

Normal Subgroups

Definition: Let G be a group. A subgroup S is a normal subgroup of G (symbolized $S \triangleleft G$) iff for every $x \in G$ and every $s \in S$, $xsx^{-1} \in S$.

Notice that S must first be a subgroup of G before we can consider whether or not it is normal. The condition $xsx^{-1} \in S$ will be called the **Normal Test Condition** and abbreviated **NTC**. We will discover that the normal subgroups of a group are vitally important. The subgroups that are not normal will be of little use to us in future chapters.

Example 1

Let $G = D_3$. Let $S = \{e, a\}$. To prove that S is a subgroup, we must show that S is closed. This is a rather routine exercise. Let's find out if S is a normal subgroup of D_3 . To apply the NTC we are required to compute 12 products:

<u>e</u>	<u>a</u>
$eee^{-1} = e$	$eaee^{-1} = a$
$aea^{-1} = e$	$aaa^{-1} = a$
$beb^{-1} = e$	$bab^{-1} = ab$ ←
$b^2e(b^2)^{-1} = e$	$b^2a(b^2)^{-1} = ?$
$abe(ab)^{-1} = e$	$aba(ab)^{-1} = ?$
$ab^2e(ab^2)^{-1} = e$	$ab^2a(ab^2)^{-1} = ?$

We have stopped testing the condition on the third product of the "a" column. The result ab is not an element of S . The rest of the computations in this column are no longer material. $S \not\triangleleft D_3$.

Example 2

The set $S = \{e, b^2, b^4\}$ is a subgroup of D_6 . You can verify this by testing closure. Is S a normal subgroup? Let's attempt the NTC. Before we do, let's consider xex^{-1} for any x in any group.

$$\begin{aligned} xex^{-1} &= \\ xx^{-1} &= \\ e & \end{aligned}$$

Since e is an element of any subgroup, no violation of the NTC can occur in the e column.

Now let's complete the test:

e	b^2	b^4
all	$eb^2e^{-1} = b^2$	$eb^4e^{-1} = b^4$
answers	$ab^2a^{-1} = b^4$	$ab^4a^{-1} = b^2$
are	$bb^2b^{-1} = b^2$	$bb^4b^{-1} = b^4$
e	$b^2b^2(b^2)^{-1} = b^2$	$b^2b^4(b^2)^{-1} = b^4$
	$b^3b^2(b^3)^{-1} = b^2$	$b^3b^4(b^3)^{-1} = b^4$
	$b^4b^2(b^4)^{-1} = b^2$	$b^4b^4(b^4)^{-1} = b^4$
	$b^5b^2(b^5)^{-1} = b^2$	$b^5b^4(b^5)^{-1} = b^4$
	$abb^2(ab)^{-1} = b^4$	$abb^4(ab)^{-1} = b^2$
	$ab^2b^2(ab^2)^{-1} = b^4$	$ab^2b^4(ab^2)^{-1} = b^2$
	$ab^3b^2(ab^3)^{-1} = b^4$	$ab^3b^4(ab^3)^{-1} = b^2$
	$ab^4b^2(ab^4)^{-1} = b^4$	$ab^4b^4(ab^4)^{-1} = b^2$
	$ab^5b^2(ab^5)^{-1} = b^4$	$ab^5b^4(ab^5)^{-1} = b^2$

Every answer was an element of S . $\therefore S \triangleleft D_6$.

Example 3

Let G be the group of all 2×2 nonsingular matrices. Let S be the set of all nonsingular matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Notice that a can not equal 0. Let's find out if S is a normal subgroup of G . First we need to find out if S is a subgroup.

Closure

Let $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in S$ and $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in S$.

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix}$$

Since a and b can't be 0, $ab \neq 0$ and this result is clearly in S .

Inverse Closure

Let $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in S$.

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \text{ which is clearly in } S.$$

We have now established that S is a subgroup. The only issue remaining is normality.

NTC

Let $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in S$. Let $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be an arbitrary nonsingular 2×2 matrix. Let $D = xw - zy$, the determinant of $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Notice that $D \neq 0$.

Consider:

$$\begin{aligned}
 & \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} \\
 &= \begin{pmatrix} xa & ya \\ za & wa \end{pmatrix} \cdot \begin{pmatrix} \frac{w}{D} & \frac{-y}{D} \\ \frac{-z}{D} & \frac{x}{D} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{xaw - ya z}{D} & \frac{-xay + xay}{D} \\ \frac{zwa - wa z}{D} & \frac{-zay + waz}{D} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{aD}{D} & 0 \\ 0 & \frac{aD}{D} \end{pmatrix} \\
 &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in S \\
 &\therefore S \text{ is a normal subgroup}
 \end{aligned}$$

Normal subgroups have a characteristic that other subgroups don't possess. We have shown that $\{e, b^2, b^4\} \triangleleft D_6$. If we construct the right cosets generated by $S = \{e, b^2, b^4\}$ we obtain:

$$\begin{aligned}
 Se &= \{e, b^2, b^4\} \\
 Sa &= \{a, ab^4, ab^2\} \\
 Sb &= \{b, b^3, b^5\} \\
 Sb^2 &= \{b^2, b^4, e\} \\
 Sb^3 &= \{b^3, b^5, b\} \\
 Sb^4 &= \{b^4, e, b^2\} \\
 Sb^5 &= \{b^5, b, b^3\} \\
 Sab &= \{ab, ab^5, ab^3\} \\
 Sab^2 &= \{ab^2, a, ab^4\} \\
 Sab^3 &= \{ab^3, ab, ab^5\} \\
 Sab^4 &= \{ab^4, ab^2, a\} \\
 Sab^5 &= \{ab^5, ab^3, ab\}
 \end{aligned}$$

