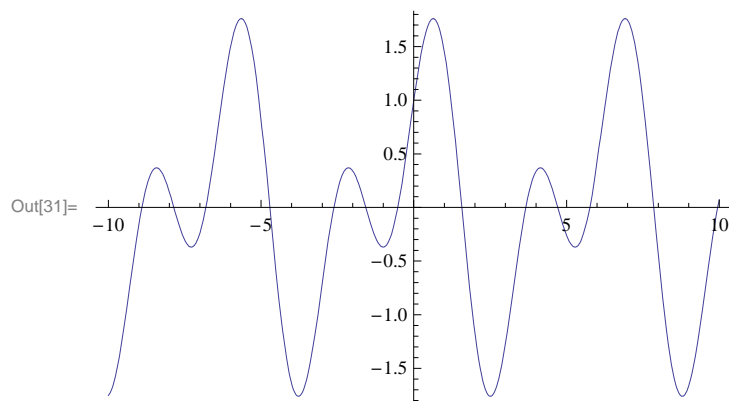
```
Out[29]= {{Cos[x] -> -Sin[2 x]}}
```

```
In[30]:= a = FindRoot[Sin[2 x] + Cos[x] == 0, {x, 0}]
```

```
Out[30]= {x -> -0.523599}
```

Thus, **Solve** does not give a solution while **FindRoot** does. In fact, **FindRoot** yields the solution $x = -0.523599$ or $x = -\pi/6$ for the equation $\sin 2x + \cos x = 0$. This solution is the one closest to $x = 0$. Similarly we can find the solutions of this equation near other points, such as 1, 2, 3, We leave this to the student as an exercise. In the next chapter, we shall learn how to draw graphs of functions. Here is the graph of $f(x) = \sin(2x) + \cos x$.

```
In[31]:= Plot[Sin[2 x] + Cos[x], {x, -10, 10}]
```



CHAPTER TWO

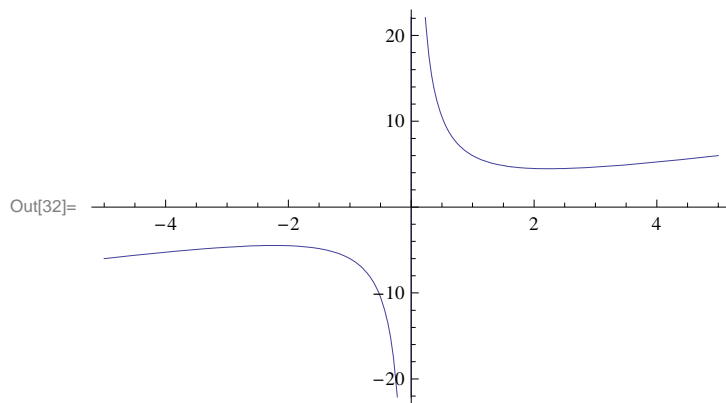
CALCULUS OF ONE VARIABLE

2.1 Graphs of Functions

Plot[**f**,{**x**,**a**,**b**}] generates a plot of **f** as a function of **x** as **x** goes from **a** to **b**. **Plot**[{ f_1, f_2, f_3, \dots }, { **x**, **a**, **b** }] plots the graphs of the functions $f_1, f_2, f_3 \dots$ over the interval [**a**,**b**]. **Plot** has many options most of which we will discuss in this chapter. In *Mathematica* 3.0 **Plot** can be entered from the menu **File-Palette-BasicCalculations-Graphics**.

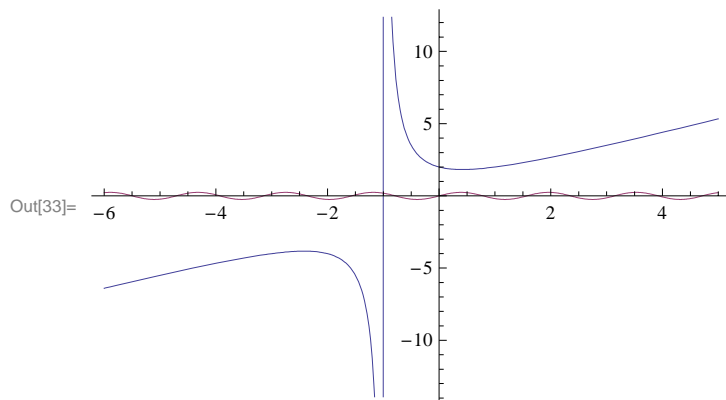
Example 1: Draw the graph of $f(x) = x + \frac{5}{x}$.

```
In[32]:= Plot[x + 5/x, {x, -5, 5}]
```



Example 2: Plot the graphs of the functions $\frac{x^2+x+2}{x+1}$ and $\frac{\sin(4x)}{4}$ on the same set of axes for $-4 \leq x \leq 4$.

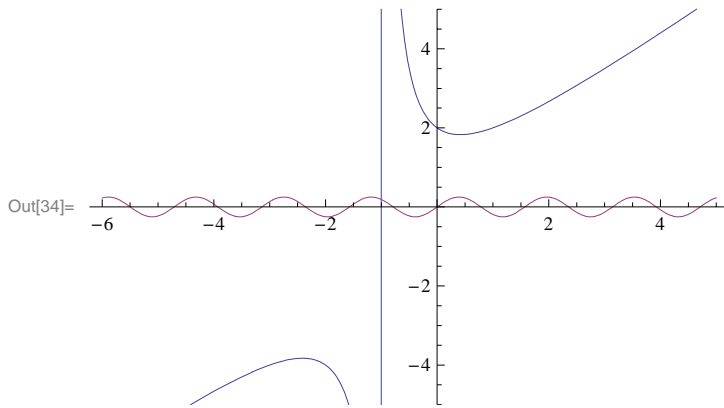
```
In[33]:= Plot[{(x^2 + x + 2)/(x + 1), 1/4 Sin[4 x]}, {x, -6, 5}]
```



Notice that the graph of $(x^2 + x + 2)/(x + 1)$ is not visible in this output since the range from -1 to 1 along the vertical axis is

too small. We can expand this range by using the **PlotRange** option:

```
In[34]:= Plot[{(x^2 + x + 2)/(x + 1), 1/4 Sin[4 x]}, {x, -6, 5}, PlotRange -> {-5, 5}]
```



Exercise 2.1. Plot the graphs of the following functions on the specified interval:

a) $f(x) = x^2 + 1$ on $[-5, 5]$ b) $g(x) = \frac{1}{x-2}$ on $[0, 4]$ c) $h(x) = \frac{\sin x}{x}$ on $[-\pi, \pi]$

Exercise 2.2. Plot the graphs of $f(x) = 0.01 x(x-3)(x+3)$ and $g(x) = 2x+1$ together on the same set of axes and over the interval $[20, 20]$. Use the **PlotRange** option to change the range so that their points of intersection are visible.

2.2 Limits

Limit[f, x -> a] finds the limit of the function **f** when the variable **x** approaches **a**. The command for the left hand-side limit is **Limit[f, x->a, Direction->1]** and for the right hand-side limit is **Limit[f, x->a, Direction->-1]**. If it encounters functions about which it has no specific information, **Limit** returns an unevaluated expression. **Limit** returns an interval to represent a range of possible values of the function when a limit does not exist.

Example 3: Evaluate $\lim_{x \rightarrow 1} \left(\frac{x^2 + x + 2}{x + 1} \right)$

```
In[35]:= Limit[(x^2+x+2)/(x+1), x->1]
```

Out[35]= 2

Example 4: Evaluate $\lim_{x \rightarrow 0} \left(\frac{\sin(4x)}{\tan x} \right)$

```
In[36]:= Limit[Sin[4x]/Tan[x], x->0]
```

Out[36]= 4

Example 5: Evaluate $\lim_{x \rightarrow \infty} \left(\frac{e^x}{x} \right)$

(Note: To enter the Euler number **e**, you can either type **E** or obtain it from the **BasicInput** palette, where you can also find ∞ among others).

In[37]:= `Limit[e^x/x,x->∞]`

Out[37]= ∞

Example 6: Evaluate $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

Note: The natural logarithmic function $\ln x$ is one of the built-in *Mathematica* functions and is entered as `Log[x]`. The logarithmic function $\log_a x$ is entered as `Log[a,x]`.

In[38]:= `Limit[(1/Log[x])-(1/(x-1)),x->1,Direction->-1]`

Out[38]= $\frac{1}{2}$

Example 7: (An example where the limit does not exist.) Evaluate $\lim_{x \rightarrow \infty} (\cos x)$

In[39]:= `Limit[Cos[x],x->Infinity]`

Out[39]= `Interval[{-1, 1}]`

Exercise 2.3. Compute the following limits:

a) $\lim_{x \rightarrow 1} x^2 - \frac{1}{x-1}$ b) $\lim_{x \rightarrow -5} \frac{100}{x+5}$ c) $\lim_{x \rightarrow \infty} \frac{1+x+x^2}{\sqrt[3]{x}10-x}$ d) $\lim_{x \rightarrow 0} \sin \frac{x}{x}$

e) $\lim_{x \rightarrow 0^+} \left(\frac{1-\ln x}{e^x} \right)$ f) $\lim_{x \rightarrow 0^+} (\sin x)^{1/x}$ g) $\lim_{x \rightarrow (\frac{\pi}{2})^-} (\sec 3x \cos 5x)$

Exercise 2.4. Find the derivative of $f(x) = \cos x$ according to the limit definition and evaluate the answer at $x = \pi$.

2.3 Differentiation

All the rules of differentiation are programmed into *Mathematica*. This makes it easy to differentiate complicated expressions. The notation `D[f, x]` calculates the derivative of the function `f` with respect to the variable `x`. We can obtain higher derivatives of order `n`, by using the command `D[f, {x, n}]`. In *Mathematica* 3.0 we can get the command for derivatives from the **File-Palette-BasicCalculations-Calculus-Common Operations**. The symbol ∂_{\square} for the derivative of a functions of one variable is $\partial_{\square} \square$.

Example 8: Find the derivative of $x \sin x$ and the value of the derivative at $x = \frac{\pi}{4}$.

Solution: There are a number of ways to find this derivative. One way is to type in `D[x*Sin[x], x]`. The second method is to go to the **Common Operations** submenu of **Calculus** and click on ∂_{\square} , and in the lower box type `x` and use the **TAB** key to move to the second box and type the function, in this case `x*Sin[x]`. The third way is to type `f=x*Sin[x]` and compute `D[f, x]`. Yet, another way to find the derivative is by defining a function. This is done by typing `f[x_]:= x*Sin[x]`. Now we can just type `f'[a]` to find the derivative of `f` at `a`. It is very important to **Clear** any previous use of the letters you are going to use to denote a function or an expression. Here is an illustration of each of the above methods. In all the methods except **Method 5**, we will use the substitution command `expr /.x-> a` which replaces `x` in `expr` by the value `a`.

Method 1:

```
In[40]:= D[x*Sin[x],x]
```

```
Out[40]= x Cos[x] + Sin[x]
```

```
In[41]:= D[x*Sin[x],x]/.x->Pi/4
```

```
Out[41]=  $\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$ 
```

```
In[42]:= N[%]
```

```
Out[42]= 1.26247
```

Method 2: (Note: $x \sin x$ should be written between parentheses. Can you guess why?)

```
In[43]:= D[x * Sin[x]]
```

```
Out[43]= x Cos[x] + Sin[x]
```

```
In[44]:= D[x * Sin[x]] /. x -> Pi / 4
```

```
Out[44]=  $\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$ 
```

Method 3:

```
In[45]:= Clear[f]
f = x*Sin[x]
```

```
Out[46]= x Sin[x]
```

```
In[47]:= Df= D[f,x]
```

```
Out[47]= x Cos[x] + Sin[x]
```

```
In[48]:= Df /. x->Pi/4
```

```
Out[48]=  $\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$ 
```

Method 4:

```
In[49]:= Clear[f]
f[x_] := x*Sin[x]
```

```
In[51]:= D[f[x],x]
```

```
Out[51]= x Cos[x] + Sin[x]
```

```
In[52]:= D[f[x],x] /. x -> Pi/4
```

```
Out[52]=  $\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$ 
```

Method 5:

```
In[53]:= f'[x]
```

```
Out[53]= x Cos[x] + Sin[x]
```

In[54]:= **f' [Pi/4]**

$$\text{Out[54]} = \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$$

Example 9: Find the second derivative of $x\sqrt{x^2-1}$

Solution: We give two methods:

Method 1: We use the command **D[f, {x, 2}]** for the second derivative of f .

In[55]:= **D[x*Sqrt[x^2-1], {x, 2}]**

$$\text{Out[55]} = \frac{2x}{\sqrt{-1+x^2}} + x \left(-\frac{x^2}{(-1+x^2)^{3/2}} + \frac{1}{\sqrt{-1+x^2}} \right)$$

In[56]:= **Together [%]**

$$\text{Out[56]} = \frac{-3x + 2x^3}{(-1+x^2)^{3/2}}$$

Method 2: We use the prime notation of the derivative by defining a function in *Mathematica* :

In[57]:= **Clear [g]**

$$\mathbf{g[x_]} := \mathbf{x \sqrt{x^2 - 1}}$$

$$\mathbf{g'' [x]}$$

$$\text{Out[59]} = -\frac{x^3}{(-1+x^2)^{3/2}} + \frac{3x}{\sqrt{-1+x^2}}$$

Example 10: Find an equation of the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Solution: We need to find the slope of the tangent line. That is, find $\frac{dy}{dx}$ at $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$. To this end, we rewrite the given equation as $x^2(x^2 + y[x]^2) = y[x]^2$ so that *Mathematica* will understand y as a function of x . We shall call this equation **eq**. Then we solve **deq=D[eq,x]** (this is the derivative of eq with respect to x) for **y'[x]** and substitute x and $y[x]$ by $\frac{\sqrt{2}}{2}$. (Note: An equation in *Mathematica* is entered as ==.) Here are the steps outlined above.

In[60]:= **Clear [eq, deq, x, y]**

In[61]:= **eq =(x^2(x^2+y[x]^2)==y[x]^2)**

$$\text{Out[61]} = x^2 (x^2 + y[x]^2) = y[x]^2$$

In[62]:= **deq=D[eq,x]**

$$\text{Out[62]} = 2x (x^2 + y[x]^2) + x^2 (2x + 2y[x] y'[x]) = 2y[x] y'[x]$$

In[63]:= **soln = Solve[deq,y'[x]]**

$$\text{Out[63]} = \left\{ \left\{ y'[x] \rightarrow \frac{-2x^3 - x y[x]^2}{(-1+x^2) y[x]} \right\} \right\}$$

In[64]:= `m= soln /. {x->Sqrt[2]/2, y[x]->Sqrt[2]/2}`

Out[64]:= $\left\{ \left\{ y' \left[\frac{1}{\sqrt{2}} \right] \rightarrow 3 \right\} \right\}$

The output $\left\{ \left\{ y' \left[\frac{1}{\sqrt{2}} \right] \rightarrow 3 \right\} \right\}$ means that the slope is 3. (Can you see why?) Finally we write the equation of the tangent line as:

In[65]:= `Clear[x, y]`

In[66]:= `y=3(x-Sqrt[2]/2)+Sqrt[2]/2`

Out[66]:= $\frac{1}{\sqrt{2}} + 3 \left(-\frac{1}{\sqrt{2}} + x \right)$

In[67]:= `Together[%]`

Out[67]:= $-\sqrt{2} + 3x$

Thus, the equation of the tangent line is $y = 3x - \sqrt{2}$.

