

THE UNLIKELY UNION OF PARTITIONS AND DIVISORS

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In the multiplicative number theory we decompose a natural number n into prime factors $n = p_1 p_2 p_3 \dots p_k$ and consider the consequences. In the additive theory we decompose a natural number into a sum of elements from some set. For example we could try to express n as a sum of squares. In [3], and [4], the authors treated the properties of the partition function, which is a good example of additive number theory.

A good example of functions studied in multiplicative theory is the divisor function $\sigma(n)$. $\sigma(n)$ is defined as the *sum of the positive divisors of n* . For example the divisors of 6 are 1, 2, 3, and 6. Thus the sum of the divisors of 6 is $\sigma(6) = 1 + 2 + 3 + 6 = 12$. Now divisors of numbers are related to primes, and primes seem unrelated to partitions. We are not surprised that partitions satisfy a recursion relation. We do not expect $\sigma(n)$ to satisfy a recursion relation. What do the divisors of n have to do with the divisors of $n-1, n-2, \dots$? Yet Euler showed that $\sigma(n)$ satisfies the same recursion relation as does $p(n)$, the partition of n . (See [1]) Only $\sigma(0)$ is different from $p(0)$. Euler was astonished at this result, and you can read a translation of his own words in Polya [5] and in Young [8]. There are even relations “marrying” the two functions such as (Schroeder [6])

$$n p(n) = \sum_{k=1}^n \sigma(k) p(n-k).$$

In this paper we will examine this and other properties of $\sigma(n)$ and $p(n)$.

1. BASIC PROPERTIES OF THE DIVISOR FUNCTION

We use the convention that $\sigma(0) = 0$. It is clear that $\sigma(p) = p + 1$ for any prime number p , since the only positive divisors of p are 1 and p . Also the only divisors of p^2 are $1, p$ and p^2 .

Thus $\sigma(p^2) = p^2 + p + 1$ and little bit of algebra shows that this can be expressed as

$$\sigma(p^2) = p^2 + p + 1 = \frac{p^3 - 1}{p - 1}. \text{ It should now be easy to prove}$$

Lemma 1. If p is a prime and k is a nonnegative integer, then

$$\sigma(p^k) = p^k + \cdots + p + 1 = \frac{p^{k+1} - 1}{p - 1}$$

The following is a standard theorem in Number Theory textbooks. (See [9]).

Lemma 2. $\sigma(n)$ is multiplicative, that is, if m and n are relatively prime (they have no common divisor other than 1), then $\sigma(nm) = \sigma(n)\sigma(m)$.

An immediate consequence of Lemma 1 and 2 is the following formula that can be used to evaluate the sum of the divisors of a given integer n .

Proposition 1. If $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, where p_1, p_2, \dots, p_m are primes and k_1, k_2, \dots, k_m are nonnegative integers, then

$$\sigma(n) = \sigma(p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdots \frac{p_m^{k_m+1} - 1}{p_m - 1}$$

Proposition 1 has a misleading simplicity. Suppose we take $n = 2^{231} + 1$. What are the prime factors of n ? In other words, if n is a large number whose prime factors cannot easily be obtained, the formula becomes less useful. It is in this sense that we now look at the recurrence relations that $\sigma(n)$ satisfies.

2. RECURRENCE RELATIONS INVOLVING THE DIVISOR FUNCTION

Let $p(n)$ be the partition of n , that is, the number of ways we can write n as a sum of positive integers. We define $p(n) = 0$ if $n \leq 0$. Let $f(k) = \frac{k(3k-1)}{2}$. Euler proved the following recurrence relation for $p(n)$.

$$(1.1) \quad p(n) = \sum_{k=0}^{n-1} (-1)^{k+1} \{p(n-f(k)) + p(n-f(-k))\}$$

The main tool that Euler used to prove this formula was the function given by the infinite product $g(x) = \prod_{n=1}^{\infty} (1-x^n)$. He showed that this function has the following two remarkable properties:

$$(A) \quad g(x) = \prod_{n=1}^{\infty} (1-x^n) = \sum_{k=-\infty}^{\infty} (-1)^{f(k)} x^{f(k)}$$

$$(B) \quad \frac{1}{g(x)} = \frac{1}{\prod_{n=1}^{\infty} (1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n$$

For more on this, see [3]. We now define the so-called Lambert series and look at the relation of this series with partition and divisor functions. Let $\{a(n)\}$ be a sequence of real number.