Exercise 2.5. Compute the derivatives of the following functions:

a) $f(x) = 3x^2 + 1$ b) $g(x) = \frac{1}{x^3}$ c) $h(x) = \frac{\sin x}{\cos x}$ d) $y = \ln(\cos(e^x))$ e) $y = e^x \operatorname{arcsec} x$

Exercise 2.6. Evaluate the derivatives of the following functions at the specified values of x :

a) $f(x) = (x-1)(x+1)$ at $x = 1$ b) $g(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$ at $x = 9$

Exercise 2.7. Compute the second derivatives of the functions given in Exercise 2.5.

Exercise 2.8. Find an equation of the line tangent to the graph of $x + y^2 = 0$ at the point $P(9, -3)$.

Exercise 2.9. Find all relative extrema of the function: a) $f(x) = x^4 - 7x^2 + 10$ and b) $f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1}$.

Exercise 2.10. Find all relative extrema of $f(x) = \sqrt{x^2 + 1} - 1/2 x$. Also, find the absolute minimum and absolute maximum values of $f(x)$ on $[0, 2]$.

Exercise 2.11. Use implicit differentiation to find $\frac{dy}{dx}$ if $x^2 y + 3x y^2 - x = 3$. Also find the equation of the tangent line to the graph of the equation at the point $(1, 1)$.

Exercise 2.12. Let $f(x) = e^{\sin x}$ for $-2\pi \leq x \leq 2\pi$

- Define f in *Mathematica*
- Find the solutions of $f'(x) = 0$. (Hint: Use **FindRoot**)
- Identify the relative extrema of f .
- Find the inflection points of f .
- Verify your answers in (b), (c), and (d) by drawing the graph of f .

2.3 Integration

Integrate[**f**, **x**] gives the indefinite integral of **f**. **Integrate**[**f**, {**x**, **a**, **b**}] gives the definite integral of **f** over the interval [**a**, **b**]. Integrate can evaluate integrals of rational functions. It can also evaluate integrals that involve exponential, logarithmic, trigonometric and inverse trigonometric functions, so long as the result comes out in terms of the same set of functions. **Integrate**[**f**, **x**] and **Integrate**[**f**, {**x**, **a**, **b**}] can be entered from (what should by now be) the familiar menu **File-Palette-BasicCalculations-Calculus-Common Operations**. We can get a numerical result by applying **N** to a definite integral. That is, **NIntegrate**[**f**, {**x**, **a**, **b**}] gives a numerical approximation to the definite integral $\int_a^b f(x) dx$.

Example 11: Evaluate $\int x(x^2 + 1)^2 dx$

Solution: We give two methods. In the first method, the input will be entered without using a palette. In the second method we will use a palette for the input.

Method 1:

In[68]:= **Integrate**[**x(x²+1)²,x**]

$$\text{Out[68]} = \frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{6}$$

Method 2:

$$\text{In[69]} := \int \mathbf{x} (\mathbf{x}^2 + \mathbf{1})^2 d\mathbf{x}$$

$$\text{Out[69]} = \frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{6}$$

Question: If you use the substitution method the answer you get is $\frac{1}{6}(1+x^2)^3 + C$. How do we reconcile these differences?

Example 12: Evaluate $\int \frac{x^5 + x^2 + x + 2}{(x^2 + 1)^2} dx$

Solution:

$$\text{In[70]} := \int \frac{\mathbf{x}^5 + \mathbf{x}^2 + \mathbf{x} + 2}{(\mathbf{x}^2 + \mathbf{1})^2} d\mathbf{x}$$

$$\text{Out[70]} = \frac{1}{2} \left(x^2 + \frac{-2+x}{1+x^2} + 3 \text{ArcTan}[x] - 2 \text{Log}[1+x^2] \right)$$

Note: All functions in an output appear in *Mathematica's* notation. To convert a function in an output to a more familiar form use the command **TraditionalForm**. For the above example the "traditional" form is obtained as follows. (Note that $\log x$ is the same as $\ln x$.)

$$\text{In[71]:= } \int \frac{x^5 + x^2 + x + 2}{(x^2 + 1)^2} dx // \text{TraditionalForm}$$

Out[71]/TraditionalForm=

$$\frac{1}{2} \left(x^2 + 3 \tan^{-1}(x) - 2 \log(x^2 + 1) + \frac{x-2}{x^2 + 1} \right)$$

Example 13: Approximate $\int_0^1 \tan x^2 dx$

Solution: The first thing we want to do is let *Mathematica* evaluate the integral. If *Mathematica* returns an unevaluated integral, we use either `N[%]` (a command for numerical approximation of the previous output) or `NIntegrate[f, {x, a, b}]`.

$$\text{In[72]:= } \int_0^1 \text{Tan}[x^2] dx$$

$$\text{Out[72]= } \int_0^1 \text{Tan}[x^2] dx$$

In[73]:= `N[%]`

Out[73]= 0.398414

The above two steps can be combined as follows:

In[74]:= `NIntegrate[Tan[x^2], {x, 0, 1}]`

Out[74]= 0.398414

Exercise 2.13. Integrate each of the following and check your work by differentiating the answer.

$$\text{a) } \int (x^2 + 2) dx \quad \text{b) } \int \cos 3x dx \quad \text{c) } \int \frac{x}{x^2+1} dx \quad \text{d) } \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

Exercise 2.14. Evaluate each of the following integrals.

$$\text{a) } \int_0^1 \sqrt{1-x^2} dx \quad \text{b) } \int_{-\pi}^{\pi} \sin^2 x dx \quad \text{c) } \int_2^{\infty} \frac{1}{\sqrt[3]{x(x-1)}} dx$$

Exercise 2.15. Approximate the area under the curve $y = \sqrt{1-x^2}$ and above the interval $[0, 1]$. (How does this answer compare with the answer in part c, Exercise 2.14 a)?)

Exercise 2.16. Compute the arc length of the curve:

$$\text{a) } y = \sqrt{1-x^2} \text{ along the interval } [0, 1]. \quad \text{b) } y = 1/6 x^3 + 1/(2x) \text{ on } [1,3].$$

Exercise 2.17. a) Compute the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

b) Sketch the region that is bounded by the graphs of $f(x) = 3^x$ and $g(x) = 2x+1$ and find the area of the region.

CHAPTER THREE

PARAMETRIC EQUATIONS AND POLAR GRAPHS

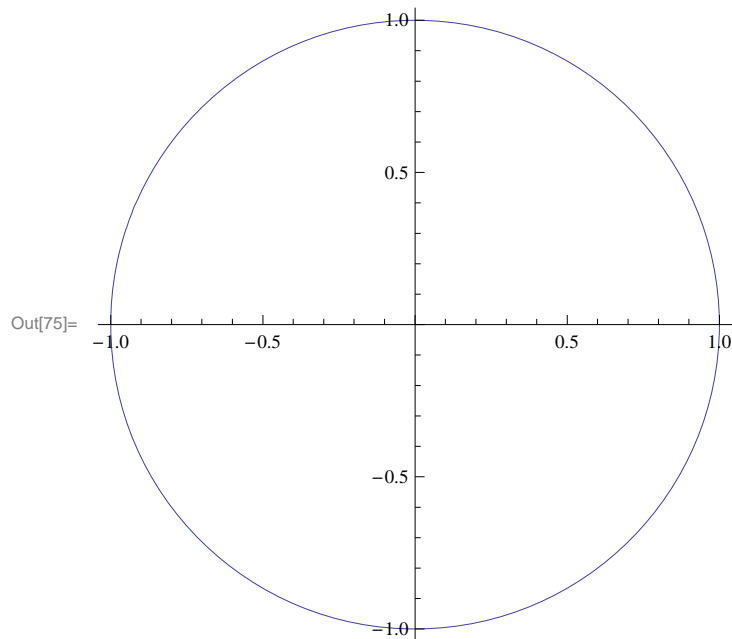
3.1 PARAMETRIC CURVES

Example 1: Sketch the curve described by the parametric equation

$$x = \cos t; \text{ and } y = \sin t, \quad 0 \leq t \leq 2\pi.$$

Solution :

```
In[75]:= ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 π}]
```



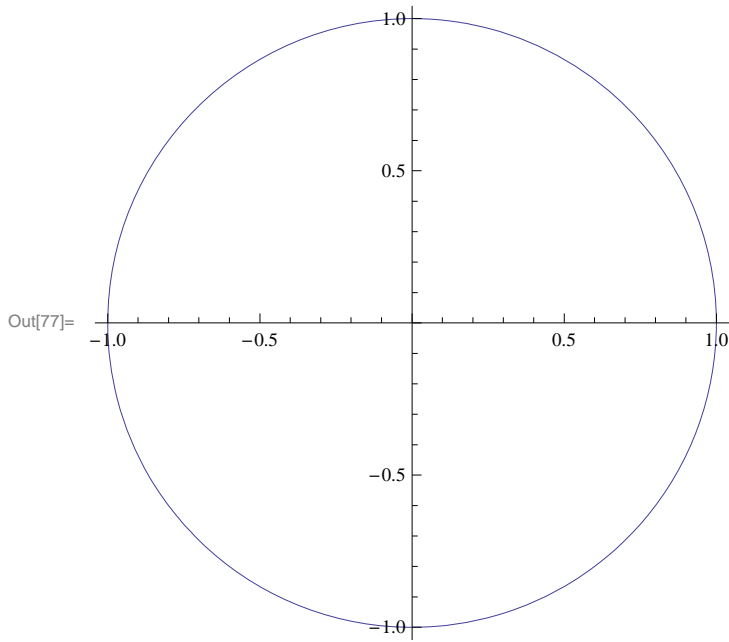
Recall that the above parametric equations represent the unit circle. However, *Mathematica* produced a graph that visually looks like an ellipse due to the different settings of the x and y-axes. An important option to fix this is to use the option **AspectRatio**. The default value of aspect ratio, which is the ratio of the height to the width of a plot, is **1/GoldenRatio** and its value is

```
In[76]:= N[1 / GoldenRatio]
```

```
Out[76]= 0.618034
```

This value was chosen because it produces plots that are pleasing to the eye. However, to get a plot that accurately represents the graph of the given equations we need to set aspect ratio to automatic. We do this as follows.

```
In[77]:= ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 π}, AspectRatio -> Automatic]
```

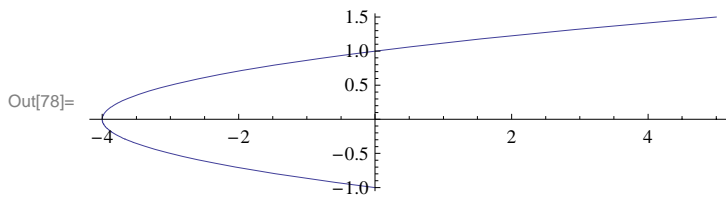


Example 2: Sketch the curve described by the parametric equation

$$x = t^2 - 4; \text{ and } y = \frac{t}{2}, -2 \leq t \leq 3.$$

Solution :

```
In[78]:= ParametricPlot[{t^2 - 4, t/2}, {t, -2, 3}]
```

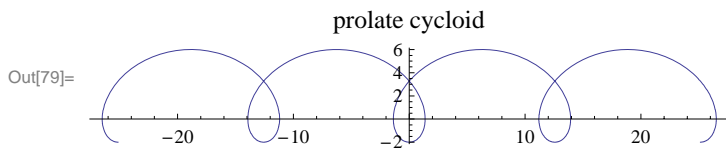


Example 3: Sketch the curve (*prolate cycloid*) described by the parametric equations

$$x = 2\theta - 4 \sin\theta \text{ and } y = 2 - 4 \cos\theta.$$

Solution:

```
In[79]:= ParametricPlot[{2 θ - 4 Sin[θ], 2 - 4 Cos[θ]}, {θ, -4 π, 4 π}, PlotLabel -> "prolate cycloid"]
```



Note that in the above input we have used the the command **PlotLabel**. In general, the command **PlotLabel ->"text"** prints the title **text** for the given plot.

Example 4: *Folium of Descartes* Given the parametric equations

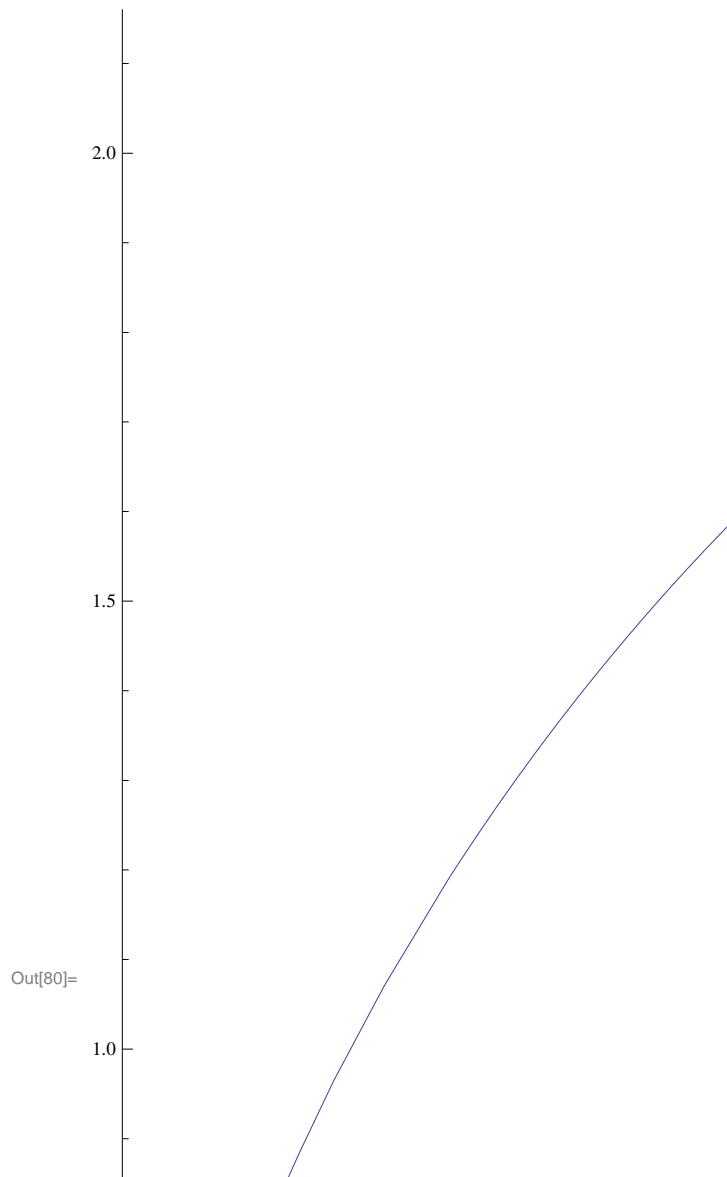
$$x = \frac{4t}{1+t^3} \text{ and } y = \frac{4t^2}{1+t^3}, \text{ perform each of the following.}$$

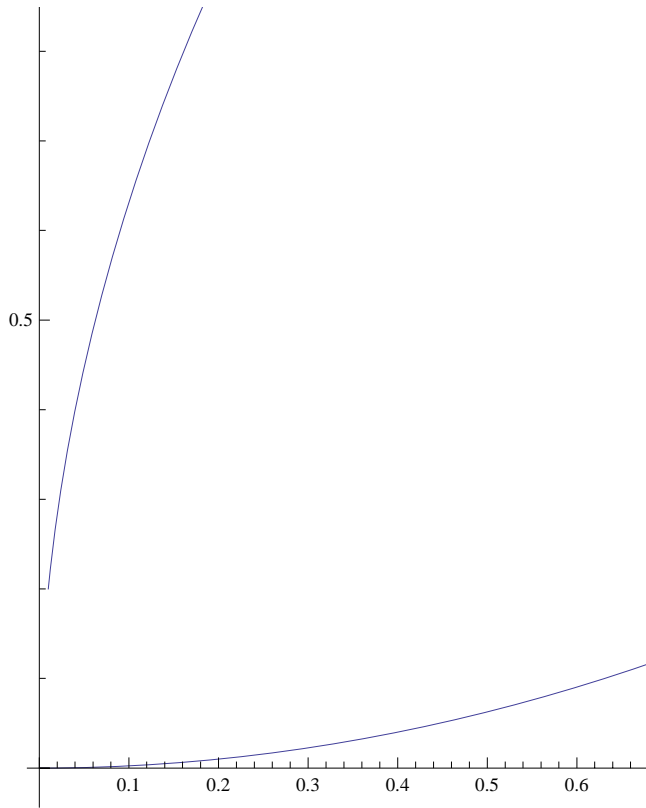
- Sketch the curve described by the parametric equations.
- Find the points of horizontal tangency to the curve.

Solution :

```
In[80]:= ParametricPlot[{{ $\frac{4t}{1+t^3}$ ,  $\frac{4t^2}{1+t^3}$ }, {t, 0, 20}},
  AspectRatio -> Automatic, PlotLabel -> "Folium of Descartes"]
```

Folium of Descartes





b) In order to find the points of horizontal tangency, we find the points where the slope of the tangent line is zero. Hence, we evaluate

$$\text{In[81]:= Solve}\left[D\left[\frac{4 t^2}{1+t^3}, t\right] == 0, t\right]$$

$$\text{Out[81]= } \left\{ \{t \rightarrow 0\}, \{t \rightarrow -(-2)^{1/3}\}, \{t \rightarrow 2^{1/3}\}, \{t \rightarrow (-1)^{2/3} 2^{1/3}\} \right\}$$

Thus, there are two points, corresponding to $t = 0$ and $t = 2^{1/3}$, at which the tangent lines are horizontal.

Example 5: Find the arc length of the curve $x = e^{-t} \cos t$, $y = e^{-t} \sin t$; $0 \leq t \leq \frac{\pi}{2}$.

Solution:

$$\text{In[82]:= } \int_0^{\pi/2} \sqrt{\left(\left(D[E^{-t} \cos[t], t]\right)^2 + \left(D[E^{-t} \sin[t], t]\right)^2\right) dt}$$

$$\text{Out[82]= } \sqrt{2} \left(1 - e^{-\pi/2}\right)$$

$$\text{In[83]:= N[\%]}$$

$$\text{Out[83]= } 1.12023$$

Exercise 3.1. Sketch the curve represented by the parametric equations.

a) $x = t^3$, $y = t^2/2$ b) $x = 2(\theta - \sin \theta)$, $y = 1 - \cos \theta$ c) $x = 3\cos^3 \theta$, $y = 3\sin^3 \theta$

Exercise 3.2. Find all points of horizontal and vertical tangency to the curve $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$, $0 \leq \theta \leq 2\pi$.

Exercise 3.3. Consider the parametric equations given by $x = 3 \cos(t/3) - \cos t$ and $y = 3 \sin(t/3) - \sin t$.

- a) Graph the curve represented by the parametric equations.
- b) Find the slope of the tangent line at the point where $t = \pi/4$.
- c) Find the length of the arc from $t=0$ to $t = 3\pi/2$.

3.2 POLAR CURVES

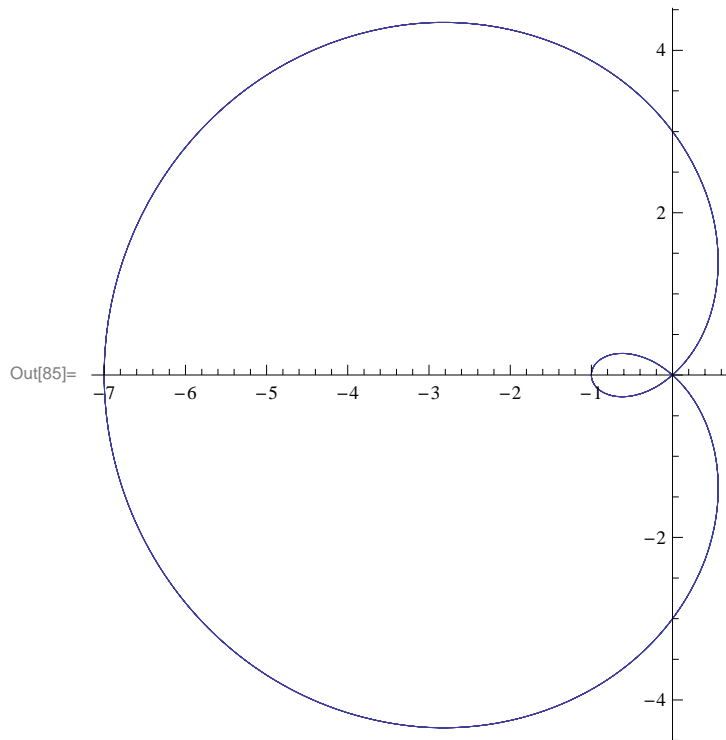
In order to graph polar equations in *Mathematica*, we must load the package for polar plotting, We do this prior to plotting and by typing:

```
In[84]:= Needs["BarCharts`"]; Needs["Histograms`"]; Needs["PieCharts`"]
```

Example 6: Sketch the graph of $r = 3 - 4 \cos\theta$.

Solution:

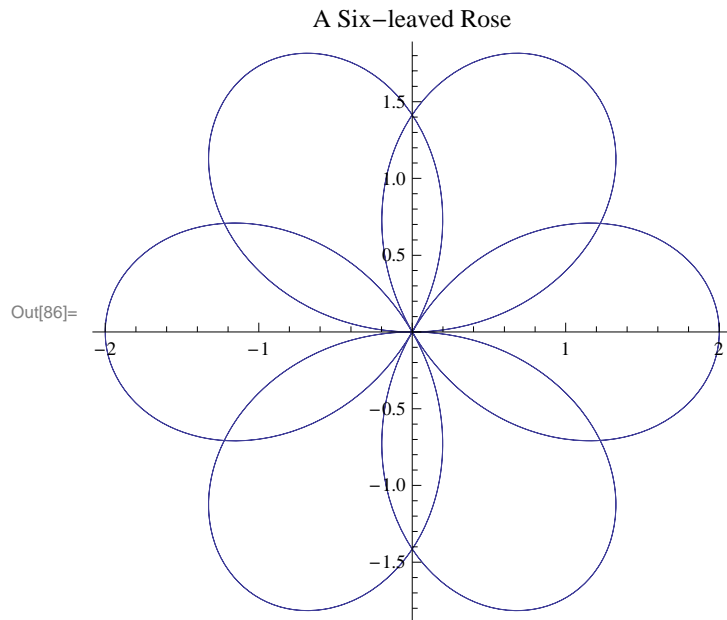
```
In[85]:= PolarPlot[3 - 4 Cos[\theta], {\theta, -4 \pi, 4 \pi}, AspectRatio -> Automatic]
```



Example 7: Sketch the graph of the six – leaf rose $r = 2 \cos\left(\frac{3\theta}{2}\right)$.

Solution :

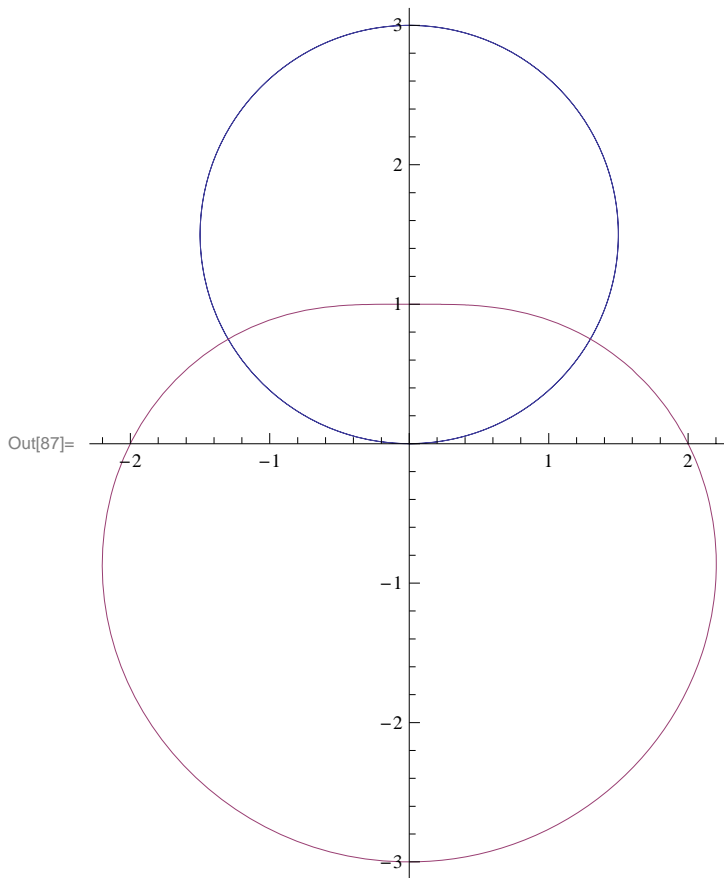
```
In[86]:= PolarPlot[2 Cos[ $\frac{3 \theta}{2}$ ], { $\theta$ , -4  $\pi$ , 4  $\pi$ },
  PlotLabel -> "A Six-leaved Rose", AspectRatio -> Automatic]
```



Example 8: Find the area of the region that is inside the circle $r = 3 \sin \theta$ and outside the convex limaçon $r = 2 - \sin \theta$.

Solution: First, we plot the two polar curves on the same axes.

```
In[87]:= PolarPlot[{3 Sin[θ], 2 - Sin[θ]}, {θ, 0, 2 π}, AspectRatio → Automatic]
```



Next, we find the values of θ that correspond to the two points of intersection of the above curves.

```
In[88]:= Solve[3 Sin[θ] == 2 - Sin[θ], θ]
```

```
Solve::ifun :
```

```
Inverse functions are being used by Solve, so some solutions may not be found;  
use Reduce for complete solution information. >>
```

```
Out[88]= {{θ →  $\frac{\pi}{6}$ }}
```

Note that *Mathematica* gives only the solution $\theta = \frac{\pi}{6}$ which lies in the first quadrant since inverse functions are being used. We can see from the above graph that the other point of intersection must be at $\theta = \frac{5\pi}{6}$. Thus, the area enclosed is given by (using symmetry)

```
In[89]:= Integrate[(3 Sin[θ])^2, {θ, Pi / 6, Pi / 2}] -  
Integrate[(2 - Sin[θ])^2, {θ, Pi / 6, Pi / 2}]
```

```
Out[89]=  $-\frac{3}{8}(-5\sqrt{3} + 4\pi) + \frac{3}{8}(3\sqrt{3} + 4\pi)$ 
```

```
In[90]:= N [%]
```

```
Out[90]= 5.19615
```

Exercise 3.4. Graph the given polar equation and find an interval for θ over which the graph is traced only once.

a) $r = 3 - 4 \cos \theta$ b) $r = 2 + \sin \theta$ c) $r = 3 \cos(3\theta/2)$ d) $r = 5 \sin 2\theta$

Exercise 3.5. Consider the rose curve $r = \cos(2\theta)$ for $-2\pi \leq \theta \leq 2\pi$.

- Graph the curve.
- Find the area of one petal of the curve.

Exercise 3.6. Generate the butterfly curve $r = e^{\cos \theta} - 2 \cos(4\theta) + \sin^5(\theta/12)$.

Exercise 3.7. Graph and find the area of each of the following regions.

- The common interior of $r = 3 - 2 \sin \theta$ and $r = -3 + 2 \sin \theta$
- Inside $r = 2(1 + \cos \theta)$ and outside $r = 2 \cos \theta$
- Inner loop of $r = 3 + 4 \sin \theta$

Exercise 3.8. Find the length of the given curve on the specified interval.

a) $r = 1 + \sin \theta$, $0 \leq \theta \leq 2\pi$. b) $r = 6(1 + \cos \theta)$, $0 \leq \theta \leq 2\pi$.

Exercise 3.9. Consider the polar equations $r = 4 \sin \theta$ and $r = 2(2 - \sin^2 \theta)$.

- Graph the polar equations on the same axes.
- Find the points of intersection of the curves.
- Find the circumference of each curve.

CHAPTER FOUR

VECTOR CALCULUS

4.1 VECTOR ALGEBRA

Vectors are typed in braces with commas between components. For example if \mathbf{a} is the vector

$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ we type

```
In[91]:= a={1, 3, -5}
```

```
Out[91]= {1, 3, -5}
```

The j th components of \mathbf{a} is denoted by $\mathbf{a}[[j]]$. Vector addition, subtraction and scalar multiplications are performed in a similar way as with numbers. The **dot product** of vectors \mathbf{a} and \mathbf{b} is obtained by $\mathbf{a} \cdot \mathbf{b}$ and the length of \mathbf{a} is obtained by $\text{Sqrt}[\mathbf{a} \cdot \mathbf{a}]$.

Example 1: Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Find $3\mathbf{a} - 2\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$, $\|\mathbf{b}\|$, $\|\mathbf{a}\|$, and the third component of $5\mathbf{a} + 4\mathbf{b}$.