Then Lambert's series associated with $\{a(n)\}$ is defined by

$$L_a(x) = \sum_{n=1}^{\infty} a(n) \frac{x^n}{1-x^n}$$

This Lambert series is Taylor series expansion given by

$$L_a(x) = \sum_{n=1}^{\infty} A(n)x^n,$$

where

$$A(n) = \sum_{d|n} a(d)$$

(the summation is taken over the positive divisors of n .) Note then that if $a(n) = n$ then $A(n) = \sigma(n)$. Thus we have the following result.

Lemma 3.
$$\sum_{n=1}^{\infty} n \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n$$

For more interesting properties and the many other applications of Lambert series the reader is hereby invited to indulge in the classic book of Knopp [2]. We are now in a position to prove our first recurrence relation that connects $\sigma(n)$ and $p(n)$. Let us rewrite the infinite

product of Euler as
$$g(x) = \prod_{n=1}^{\infty} (1-x^n) = \sum_{k=-\infty}^{\infty} (-1)^{f(k)} x^{f(k)} = \sum_{n=0}^{\infty} e_n x^n .$$

Theorem 1. The sum of divisor and partition function are related by the following formula

(1.2)
$$\sigma(n) = -\sum_{k=0}^n k e_k p(n-k)$$

Proof: If we take the logarithmic differentiation of $g(x) = \prod_{n=1}^{\infty} (1-x^n)$, and multiply the resulting equation by x , we get

(1.3)
$$-\frac{xg'(x)}{g(x)} = \sum_{n=1}^{\infty} n \frac{x^n}{1-x^n}$$

On the other hand, $g(x) = \sum_{n=0}^{\infty} e_n x^n$ implies that $xg'(x) = \sum_{n=0}^{\infty} n e_n x^n$. We now recall that

$$\frac{1}{g(x)} = \sum_{n=0}^{\infty} p(n)x^n .$$
 Using Lemma 3 and substituting these last two series for

$xg'(x)$ and $\frac{1}{g(x)}$ in (1.3), we get

(1.4)
$$\left(\sum_{n=0}^{\infty} p(n)x^n \right) \cdot \left(\sum_{n=0}^{\infty} n e_n x^n \right) = -\sum_{n=1}^{\infty} n \frac{x^n}{1-x^n} = -\sum_{n=1}^{\infty} \sigma(n)x^n$$

Multiplying out the two series on the left side of (1.4) and comparing coefficients proves the theorem.

In the same line of reasoning we can show the following recurrence.

Theorem 2. The divisor function satisfies the recurrence relation

(1.5)
$$\sigma(n) = -n e_n - \sum_{k=0}^{n-1} e_{n-k} \sigma(k)$$

Proof: We have noted that $-\frac{xg'(x)}{g(x)} = \sum_{n=1}^{\infty} \sigma(n)x^n$. Multiply both sides by $g(x)$ to get

$$-xg'(x) = g(x) \cdot \sum_{n=1}^{\infty} \sigma(n)x^n .$$
 Now use the fact that $g(x) = \sum_{n=0}^{\infty} e_n x^n$ and $xg'(x) = \sum_{n=0}^{\infty} n e_n x^n$ to get

$$-\sum_{n=0}^{\infty} n e_n x^n = \left(\sum_{n=0}^{\infty} e_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} \sigma(n)x^n \right) .$$
 Since $e_0 = 1$, the theorem follows from comparing

coefficients of the two power series.