Solution: First we will enter the components of \mathbf{a} and \mathbf{b} as lists. Then we enter the solutions for all the problems in one cell: (Note: It is possible to enter more than one input under the same cell but on different lines. To go to a new line press the [RETURN] key. When *Mathematica* computes such inputs the outputs will appear in rows with consecutive output numbers.)

```
In[92]:= a={1,3,-5}
         b={-1,2,3}

Out[92]= {1, 3, -5}

Out[93]= {-1, 2, 3}

In[94]:= 3 a - 2 b
         a.b
         Sqrt[a.a]
         Sqrt[b.b]
         (5 a + 4 b) [[3]]

Out[94]= {5, 5, -21}

Out[95]= -10

Out[96]=  $\sqrt{35}$ 

Out[97]=  $\sqrt{14}$ 

Out[98]= -13
```

Example 2: Show that the dot product of vectors is commutative and that it is distributive over addition, that is, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ and $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ for any vectors \mathbf{a} , \mathbf{b} and \mathbf{c} .

Solution: We will show commutativity of dot product for vectors in space.

```
In[99]:= a={a1,a2,a3}
         b={b1,b2,b3}
         c={c1,c2,c3}
         a.b-b.a
         a.(b+c)-(a.b+a.c) (*if this is equal to 0, this proves the
                             distributive property*)

Out[99]= {a1, a2, a3}

Out[100]= {b1, b2, b3}

Out[101]= {c1, c2, c3}

Out[102]= 0

Out[103]= -a1 b1 - a2 b2 - a3 b3 - a1 c1 + a1 (b1 + c1) - a2 c2 + a2 (b2 + c2) - a3 c3 + a3 (b3 + c3)
```

It is not clear why the last output is zero. We check this by entering:

```
In[104]:= Simplify[%]

Out[104]= 0
```

Example 3: Let $\mathbf{u} = i + 3j + 2k$, $\mathbf{v} = 2i + 3j$ and $\mathbf{w} = -2i + 3j + 4k$. Find $\mathbf{u} \times \mathbf{v}$, $\mathbf{i} \times \mathbf{w}$ and the scalar triple product of \mathbf{u} , \mathbf{v} , and \mathbf{w} (that is $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$)

Solution: In order to calculate the **cross product** of two vectors we need a package, called `Calculus`VectorAnalysis``. It is recommended that you load this package before starting your work. The command to load this package is `Needs["Calculus`VectorAnalysis`"]`. To compute the cross product $\mathbf{u} \times \mathbf{v}$ of \mathbf{u} and \mathbf{v} we type `CrossProduct[u,v]`. In *Mathematica 3.0*, there is

a palette that contains the command for cross product. It is in **File-Palettes-Lists and Matrices-Matrix Operations** and is denoted by `Cross[□,□]`. First, we load the package:

```
In[105]:= Needs["VectorAnalysis`"]
```

```
In[106]:= u={1,3,2}
          v={-2,3,0}
          w={-2,3,4}
          i={1,0,0}
```

```
Out[106]= {1, 3, 2}
```

```
Out[107]= {-2, 3, 0}
```

```
Out[108]= {-2, 3, 4}
```

```
Out[109]= {1, 0, 0}
```

```
In[110]:= prod= CrossProduct[u,v]
          CrossProduct[i,w]
          prod.w
```

```
Out[110]= {-6, -4, 9}
```

```
Out[111]= {0, -4, 3}
```

```
Out[112]= 36
```

Example 4: Consider the parallelepiped with sides $\mathbf{a}=\mathbf{i}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $\mathbf{c}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

a) Find the volume. b) Find the area of the face determined by \mathbf{b} and \mathbf{c} .

Solution:

```
In[113]:= a = {1, 0, 1}
```

```
          b = {2, 1, 3}
```

```
          c = {1, 1, -2}
```

```
          volume = Abs[a.Cross[b, c]]
```

```
          area =  $\frac{\sqrt{\text{Cross}[b, c].\text{Cross}[b, c]}}{2}$ 
```

```
Out[113]= {1, 0, 1}
```

```
Out[114]= {2, 1, 3}
```

```
Out[115]= {1, 1, -2}
```

```
Out[116]= 4
```

```
Out[117]=  $\frac{5\sqrt{3}}{2}$ 
```

Exercise 4.1. Let $\mathbf{u}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{v}=5\mathbf{i}+4\mathbf{j}-\mathbf{k}$.

a) Determine each of the following. i) $6\mathbf{u}-4\mathbf{v}$ ii) $\|\mathbf{u}+\mathbf{v}\|$ iii) $\mathbf{u}\cdot\mathbf{v}$

b) Find $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$.

c) Find a unit vector that is orthogonal to both \mathbf{u} and \mathbf{v} .

c) Find the area of the triangle determined by \mathbf{u} and \mathbf{v} .

Exercise 4.2. Find the volume of the parallelepiped that is determined by the vectors $\mathbf{u}=\mathbf{i}+\mathbf{j}$ and $\mathbf{v}=\mathbf{j}+\mathbf{k}$ and $\mathbf{w}=\mathbf{i}+\mathbf{k}$.

Exercise 4.3. Let $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$, $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$ be vectors in space. Prove that:

a) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$

4.2 VECTOR VALUED FUNCTIONS

The derivative of a vector valued function $\mathbf{r}(t) = \{ \mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t) \}$ of a single variable t can be obtained by entering $\mathbf{D}[\mathbf{r}, t]$. Another way is to type $\mathbf{MapThread}[\mathbf{D}, \{\mathbf{r}, \{t, t, t\}\}]$. $\mathbf{MapThread}$ applies the operator \mathbf{D} (i.e., differentiation) to each of the three components of \mathbf{r} with respect to the corresponding parameter in $\{t, t, t\}$, so that we get $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$.

Example 5: Find the first and second derivative of $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k}$.

Solution: We will give two different methods. First, we want to make an observation. Once you use letters like a, b, x, y , etc. to represent a certain quantity, *Mathematica* will keep using those values for every input containing those letters. Thus, before using the same letter for another representation we should tell *Mathematica* to clear the previous usage of the letters. The command for this is (naturally) $\mathbf{Clear}[\dots]$. For both methods we define \mathbf{r} in *Mathematica*.

Method 1:

```
In[118]:= Clear[a, b]
In[119]:= r[t_] = {a Cos[t], a Sin[t], b t}
Out[119]= {a Cos[t], a Sin[t], b t}
In[120]:= r'[t]
           r''[t]
Out[120]= {-a Sin[t], a Cos[t], b}
Out[121]= {-a Cos[t], -a Sin[t], 0}
```

Method 2:

```
In[122]:= r1 = MapThread[D, {r[t], {t, t, t}}]
           r2 = MapThread[D, {r1, {t, t, t}}]
Out[122]= {-a Sin[t], a Cos[t], b}
Out[123]= {-a Cos[t], -a Sin[t], 0}
```

Example 6: Represent the hyperbola $4x^2 - 9y^2 = 36$, $x \geq 3$, in parametric form and find the equation of the tangent line at $(3, 0)$.

Solution: One way of representing the curve in parametric form is by using hyperbolic functions and letting $x = 3 \cosh t$, and $y = 2 \sinh t$.

To find the equation of the tangent line we proceed as follows:

```
In[124]:= Clear[r, t]
In[125]:= r[t_] := {3 Cosh[t], 2 Sinh[t]}
```

We solve $r(t) = (3,0)$ to find the value of t that corresponds to the point $(3,0)$. Once we find this value, say t_0 , we use the formula $q(w) = r(t_0) + w r'(t_0)$ for the tangent line.

```
In[126]:= Solve[r[t] == {3, 0}, t]
```

```
Solve::ifun :
```

```
Inverse functions are being used by Solve, so some solutions may not be found;  
use Reduce for complete solution information. >>
```

```
Out[126]= {{t -> 0}}
```

```
In[127]:=
```

```
Clear[w]  
r[0]  
r'[0]  
q = r[0] + w r'[0]
```

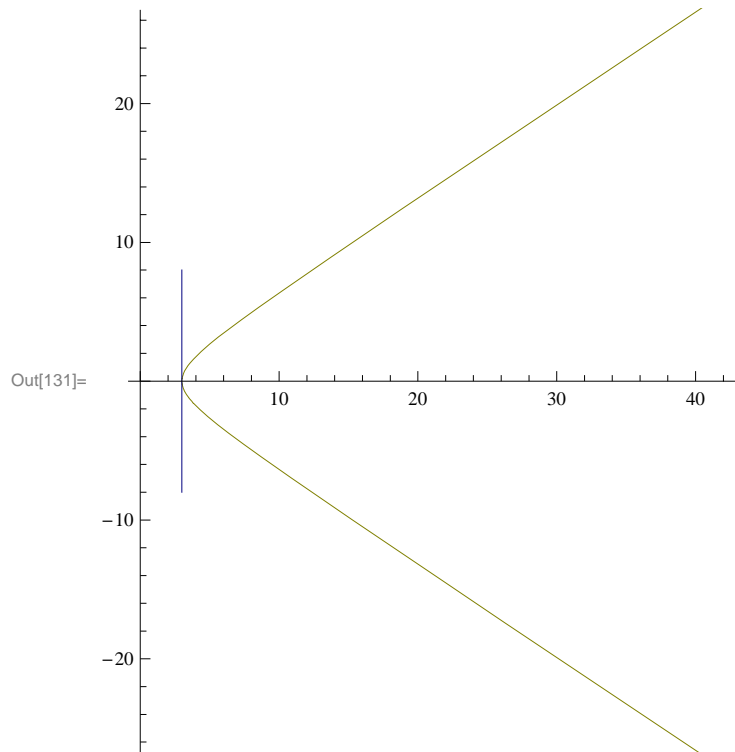
```
Out[128]= {3, 0}
```

```
Out[129]= {0, 2}
```

```
Out[130]= {3, 2 w}
```

Thus, a parametric equation of the tangent line is $x=3$, and $y=2w$. The following is the graph of the curve and the tangent line at $(3,0)$.

```
In[131]:= ParametricPlot[{{3 Cosh[t], 2 Sinh[t]}, {3, 2 t}}, {t, -4, 4},  
PlotStyle -> {{RGBColor[0.500008, 0.500008, 0]}, {RGBColor[0, 0, 0.500008]}},  
AspectRatio -> Automatic]
```



Example 7: Consider the curve given by $\mathbf{r}(t)=a \cos t \mathbf{i}+a \sin t \mathbf{j}+c t \mathbf{k}$.

- a) Find the length of the curve from $(a,0,0)$ to $(a,0,2\pi c)$.
- b) Find the arc length parametrization of the curve.
- c) Find its curvature.

Solution: We begin by defining $\mathbf{r}(t)$ in *Mathematica*.

In[132]:= `Clear[a, c]`

In[133]:= `r[t_] := {a Cos[t], a Sin[t], c t}`

a) Note that when we move from $(a,0,0)$ to $(a,0,2\pi c)$, t goes from $t=0$ to $t=2\pi$. Thus the length of the curve between these two points is

In[134]:=
$$l = \int_0^{2\pi} \sqrt{\mathbf{r}'[t] \cdot \mathbf{r}'[t]} dt$$

Out[134]= $2 \sqrt{a^2 + c^2} \pi$

b) The arc length parameter s is given by:

In[135]:= `Clear[t, u]`

$$s = \int_0^t \sqrt{\mathbf{r}'[u] \cdot \mathbf{r}'[u]} du$$

Out[136]= $\sqrt{a^2 + c^2} t$

Solving for t we see that $t = \frac{s}{\sqrt{a^2+c^2}}$. Thus the arc length parametrization of the curve is

$$\mathbf{r}(s) = a \cos\left(\frac{s}{\sqrt{a^2+c^2}}\right) \mathbf{i} + a \sin\left(\frac{s}{\sqrt{a^2+c^2}}\right) \mathbf{j} + \frac{cs}{\sqrt{a^2+c^2}} \mathbf{k}.$$

c) We first use the arc length parametrization of the curve to compute its curvature.

In[137]:= `Clear[a, c, s]`

In[138]:= `r[s_] := {a Cos[$\frac{s}{\sqrt{a^2+c^2}}$], a Sin[$\frac{s}{\sqrt{a^2+c^2}}$], $\frac{c s}{\sqrt{a^2+c^2}}$ }`

In[139]:= `T[s_] := $\frac{\mathbf{r}'[s]}{\sqrt{\mathbf{r}'[s] \cdot \mathbf{r}'[s]}}$`

Thus, the curvature is given by:

In[140]:= `κ = $\sqrt{\mathbf{T}'[s] \cdot \mathbf{T}'[s]}$`

Out[140]=
$$\sqrt{\left(\frac{a^2 \cos^2\left[\frac{s}{\sqrt{a^2+c^2}}\right]^2}{(a^2+c^2)^2 \left(\frac{c^2}{a^2+c^2} + \frac{a^2 \cos^2\left[\frac{s}{\sqrt{a^2+c^2}}\right]^2}{a^2+c^2} + \frac{a^2 \sin^2\left[\frac{s}{\sqrt{a^2+c^2}}\right]^2}{a^2+c^2} \right)} + \frac{a^2 \sin^2\left[\frac{s}{\sqrt{a^2+c^2}}\right]^2}{(a^2+c^2)^2 \left(\frac{c^2}{a^2+c^2} + \frac{a^2 \cos^2\left[\frac{s}{\sqrt{a^2+c^2}}\right]^2}{a^2+c^2} + \frac{a^2 \sin^2\left[\frac{s}{\sqrt{a^2+c^2}}\right]^2}{a^2+c^2} \right)} \right)}$$

In[141]:= **Simplify[%]**

$$\text{Out[141]= } \sqrt{\frac{a^2}{(a^2 + c^2)^2}}$$

Method 2: A formula for computing the curvature of a curve that is not parametrized by arc length is given by :

$$\kappa(\mathbf{t}) = \frac{\| \mathbf{r}'(\mathbf{t}) \times \mathbf{r}''(\mathbf{t}) \|}{\| \mathbf{r}'(\mathbf{t}) \|^3}. \text{ We evaluate this below.}$$

In[142]:= $\mathbf{r}[\mathbf{t}_] := \{\mathbf{a} \text{Cos}[\mathbf{t}], \mathbf{a} \text{Sin}[\mathbf{t}], \mathbf{c} \mathbf{t}\}$ $\mathbf{k} =$

$$\frac{1}{(\mathbf{r}'[\mathbf{t}] \cdot \mathbf{r}'[\mathbf{t}])^{\frac{3}{2}}} \left(\sqrt{(\text{CrossProduct}[\mathbf{r}'[\mathbf{t}], \mathbf{r}''[\mathbf{t}]] \cdot \text{CrossProduct}[\mathbf{r}'[\mathbf{t}], \mathbf{r}''[\mathbf{t}]])} \right)$$

$$\text{Out[143]= } \frac{\sqrt{a^2 c^2 \text{Cos}[\mathbf{t}]^2 + a^2 c^2 \text{Sin}[\mathbf{t}]^2 + (a^2 \text{Cos}[\mathbf{t}]^2 + a^2 \text{Sin}[\mathbf{t}]^2)^2}}{(c^2 + a^2 \text{Cos}[\mathbf{t}]^2 + a^2 \text{Sin}[\mathbf{t}]^2)^{3/2}}$$

In[144]:= **Simplify[%]**

$$\text{Out[144]= } \frac{\sqrt{a^4 + a^2 c^2}}{(a^2 + c^2)^{3/2}}$$

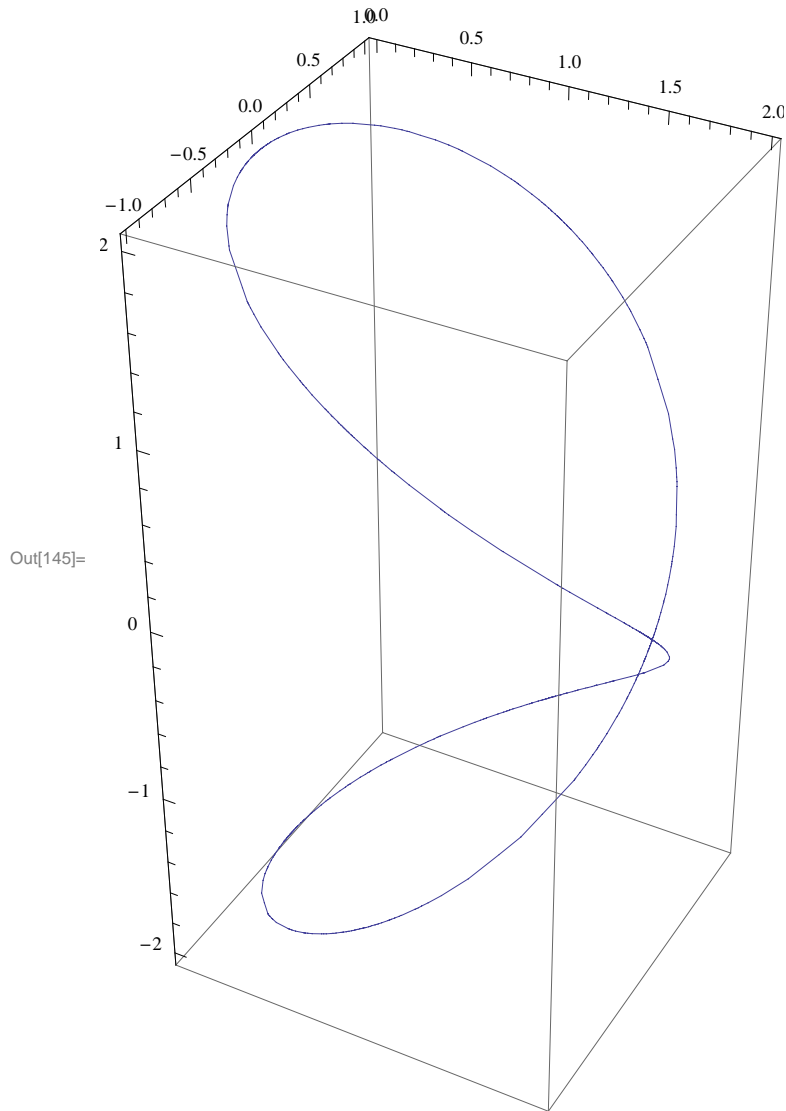
From this, it can easily be shown that $\kappa = \frac{a}{a^2 + c^2}$

Example 8: The curve that is represented by the vector equation $\mathbf{r}(t) = a(1 + \cos(t)) \mathbf{i} + a \sin(t) \mathbf{j} + 2a \sin(t/2) \mathbf{k}$ is known as *Vidiani's curve*. Consider the curve for $a=1$.

- Plot the curve.
- Compute the length of the curve from $t=0$ to $t=\pi$.
- Calculate the curvature of the curve and compute the curvature at $t=0$.

Solution: To plot a curve in three dimensions, we use the command `ParametricPlot3D`. The syntax for this is **ParametricPlot3D[{x,y,z},{t, tmin,tmax}]**.

In[145]:= ParametricPlot3D[$\left\{1 + \cos[t], \sin[t], 2 \sin\left[\frac{t}{2}\right]\right\}, \{t, 0, 4 \pi\}$]



In[146]:= $\mathbf{r}[t_] := \{1 + \cos[t], \sin[t], 2 \sin[t/2]\}$

$$l = \int_0^{\pi} \sqrt{\mathbf{r}'[t] \cdot \mathbf{r}'[t]} dt$$

Out[147]= $2\sqrt{2} \text{EllipticE}\left[\frac{1}{2}\right]$

In[148]:= N[%]

Out[148]= 3.8202

In[149]:=

 $\kappa =$

$$\frac{1}{(\mathbf{r}'[t] \cdot \mathbf{r}'[t])^{3/2}} \left(\sqrt{(\text{CrossProduct}[\mathbf{r}'[t], \mathbf{r}''[t]] \cdot \text{CrossProduct}[\mathbf{r}'[t], \mathbf{r}''[t]])} \right);$$

In[150]:= **Simplify[%]**

$$\text{Out[150]= } \frac{\sqrt{13 + 3 \text{Cos}[t]}}{(3 + \text{Cos}[t])^{3/2}}$$

Thus, the curvature at $t=0$ is given by

In[151]:= **% /. t -> 0**

$$\text{Out[151]= } \frac{1}{2}$$

Exercise 4.4. Graph the space curve that is represented by the vector-valued function.

a) $\mathbf{r}(t) = 1/2 t \mathbf{i} + \sqrt{t} \mathbf{j} + 1/4 t^3 \mathbf{k}$ b) $\mathbf{r}(t) = t \mathbf{i} + \ln t \mathbf{j} + 1/2 t^2 \mathbf{k}$

Exercise 4.5. Sketch and find the length of the curve:

$$\mathbf{r}(t) = (\sin t - t \cos t) \mathbf{i} + (\cos t + t \sin t) \mathbf{j} + t^2 \mathbf{k}.$$

Exercise 4.6. Find the curvature of the curve $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + 1/2 t^2 \mathbf{k}$.

Exercise 4.7. Plot the following pairs of intersecting lines in space and find the point of intersection.

$$L_1: x = 2t + 3, y = 5t - 2, z = -t + 1; \quad L_2: x = -2t + 7, y = t + 8, z = 2t - 1.$$

CHAPTER FIVE

FUNCTIONS OF SEVERAL VARIABLES

5.1 GRAPHS OF FUNCTIONS OF TWO VARIABLES

The following command loads all graphics packages of *Mathematica*.

In[152]:= **Needs["Graphics`Master`"]**

Get::noopen : Cannot open Graphics`Master`. >>

Needs::nocont : Context Graphics`Master` was not created when Needs was evaluated. >>

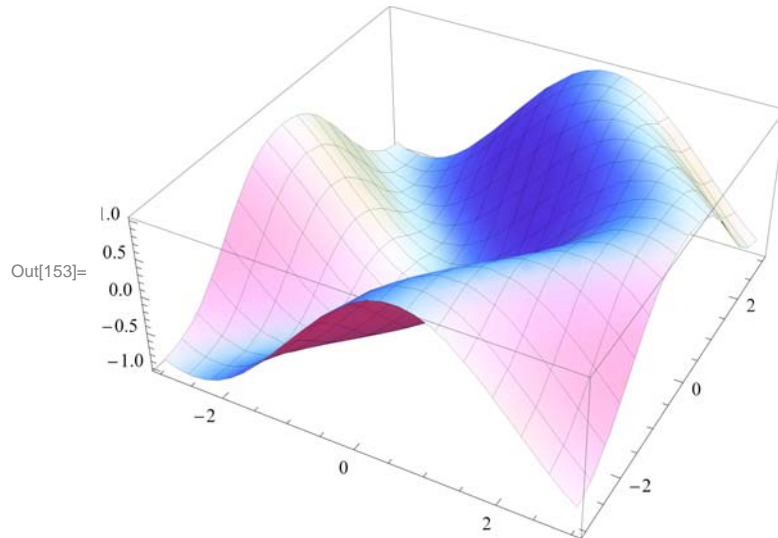
Out[152]= \$Failed

5.1.1 Plot3D

Plot3D is the three-dimensional analog of the **Plot** command. Given an expression in two variables and the domains for those two variables, **Plot3D** produces a surface plot. **Plot3D** has many options. We will discuss some of these options below. We begin with the simplest way of plotting a graph of a function of two variables.

Example 1: Plot $f(x,y) = \sin(x - \cos y)$.

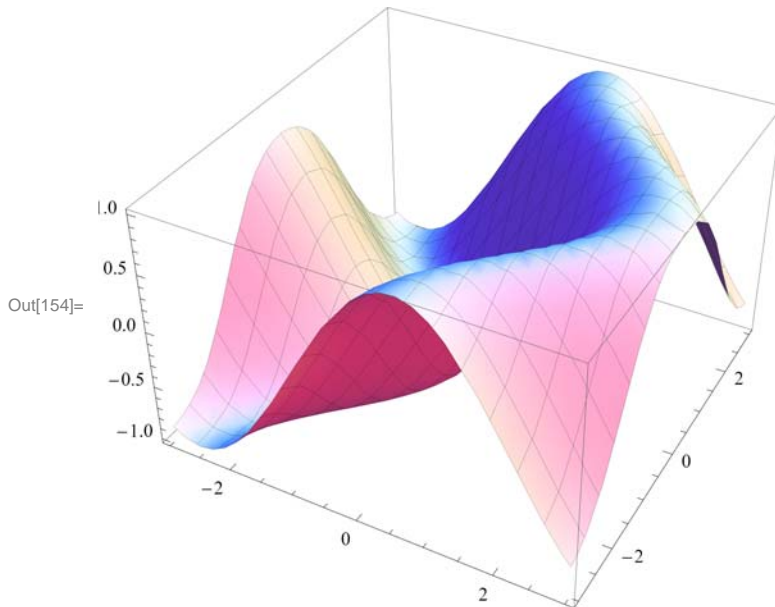
```
In[153]:= Plot3D[Sin[x - Cos[y]], {x, -3, 3}, {y, -3, 3}]
```



The first option we consider is **BoxRatios**. **BoxRatios** is an option in **Plot3D** used to specify the ratio of the lengths of the sides of the box. This is analogous to specifying the **AspectRatio** of a two-dimensional plot. For **Plot3D** the default is **BoxRatios->Automatic**. Below, we plot the graph of the function in example 1 with a different **BoxRatio**.

Example 2:

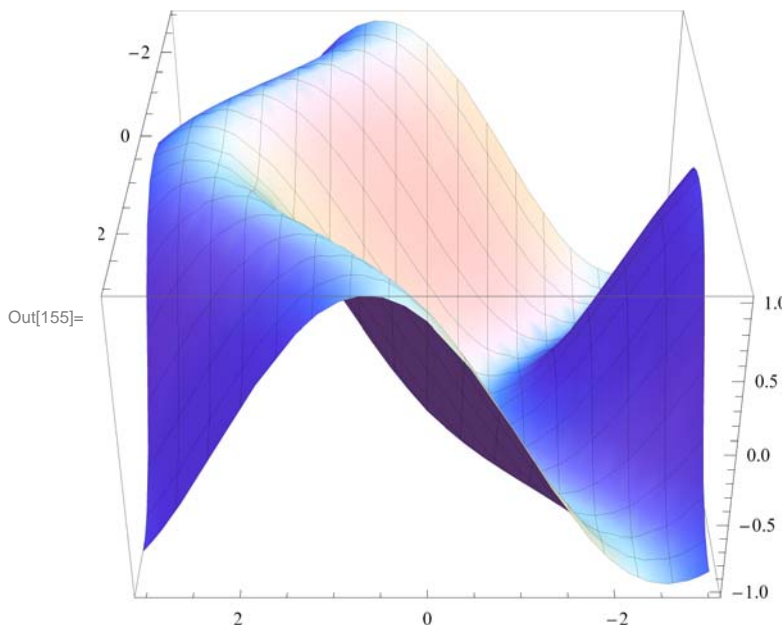
```
In[154]:= Plot3D[Sin[x - Cos[y]], {x, -3, 3}, {y, -3, 3}, BoxRatios -> {1, 1, 0.62`}]
```



Next, we consider the **ViewPoint** option. **ViewPoint** selects the point in space from which the observer looks at the object. Its default value is $\{1.3, -2.4, 2.0\}$. This may be changed by entering a new value directly as an option or can be entered from **3D ViewPoint Selector** of the **Input** menu. We give an example below.

Example 3:

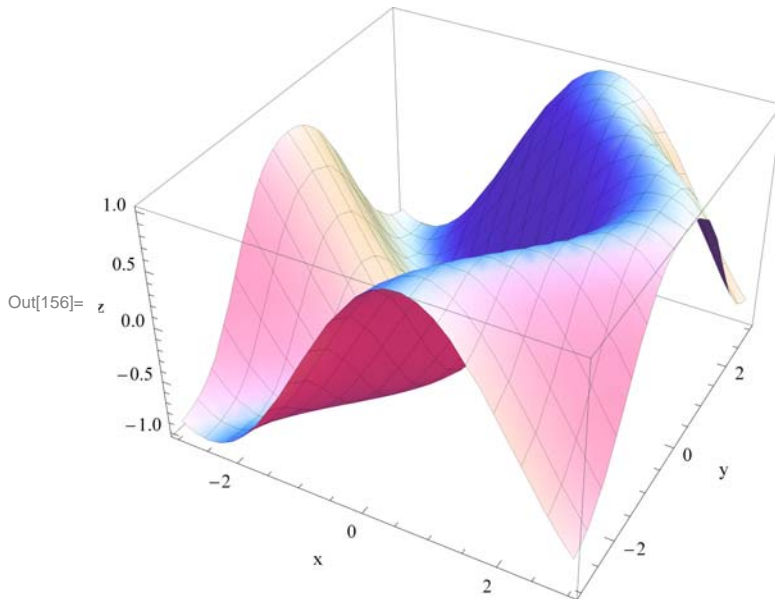
```
In[155]:= Plot3D[Sin[x - Cos[y]], {x, -3, 3}, {y, -3, 3}, ViewPoint -> {0, 3, 2}, BoxRatios -> {1, 1,
```



Our third option is **AxesLabel**. It is a command used to label the axes in plotting.