Theorem 3. The partition function and the divisor function are also related by the formula

$$(1.6) \quad np(n) = \sum_{k=0}^n \sigma(k)p(n-k)$$

Proof: Define $F(x) = \frac{1}{g(x)} = \frac{1}{\prod_{n=1}^{\infty} (1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n$. Then

$$(1.7) \quad F'(x) = -\frac{g'(x)}{(g(x))^2} = -\frac{g'(x)}{g(x)} \cdot \frac{1}{g(x)} = -\frac{g'(x)}{g(x)} \cdot F(x)$$

On the other hand, we have $-\frac{g'(x)}{g(x)} = \sum_{n=1}^{\infty} \sigma(n)x^{n-1}$. Also $F(x) = \sum_{n=0}^{\infty} p(n)x^n$ implies that

$F'(x) = \sum_{n=0}^{\infty} n \cdot p(n)x^{n-1}$. Substituting all these in (1.7), we get

$$(1.8) \quad \sum_{n=1}^{\infty} n \cdot p(n)x^{n-1} = \left(\sum_{n=1}^{\infty} \sigma(n)x^{n-1} \right) \left(\sum_{n=0}^{\infty} p(n)x^n \right)$$

Multiply (1.8) by x to get

$$\sum_{n=0}^{\infty} n \cdot p(n)x^n = \left(\sum_{n=0}^{\infty} \sigma(n)x^n \right) \left(\sum_{n=0}^{\infty} p(n)x^n \right).$$

Multiplying the infinite series on the right and compare coefficient to get the formula in the theorem.

Remark: [3] contains a combinatorial proof of Theorem 3. He first shows by a combinatorial

argument that $n \cdot p(n) = \sum_{m=1}^n \sum_{k=1}^{\lfloor \frac{n}{m} \rfloor} m \cdot p(n-km)$. Converting the double sum into a single sum

then gives the formula in Theorem 3.

3. QBASIC PROGRAM FOR THE SUM OF DIVISORS

Formula (1.1) can be rewritten as

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7)p(n-12) + p(n-15) - p(n-22) - p(n-26) + \dots$$

The authors used this recurrence formula in [3] to write a simple QBASIC program to construct a table for the values of $p(n)$. We can use the same program with a slight modification to generate a table of values for $\sigma(n)$. Here is the modified program of [Osler 1] that uses formula (1.2).

Program 1: Calculate Partitions and Sum of Divisor

'Calculate partitions of N , $P(n)=p(N)$, and the sum of divisors $D(N) = \sigma(N)$
 'exactly up to $P(301)$ and $D(301)$.
 'Set double precision, dimension array P and D , initialize P and D
 'Main loop, for each N find $P(N)$ and $D(N)$.

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90 CLS
100 DEFDBL A-Z
110 DIM P(400)
111 DIM D(400)
120 P(0) = 1
121 D(0) = 0

INPUT "ENTER A POSITIVE INTEGER", J
'Main loop, for each N find P(N) and D(N)

200 FOR N = 1 TO J
210 SIGN = 1
215 P(N) = 0
216 D(N) = 0

220 FOR K = 1 TO N
'Calculate two terms in recursion relation for P(N) and D(N)
230 F = K * (3 * K - 1) / 2
'Exit loop if argument negative
240 IF N - F < 0 THEN GOTO 400
250 P(N) = P(N) + SIGN * P(N - F)
251 D(N) = D(N) + F * SIGN * P(N - F)
260 F = K * (3 * K + 1) / 2
'Exit loop if argument negative
270 IF N - F < 0 THEN GOTO 400
280 P(N) = P(N) + SIGN * P(N - F)
281 D(N) = D(N) + F * SIGN * P(N - F)
290 SIGN = -SIGN
300 NEXT K
'Print results
400 PRINT N, P(N), D(N)
'Pause after printing 20 lines on the screen
450 IF 20 * INT(N / 20) = N THEN INPUT A$: CLS
500 NEXT N

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