Example 4:

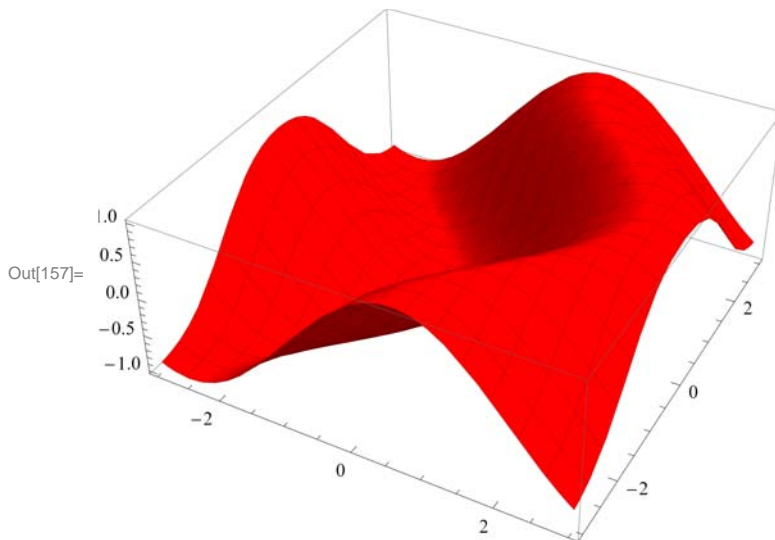
```
In[156]:= Plot3D[Sin[x - Cos[y]], {x, -3, 3}, {y, -3, 3}, BoxRatios -> {1, 1, 0.62`}, AxesLabel -> {'
```



The fourth option we discuss is **ColorFunction**. **ColorFunction** is an option for various graphics functions which specifies a function to apply to z values to determine the color to use for a particular x, y region. It is an option for **Plot3D**, **ListPlot3D**, **DensityPlot**, **ContourPlot** and related functions. The arguments provided for the function specified by **ColorFunction** are always in the range 0 to 1. **ColorFunction -> Automatic** yields a range of gray levels. **ColorFunction -> Hue** yields a range of colors. Another way of applying color to a surface is to select the color you want from **Color Selector** of the **Input** menu.

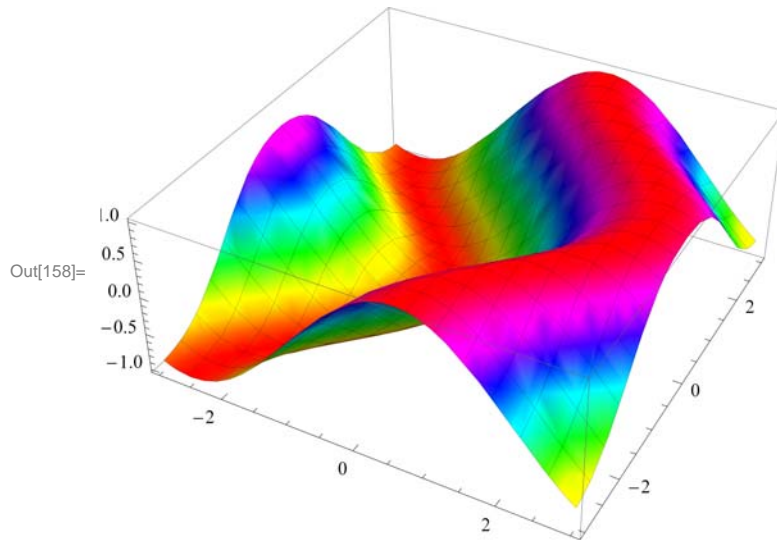
Example 5:

```
In[157]:= Plot3D[Sin[x - Cos[y]], {x, -3, 3}, {y, -3, 3}, ColorFunction -> (RGBColor[0.996109`, 0,
```



Example 6:

```
In[158]:= Plot3D[Sin[x - Cos[y]], {x, -3, 3}, {y, -3, 3}, ColorFunction -> Hue]
```



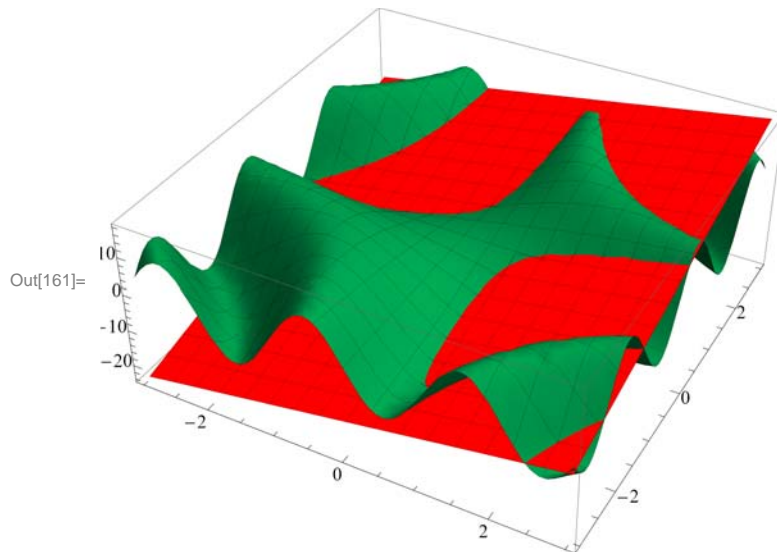
Show is another option of **Plot3D**. **Show[graphics, options]** displays two- and three-dimensional graphics, using the options specified. **Show["\(\(*StyleBox["\(\(\(g\)\)\", "TI\)\)"_1, "\(\(*StyleBox["\(\(\(g\)\)\", "TI\)\)"_2, ...]** shows several plots combined.

Example 7: Plot the graphs of $f(x,y) = 3x + 4y - 3$ and $g(x,y) = 10 \sin(xy)$ on the same axes.

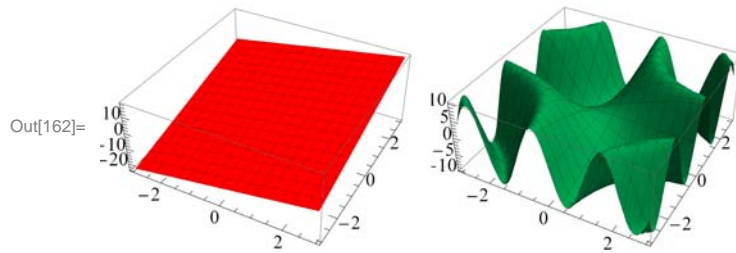
```
In[159]:= a = Plot3D[3 x + 4 y - 3, {x, -3, 3}, {y, -3, 3}, ColorFunction -> (RGBColor[1.`, 0.`, 0.`] .
```

```
In[160]:= b = Plot3D[10 Sin[x y], {x, -3, 3}, {y, -3, 3}, ColorFunction -> (RGBColor[0.`, 0.502`, 0
```

```
In[161]:= Show[a, b, DisplayFunction -> $DisplayFunction]
```



```
In[162]:= Show[GraphicsRow[{a, b}]]
```



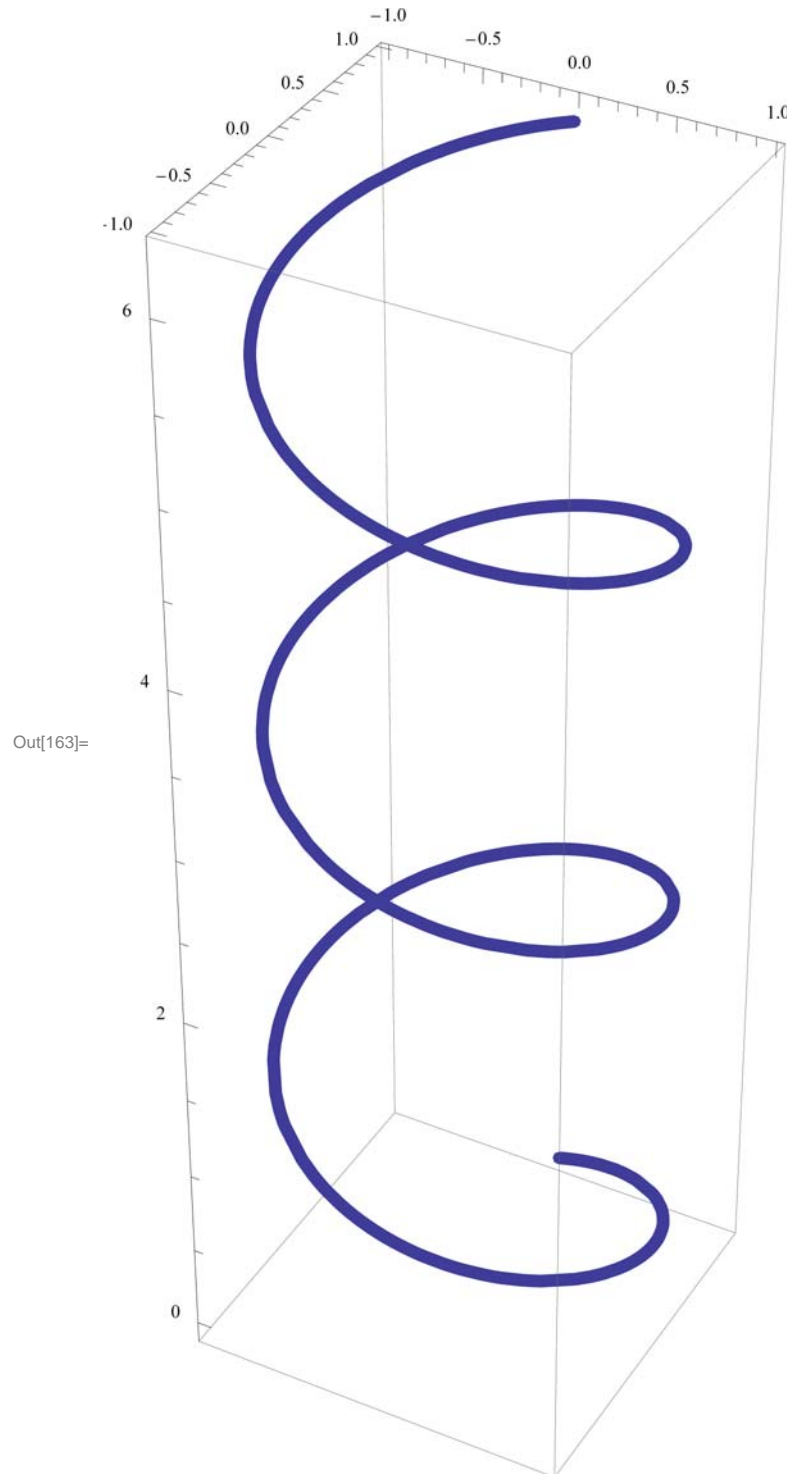
5.1.2 ParametricPlot3D

ParametricPlot3D is a direct analog of **ParametricPlot**. Depending on the input, **ParametricPlot3D** produces a space curve or a surface. **ParametricPlot3D**[{f, g, h}, {t, a, b}] produces a three dimensional space curve parametrized by a variable **t** which runs from **a** to **b**. **ParametricPlot3D**[{f, g, h}, {t, a, b}, {u, c, d}] produces a two-dimensional surface parametrized by **t** and **u**. Options are given to **ParametricPlot3D** the same way as for **Plot3D**. Most of the options are the same.

Example 8: Plot the curve that is parametrized by $x = \sin t$, $y = \cos t$, and $z = t/3$.

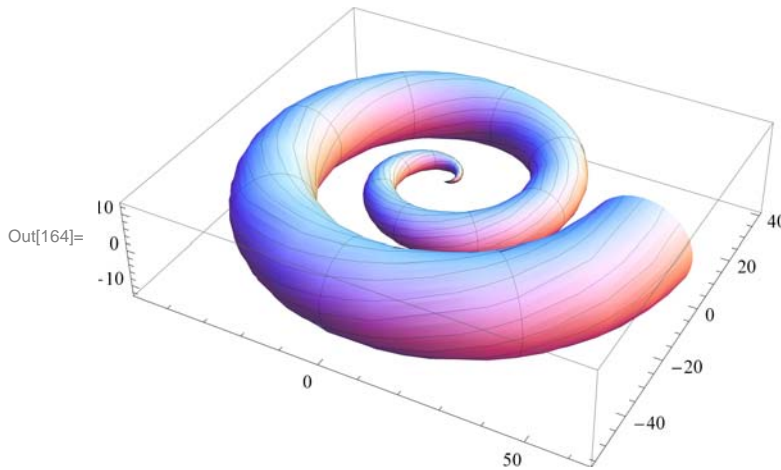
```
In[163]:= ParametricPlot3D[{Sin[t], Cos[t],  $\frac{t}{3}$ , Thickness[0.02`]}, {t, 0, 6  $\pi$ }
```

ParametricPlot3D::legacycolfunc : Use ColorFunction to specify coloring. >>



Example 9: Plot the surface that is parametrized by $x = u \cos u (4 + \cos(u + v))$, $y = u \sin u (4 + \cos(u + v))$, and $z = u \sin(u + v)$.

```
In[164]:= ParametricPlot3D[{u Cos[u] (4 + Cos[u + v]), u Sin[u] (4 + Cos[u + v]), u Sin[u + v]},
    {u, 0, 4 π}, {v, 0, 2 π}]
```



Exercise 5.1. Plot each of the following.

- a) $z = x^2$, $-3 \leq x \leq 3$, $-3 \leq y \leq 3$.
- b) $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.
- c) $f(x,y) = e^{\sin(x - 2 \cos y)}$, $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.
- d) $z(x,y) = \sin(x + \sin(y))$ for $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$.

Exercise 5.2. Graph the portion of the helicoid (spiral ramp) that is defined by:
 $x = u \cos v$, $y = u \sin v$, $z = v$ for $0 \leq u \leq 2$ and $0 \leq v \leq \pi$.

Exercise 5.3. Use parametric plot to graph the surface $f(x,y) = \frac{1}{12} \sqrt{144 - 16x^2 - 9y^2}$.

[Hint: $z = \sqrt{1 - (x/3)^2 - (y/4)^2}$ and so let $x = 3 r \cos \theta$, $y = 4 r \sin \theta$. Then, $z = \sqrt{1 - r^2}$.]

5.2 Partial Derivatives and Gradient

5.2.1 Partial Derivatives

The command for the partial derivative of f with respect to x is $D[f, x]$, and $D[f, \{x, n\}]$ gives the n th partial derivative of f with respect to x . The multiple(mixed) partial derivative of f with respect to x_1, x_2, x_3, \dots is obtained by $D[f, x_1, x_2, x_3, \dots]$. In *Mathematica* 3.0 we can access this command from **File- Palettes- BasicInputs** or **File- Palettes- BasicCalculations- Calculus- Common Operations**. The symbols are ∂_{\square} and $\partial_{\square, \square}$.

Example 10: Find the first partial derivatives of $x^3 + y^2$ with respect to x and y .

Solution: We give two methods.

Method 1: We can type all the inputs and the command as follows:

```
In[165]:= Clear[x, y]
          D[x^3 + y^2, x]
```

```
Out[166]= 3 x^2
```

```
In[167]:= D[x^3 + y^2, y]
```

```
Out[167]= 2 y
```

Method 2: We can use the **BasicInput** palette to enter the inputs.

```
In[168]:=  $\partial_x (x^3 + y^2)$ 
```

```
Out[168]= 3 x^2
```

```
In[169]:=  $\partial_y (x^3 + y^2)$ 
```

```
Out[169]= 2 y
```

Example 11: Find the four second partial derivatives of $x^3 \sin y + e^{xy}$.

Solution:

Method 1:

```
In[170]:= D[x^3 * Sin[y] + E^(x * y), {x, 2}]
```

```
Out[170]= e^{x y} y^2 + 6 x Sin[y]
```

```
In[171]:= D[x^3 * Sin[y] + E^(x * y), x, y]
```

```
Out[171]= e^{x y} + e^{x y} x y + 3 x^2 Cos[y]
```

```
In[172]:= D[x^3 * Sin[y] + E^(x * y), y, x]
```

```
Out[172]= e^{x y} + e^{x y} x y + 3 x^2 Cos[y]
```

```
In[173]:= D[x^3 * Sin[y] + E^(x * y), {y, 2}]
```

```
Out[173]= e^{x y} x^2 - x^3 Sin[y]
```

Method 2:

```
In[174]:=  $\partial_{x,x} (x^3 * \text{Sin}[y] + e^{x*y})$ 
```

```
Out[174]= e^{x y} y^2 + 6 x Sin[y]
```

```
In[175]:=  $\partial_{x,y} (x^3 * \text{Sin}[y] + e^{x*y})$ 
```

```
Out[175]= e^{x y} + e^{x y} x y + 3 x^2 Cos[y]
```

```
In[176]:=  $\partial_{y,x} (x^3 * \text{Sin}[y] + e^{x*y})$ 
```

```
Out[176]= e^{x y} + e^{x y} x y + 3 x^2 Cos[y]
```

```
In[177]:=  $\partial_{y,y} (x^3 * \text{Sin}[y] + e^{x*y})$ 
```

```
Out[177]= e^{x y} x^2 - x^3 Sin[y]
```

Example 12: Evaluate the first partial derivatives of $xy + yz^2 + xz$ at $(-1, 2, 3)$.

Solution: Recall that `Expr /. {var1 -> val1, var2 -> val2, var3 -> val3, ... }` is the command for substituting `var1` by `val1`, `var2` by `val2`, `var3` by `val3`, in `Expr`.

```
In[178]:= Clear[x, y, z]
          D[x*y + y*z^2 + x*z, x] /. {x -> -1, y -> 2, z -> 3}
```

Out[179]= 5

```
In[180]:= D[x*y + y*z^2 + x*z, y] /. {x -> -1, y -> 2, z -> 3}
```

Out[180]= 8

```
In[181]:= D[x*y + y*z^2 + x*z, z] /. {x -> -1, y -> 2, z -> 3}
```

Out[181]= 11

Example 13: Let $f(x,y,z) = ye^{-x} + x \ln z e^{-y}$. Find f_{xxx} , f_{xyz} , f_{xzz} , f_{zxx} , and f_{zzx}

Solution: First we define $f(x,y,z)$ in *Mathematica*. Then we use the $\partial_{\square, \square}$ notation. Since the palette gives only two boxes for the variables, we need to add one more box. This can be done by using **CTRL + .**

```
In[182]:= Clear[x, y, z, f]
```

```
In[183]:= f[x_, y_, z_] := y * e^x + x * Log[z] * e^-y
```

```
In[184]:= D[f[x, y, z], {x, x, x}]
```

Out[184]= $e^x y$

```
In[185]:= D[f[x, y, z], {x, y, z}]
```

Out[185]= $-\frac{e^{-y}}{z}$

```
In[186]:= D[f[x, y, z], {x, z, z}]
```

Out[186]= $-\frac{e^{-y}}{z^2}$

```
In[187]:= D[f[x, y, z], {z, x, z}]
```

Out[187]= $-\frac{e^{-y}}{z^2}$

```
In[188]:= D[f[x, y, z], {z, z, x}]
```

Out[188]= $-\frac{e^{-y}}{z^2}$

Example 14: Let $x = t^2 + s$, $y = t + s^2$ and $z = x \sin y$. Find the first partial derivatives of z with respect to s and t .

Solution:

```

In[189]:= x = t2 + s
          y = t + s2
          z = x * Sin[y] (* This gives Z as a function of t and s explicitly *)

Out[189]= s + t2

Out[190]= s2 + t

Out[191]= (s + t2) Sin[s2 + t]

In[192]:= D[z, s]

Out[192]= 2 s (s + t2) Cos[s2 + t] + Sin[s2 + t]

In[193]:= D[z, t]

Out[193]= (s + t2) Cos[s2 + t] + 2 t Sin[s2 + t]

```

5.2.2 Directional Derivative and Gradient

The notation for a vector such as $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$ in *Mathematica* is $\{2,5,-6\}$. The command for the **dot product** of two vectors \mathbf{u} and \mathbf{v} is obtained by typing $\mathbf{u.v}$

Example 15: Find the gradient and directional derivative of $f(x, y) = x^2 \sin 2y$ at the point $(1, \frac{\pi}{2}, 0)$ in the direction of $\mathbf{v} = (\frac{3}{5}, \frac{-4}{5})$

Solution: The gradient of f , ∇f , at (a,b) can be obtained by evaluating $\nabla f = (\partial_x f, \partial_y f)(a, b)$. We will discuss other commands for obtaining the gradient in the near future. The directional derivative of f at (\mathbf{a}, \mathbf{b}) in the direction of a **unit** vector \mathbf{v} is given by $\mathbf{v} \cdot \nabla f(\mathbf{a}, \mathbf{b})$.

Method 1:

```

In[194]:= Clear[x, y]

In[195]:= f=x2*Sin[2y]
          v={3/5, -4/5}

Out[195]= x2 Sin[2 y]

Out[196]= { $\frac{3}{5}, -\frac{4}{5}$ }

In[197]:= ∇f = {D[f, x], D[f, y]} /. {x -> 1, y -> Pi/2}

Out[197]= {0, -2}

In[198]:= direcderiv = ∇f.v

Out[198]=  $\frac{8}{5}$ 

```

Method 2:

In[199]:= `Clear[f]`

In[200]:= `f[x_, y_] := x^2 * Sin[2 y]`

$$\mathbf{v} = \left\{ \frac{3}{5}, \frac{-4}{5} \right\}$$

Out[201]= $\left\{ \frac{3}{5}, -\frac{4}{5} \right\}$

In[202]:= `∇f = {∂x f[x, y], ∂y f[x, y]} /. {x -> 1, y -> $\frac{\pi}{2}$ }`

Out[202]= {0, -2}

In[203]:= `direcderiv = v.∇f`

Out[203]= $-\frac{8}{5}$

Example 16: Find the gradient and directional derivative of $f(x,y,z) = xy + yz + xz$ at the point $(1, 1, 1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

Solution:

In[204]:= `Clear[x, y, z]`

In[205]:= `w = x*y + y*z + x*z`
`v = {2, 1, -1}`

Out[205]= $xy + xz + yz$

Out[206]= {2, 1, -1}

In[207]:= `unitvector = v / Sqrt[v.v]`

Out[207]= $\left\{ \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\}$

In[208]:= `∇w = {D[w, x], D[w, y], D[w, z]} /. {x -> 1, y -> 1, z -> 1}`

Out[208]= {2, 2, 2}

In[209]:= `direcderiv = unitvector.∇w`

Out[209]= $2\sqrt{\frac{2}{3}}$

Example 17: Let the temperature T at a point (x,y) on a metal plate be given by $T = \frac{x}{x^2+y^2}$.

- Plot the level curves.
- Plot the graph of the temperature.
- Find the rate of change of temperature at $(3,4)$, in the direction of $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$
- Find the unit vector in the direction of which the temperature increases most rapidly at $(3,4)$.
- Find the maximum rate of increase in the temperature at $(3,4)$.

Solution: The command to plot level curves is `ContourPlot[f,{x,a,b},{y,c,d}]` and the command to plot the graph of functions of

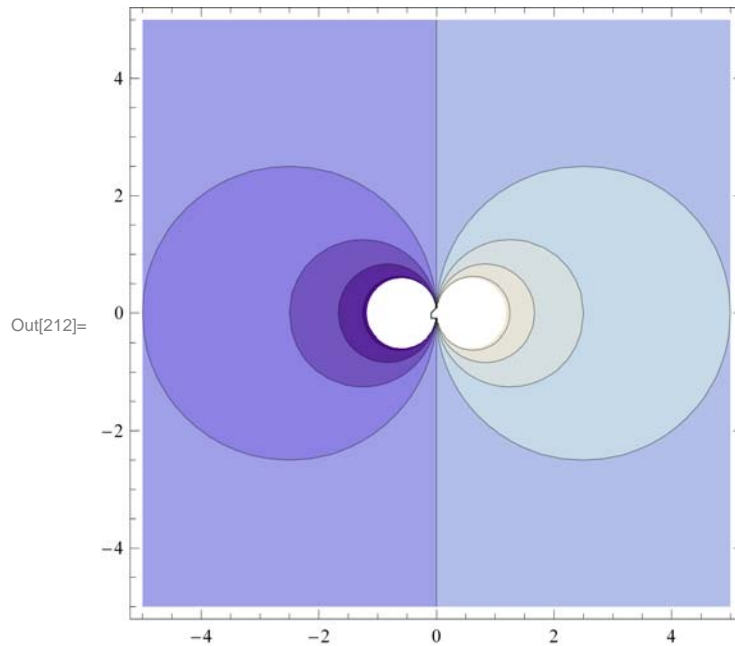
two variables is **Plot3D**[**f**,{**x**,**a**,**b**},{**y**,**c**,**d**]]. Rate of change in a given direction is nothing but directional derivative. The gradient vector is the direction of most rapid increase and its norm is the maximum rate of change.

In[210]:= **Clear**[**x**, **y**, **T**]

In[211]:= **T**[**x_**, **y_**] := $\frac{x}{x^2 + y^2}$

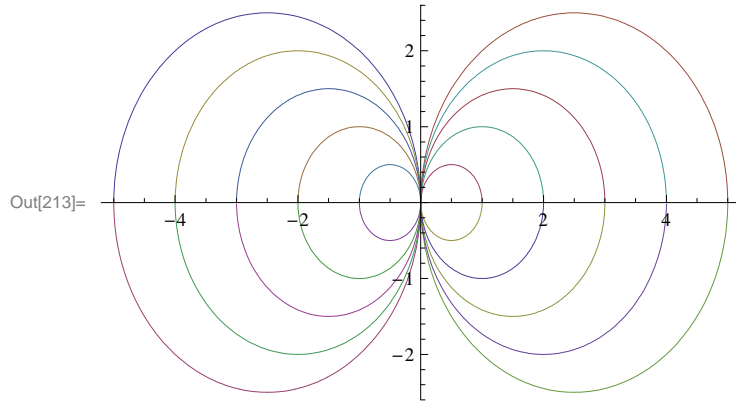
a) We give two methods of drawing level curves; one uses the **ContourPlot** command and the other uses the **Plot** command.

In[212]:= **levelcurves1** = **ContourPlot**[**T**[**x**, **y**], {**x**, -5, 5}, {**y**, -5, 5}]



- Note: 1. Use the command **ContourPlot**[**T**[**x**,**y**],{**x**,**-5**,**5**},{**y**,**-5**,**5**}, **ContourShading**->**False**]; to get unshaded contours.
 2. Solving $T(x,y)=c$ for y we get (for $c=-5,-4,-3,-2,-1,1,2,3,4,5$) the following level curves. (We leave it as an exercise to plot level curves for smaller values of $|c|$.)

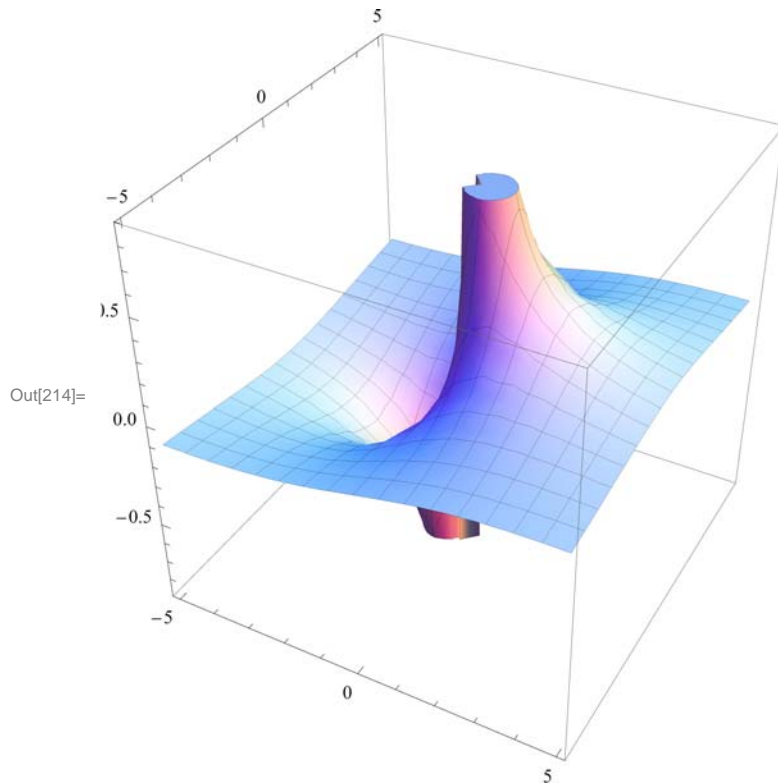
```
In[213]:= levelcurves2 = Plot[{ $\sqrt{-5x-x^2}$ ,  $-\sqrt{-5x-x^2}$ ,  $\sqrt{-4x-x^2}$ ,
 $-\sqrt{-4x-x^2}$ ,  $\sqrt{-3x-x^2}$ ,  $-\sqrt{-3x-x^2}$ ,  $\sqrt{-2x-x^2}$ ,  $-\sqrt{-2x-x^2}$ ,  $\sqrt{-x-x^2}$ ,
 $-\sqrt{-x-x^2}$ ,  $\sqrt{5x-x^2}$ ,  $-\sqrt{5x-x^2}$ ,  $\sqrt{4x-x^2}$ ,  $-\sqrt{4x-x^2}$ ,  $\sqrt{3x-x^2}$ ,
 $-\sqrt{3x-x^2}$ ,  $\sqrt{2x-x^2}$ ,  $-\sqrt{2x-x^2}$ ,  $\sqrt{x-x^2}$ ,  $-\sqrt{x-x^2}$ }, {x, -5, 5}]
```



It is a good exercise to draw these level curves using **PolarPlot**. Note that the polar form of the equations $T(x,y) = c$ is $r = \frac{\cos\theta}{c}$ or simply $r = c \cos\theta$ (since c is arbitrary).

Here is the graph of T .

```
In[214]:= graphofT = Plot3D[T[x, y], {x, -5, 5}, {y, -5, 5}, BoxRatios -> {1, 1, 1}]
```



c) Let $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$. Then \mathbf{u} is a unit vector and the rate of change in temperature at (3, 4) in the direction of \mathbf{v} is given by $\mathbf{D}_{\mathbf{u}}(\mathbf{T})(3, 4) = \text{Grad}[\mathbf{T}](3, 4) \cdot \mathbf{u}$

```
In[215]:=  $\nabla \mathbf{T} = \{\mathbf{D}[\mathbf{T}[\mathbf{x}, \mathbf{y}], \mathbf{x}], \mathbf{D}[\mathbf{T}[\mathbf{x}, \mathbf{y}], \mathbf{y}]\}$ 
 $\mathbf{v} = \{1, -2\}$ 
 $\mathbf{u} = \frac{\mathbf{v}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}$ 
 $\mathbf{u} \cdot \nabla \mathbf{T} /. \{\mathbf{x} \rightarrow 3, \mathbf{y} \rightarrow 4\} // \mathbf{N}$ 
```

```
Out[215]=  $\left\{-\frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}, -\frac{2xy}{(x^2+y^2)^2}\right\}$ 
```

```
Out[216]=  $\{1, -2\}$ 
```

```
Out[217]=  $\left\{\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\}$ 
```

```
Out[218]= 0.0393548
```

Thus, the rate of change at (3, 4) in the direction of \mathbf{v} is 0.0393548. On the last line of the above input the command `// N` converts the output to decimal form.

d) The unit vector in the direction of which the temperature increases most rapidly at (3,4) is given by

```
In[219]:=  $\frac{\nabla \mathbf{T}}{\sqrt{\nabla \mathbf{T} \cdot \nabla \mathbf{T}}} /. \{\mathbf{x} \rightarrow 3, \mathbf{y} \rightarrow 4\}$ 
```

```
Out[219]=  $\left\{\frac{7}{25}, -\frac{24}{25}\right\}$ 
```

e) The maximum rate of increase in the temperature at (3,4) is the norm of the gradient at this point. This can be obtained by:

```
In[220]:=  $\sqrt{\nabla \mathbf{T} \cdot \nabla \mathbf{T}} /. \{\mathbf{x} \rightarrow 3, \mathbf{y} \rightarrow 4\}$ 
```

```
Out[220]=  $\frac{1}{25}$ 
```

Example 18: Let $f(x, y) = 2x^2 - 3xy - x + y + y^2$ and let R be the rectangle in the xy plane whose vertices are at (0,0), (2,0), (2,2), and (0,2).

a) Find all relative extremum values of f inside R.

b) Find the maximum and minimum values of f on R.

```
In[221]:= Clear[f, x, y, d]
```

```
In[222]:= f[x_, y_] := 2 x^2 - 3 x * y - x + y + y^2
```

```
In[223]:= Solve[{ $\partial_x f[x, y] == 0$ ,  $\partial_y f[x, y] == 0$ }, {x, y}]
```

```
Out[223]= {{x -> 1, y -> 1}}
```

```
In[224]:= d[x_, y_] :=  $\partial_{x,x} f[x, y] * \partial_{y,y} f[x, y] - (\partial_{x,y} f[x, y])^2$ 
```

```
In[225]:=  $\partial_{\mathbf{x},\mathbf{x}} \mathbf{f}[\mathbf{x}, \mathbf{y}] /. \{\mathbf{x} \rightarrow 1, \mathbf{y} \rightarrow 1\}$   

 $\mathbf{d}[\mathbf{x}, \mathbf{y}] /. \{\mathbf{x} \rightarrow 1, \mathbf{y} \rightarrow 1\}$ 
```

```
Out[225]= 4
```

```
Out[226]= -1
```

Thus (1,1) is the local minimum point of f inside R and its local minimum value is $f(1,1) = 0$. Next we find the extreme values of f on the boundary of the rectangle. This is done by considering f as a function of one variable on the sides of R. Let $f1=f[x,0]$, $f2=f[x,2]$ for x between 0 and 2, and $f3=f[0,y]$ and $f4=f[2,y]$ for y between 0 and 2. We now proceed as follows:

```
In[227]:=  $\mathbf{f1} = \mathbf{f}[\mathbf{x}, 0]$   

 $\mathbf{f2} = \mathbf{f}[\mathbf{x}, 2]$   

 $\mathbf{f3} = \mathbf{f}[0, \mathbf{y}]$   

 $\mathbf{f4} = \mathbf{f}[2, \mathbf{y}]$ 
```

```
Out[227]=  $-x + 2x^2$ 
```

```
Out[228]=  $6 - 7x + 2x^2$ 
```

```
Out[229]=  $y + y^2$ 
```

```
Out[230]=  $6 - 5y + y^2$ 
```

```
In[231]:=  $\mathbf{Solve}[\mathbf{D}[\mathbf{f1}, \mathbf{x}] == 0]$ 
```

```
Out[231]=  $\left\{ \left\{ x \rightarrow \frac{1}{4} \right\} \right\}$ 
```

```
In[232]:=  $\mathbf{Solve}[\mathbf{D}[\mathbf{f2}, \mathbf{x}] == 0]$ 
```

```
Out[232]=  $\left\{ \left\{ x \rightarrow \frac{7}{4} \right\} \right\}$ 
```

```
In[233]:=  $\mathbf{Solve}[\mathbf{D}[\mathbf{f3}, \mathbf{y}] == 0]$ 
```

```
Out[233]=  $\left\{ \left\{ y \rightarrow -\frac{1}{2} \right\} \right\}$ 
```

```
In[234]:=  $\mathbf{Solve}[\mathbf{D}[\mathbf{f4}, \mathbf{y}] == 0]$ 
```

```
Out[234]=  $\left\{ \left\{ y \rightarrow \frac{5}{2} \right\} \right\}$ 
```

Thus, we have the following points on the boundary of R as critical points of f: $(\frac{1}{4}, 0)$, $(\frac{7}{4}, 2)$, $(2, \frac{5}{2})$. Observe that the point $(0, -1/2)$ is outside the rectangle R. The four vertices of R: $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$ are also critical points (Why?). We now evaluate f at each of these points and at $(1,1)$ (the relative minimum point found above) and compare the results. We do this using the substitution command.

```
In[235]:=  $\mathbf{f}[\mathbf{x}, \mathbf{y}] /. \left\{ \left\{ x \rightarrow \frac{1}{4}, y \rightarrow 0 \right\}, \left\{ x \rightarrow \frac{7}{4}, y \rightarrow 2 \right\}, \left\{ x \rightarrow 2, y \rightarrow \frac{5}{2} \right\}, \right.$   

 $\left. \left\{ x \rightarrow 0, y \rightarrow 0 \right\}, \left\{ x \rightarrow 2, y \rightarrow 0 \right\}, \right.$   

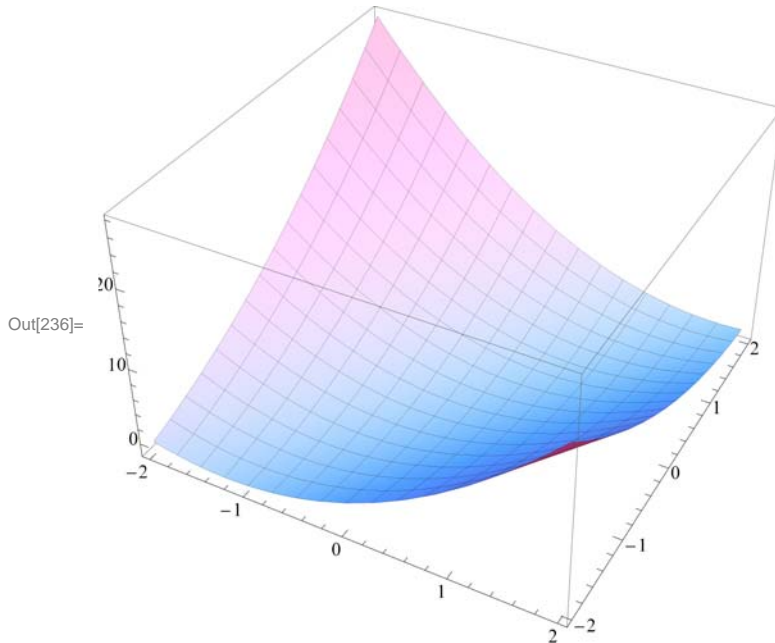
 $\left. \left\{ x \rightarrow 0, y \rightarrow 2 \right\}, \left\{ x \rightarrow 2, y \rightarrow 2 \right\}, \left\{ x \rightarrow 1, y \rightarrow 1 \right\} \right\}$ 
```

```
Out[235]=  $\left\{ -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{4}, 0, 6, 6, 0, 0 \right\}$ 
```

Thus, the minimum value of f is $-1/4$ and the maximum value of f is 6. At what points are these attained?

Here is the graph of f .

```
In[236]:= Plot3D[f[x, y], {x, -2, 2}, {y, -2, 2}, BoxRatios -> {1, 1, 0.62`}]
```



Exercise 5.4. Let $f(x, y) = \frac{(x-y)^2}{x^2 + y^2}$. Find a) $f_y(1, 0)$ b) f_{xy}

Exercise 5.5. The temperature at a point (x, y) on a metal plate is modeled by;

$$T(x, y) = 400 \text{Exp}(-x^2 + y) / 2; \quad x \geq 0, \quad y \geq 0.$$

a) Find $\nabla T(x, y)$.

b) Find the directional derivative of $T(x, y)$ at the point $(3, 5)$ in the direction of $\mathbf{u} = \frac{1}{2} \mathbf{i} + \frac{\sqrt{3}}{2} \mathbf{j}$.

c) Find the direction of greatest increase in heat from the point $(3, 5)$.

Exercise 5.6. Let $f(x, y) = \frac{4xy}{(x^2+1)(y^2+1)}$ for $-2 \leq x \leq 2$, and $0 \leq y \leq 3$.

a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point $(1, 2, 8/10)$.

b) Graph the surface, the normal line and the tangent plane found in a).

5.3 Multiple Integrals

The command for multiple integrals is more or less the same as the command for single integrals. If f is a function of two variables x and y , then `Integrate[f,x]` integrates f with respect to x while `Integrate[f,y]` integrates f with respect to y . `Integrate[f, x, y]` integrates f with respect to x and y . **It is worthwhile to keep in mind that Mathematica does not give a constant of integration.** `Integrate[f,{x,a,b},{y,c,d}]` evaluates $\int_a^b \left(\int_c^d f \, dy \right) dx = \int_a^b \int_c^d f \, dy \, dx$. Here c and d can be functions of x . In *Mathematica* 3.0, the command for multiple integrals can be entered from the **BasicInputs** palette as follows: From the palette click on $\int dx$ or $\int dy$. Use the TAB key to move to the first box inside the integral sign and then click on the integral sign of your choice from the palette.

Example 19 : Let $f(x, y) = 6xy^2$. Find a) $\int f(x, y) \, dx$ b) $\int f(x, y) \, dy$ c) $\iint f(x, y) \, dx \, dy$

Solution : We will give two methods. First we define f in *Mathematica*.

In[237]:= `Clear[f, x, y]`

In[238]:= `f[x_, y_] := 6 x * y^2`

Method 1:

In[239]:= `∫ f[x, y] dx`

Out[239]= $3x^2y^2$

In[240]:= `∫ f[x, y] dy`

Out[240]= $2xy^3$

In[241]:= `∫∫ f[x, y] dx dy`

Out[241]= x^2y^3

Method 2:

In[242]:= `Integrate[f[x, y], x]`

Out[242]= $3x^2y^2$

In[243]:= `Integrate[f[x, y], y]`

Out[243]= $2xy^3$

In[244]:= `Integrate[f[x, y], x, y]`

Out[244]= x^2y^3

In Example 19 c), notice that *Mathematica* did not give us the general solution which is $x^2y^3 + g(x) + h(y)$, where g is a function of x alone and h is a function of y alone.

Example 20 : Evaluate each of the following integrals :

$$(a) \int_0^1 \int_0^1 e^{x+y} dx dy \quad (b) \int_1^4 \int_x^{x^2} xy dy dx$$

$$(c) \int_0^1 \int_0^y (1+x^2+y^2) dx dy \quad (d) \int_0^1 \int_0^{4-x^2} e^{xy} dy dx$$

Solution: We will use the letter of the question to denote the integral:

$$\text{In[245]:= } \mathbf{a} = \int_0^1 \int_0^1 e^{x+y} dx dy$$

$$\text{Out[245]= } (-1 + e)^2$$

$$\text{In[246]:= } \mathbf{\text{Simplify}[\%]}$$

$$\text{Out[246]= } (-1 + e)^2$$

$$\text{In[247]:= } \mathbf{N[\%]}$$

$$\text{Out[247]= } 2.95249$$

$$\text{In[248]:= } \mathbf{b} = \int_1^4 \int_x^{x^2} x * y dy dx$$

$$\text{Out[248]= } \frac{2475}{8}$$

$$\text{In[249]:= } \mathbf{c} = \text{Integrate}[1 + x^2 + y^2, \{y, 0, 1\}, \{x, 0, y\}]$$

$$\text{Out[249]= } \frac{5}{6}$$

$$\text{In[250]:= } \mathbf{d} = \int_0^1 \int_0^{4-x^2} e^{x*y} dy dx$$

$$\text{Out[250]= } \int_0^1 \frac{-1 + e^{4x-x^3}}{x} dx$$

This example shows that there are double integrals that *Mathematica* can not evaluate exactly. For such cases we can get a numerical approximation by using **NIntegrate** or **N[%]**. (We will use **N[%]** below.)

$$\text{In[251]:= } \mathbf{N[\%]}$$

$$\text{Out[251]= } 11.415$$

We leave it as an exercise to the reader to sketch the regions of integration for the integrals of Example 2.

Example 21: Evaluate each of the following triple integrals:

$$a) \int_1^4 \int_0^1 \int_0^x 2z e^{-x^2} dy dx dz \quad b) \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2} x dz dy dx$$

$$\text{In}[252]:= \int_1^4 \int_0^1 \int_0^{-x} 2z * e^{-x^2} \, dy \, dx \, dz$$

$$\text{Out}[252]= \frac{15(-1 + e)}{2e}$$

$$\text{In}[253]:= \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2} x \, dz \, dy \, dx$$

$$\text{Out}[253]= \frac{128}{15}$$

Exercise 5.7. a) Evaluate the iterated integral $\int_0^2 \int_x^2 e^{-y^2} \, dy \, dx$
 b) Switch the order of integration and evaluate the integral in a).

Exercise 5.8. Approximate the integral $\int_0^2 \int_x^2 \sqrt{16 - x^3 - y^3} \, dy \, dx$

Exercise 5.9. Find the volume of the solid that lies below the surface $z = 1 - xy$ and above the plane region R bounded by $y = x$, $y = 1$, and $x = 0$.

Exercise 5.10. Evaluate the iterated integral $\int_0^{\pi/4} \int_0^{\text{Null}} r^2 \sin\theta \cos\theta \, dr \, d\theta$.

Exercise 5.11. a) Evaluate the sum of the integrals

$$\int_0^{5\sqrt{2}/2} \int_0^{\text{Null}} xy \, dy \, dx + \int_{5\sqrt{2}/2}^5 \int_0^{\text{Null}} xy \, dy \, dx$$

b) Use polar coordinates to combine the sum of the integrals in a) into a single integral and evaluate it.

Exercise 5.12. Find the surface area of $z = 2x + y^2$ over the triangle with vertices $(0,0)$, $(2,0)$, and $(2,2)$.

Exercise 5.13. Evaluate the triple integral:

$$\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} y \, dz \, dy \, dx$$