As we have proven, two right cosets are either equal or disjoint. An interesting phenomenon occurs when the left cosets are constructed:

$$\begin{aligned}
 eS &= \{e, b^2b^4\} \\
 aS &= \{a, ab^2, ab^4\} \\
 bS &= \{b, b^3, b^5\} \\
 b^2S &= \{b^2, b^4, e\} \\
 b^3S &= \{b^3, b^5, b\} \\
 b^4S &= \{b^4, e, b^2\} \\
 b^5S &= \{b^5, b, b^3\} \\
 abS &= \{ab, ab^3, ab^5\} \\
 ab^2S &= \{ab^2, ab^4, a\} \\
 ab^3S &= \{ab^3, ab^5, ab\} \\
 ab^4S &= \{ab^4, ab, ab^2\} \\
 ab^5S &= \{ab^5, ab, ab^3\}
 \end{aligned}$$

Each left coset generated by a particular element of D_6 equals the right coset generated by the same element. For example, $Sab = abS$. As the next theorem states, this would not have occurred if $s = \{e, b^2, b^4\}$ had not been a normal subgroup of D_6 .

Theorem: Let G be a group. Let S be a subgroup of G . $S \triangleleft G$ iff $xS = Sx$ for any $x \in G$.

Proof: First let's prove that if $S \triangleleft G$ and x is an arbitrary element of G then $xS = Sx$.

Let $z \in xS$. $z = xs_1$ for some $s_1 \in S$

$$\therefore zx^{-1} = xs_1x^{-1}$$

Since S is normal, xs_1x^{-1} equals an element $s_2 \in S$

$$\therefore zx^{-1} = s_2$$

$$\Rightarrow z = s_2x$$

$$\Rightarrow z \in Sx$$

Therefore, every element in xS is in Sx .

Let $w \in Sx$. $w = s_3x$ for some $s_3 \in S$.

$$\therefore x^{-1}w = x^{-1}s_3x$$

$$\Rightarrow x^{-1}w = (xs_3x^{-1})^{-1}$$

Since S is normal, xs_3x^{-1} is an element of S . Since S is a subgroup, S is inverse closed.

$\therefore (xs_3x^{-1})^{-1}$ is also an element of S . Let $(xs_3x^{-1})^{-1} = s_4$

$$\therefore x^{-1}w = s_4$$

$$\Rightarrow w = xs_4$$

$$\Rightarrow w \in xS$$

Therefore, every element in Sx is also in xS . $\therefore Sx = xS$.

This completes the first half of the proof. Now we must prove that if $xS = Sx$ for every x then S is normal.

Let $x \in G$ and $s \in S$. Construct the product xsx^{-1} . Look at the first two factors. $xs \in xS$. Since $xS = Sx$, $xs \in Sx$ $\therefore \exists s_5 \in S$ such that $xs = s_5x$

$$\therefore xsx^{-1} = s_5xx^{-1} = s_5$$

$$\therefore xsx^{-1} \in S$$

$$\therefore S \triangleleft G$$

QED

The previous result gives us an alternate method to test subgroups for normality. For example, suppose we wish to determine whether or not $S = \{e, (12)\}$ is normal in S_3 (note that $\{e, (12)\}$ is a subgroup).

Right Cosets

$$Se = S$$

$$S(12) = S$$

$$S(13) = \{(13), (132)\} \longleftarrow$$

\vdots

Left Cosets

$$eS = S$$

$$(12)S = S$$

$$(13)S = \{(13), (123)\} \longleftarrow$$

\vdots

There is no need to continue forming cosets.

$$S(13) \neq (13)S \therefore S \not\triangleleft S_3.$$

The next few theorems will help identify some normal subgroups without the need for extensive testing.

Theorem: If G is an abelian group then all subgroups of G are normal.

Proof: Let S be a subgroup of G . Let $x \in G$ and $s \in S$.

$$xsx^{-1} = sxx^{-1} = s \in S$$

$$\therefore xsx^{-1} \in S$$

$$\Rightarrow S \triangleleft G$$

QED

Suppose G is a finite group and S is one of its subgroups. All right (and left) cosets generated by S have the same number of elements as S . Two right cosets are either disjoint or equal. As we pointed out in the last chapter, the number of right (or left) cosets equals $o(G)$ divided by $o(S)$. We will call this number the index of S in G .

For example, if $o(G) = 12$ and $o(S) = 3$ then the index of S in G is 4. This means that S generates 4 distinct right cosets. This also means that S generates 4 distinct left cosets (but if S is not normal these will represent a different partition of G into 4 element subsets).

Theorem: Let G be a finite group. Let S be a subgroup of G with index 2 (i.e. $o(S) = \frac{1}{2}o(G)$). S must be a normal subgroup of G .

Proof: Let S^c represents the elements of G that are not in S . We call S^c the complement of S . Since $o(S) = \frac{1}{2}o(G)$, $o(S) = o(S^c)$. Note that S has to be one of the right cosets generated by S since $Se = S$. Any other right coset must equal S or be disjoint from S while having the same number of elements as S . This restricts all right cosets to be either S or S^c . If x is an arbitrary element of G , how can we tell whether $Sx = S$ or $Sx = S^c$? If $x \in S$, Sx has to be S because of the closure of S . If $x \notin S$, can $Sx = S$? Suppose $Sx = S$. Choose s_1 in S . s_1x has to be an element of S which we will call s_2 .

$$\begin{aligned} \therefore s_1x &= s_2 \\ \Rightarrow x &= s_1^{-1}s_2 \\ \Rightarrow x &\in S \end{aligned}$$



$$\therefore \text{If } x \notin S, Sx = S^c$$

By a similar argument for left cosets:

$$xS = S \text{ if } x \in S$$

$$xS = S^c \text{ if } x \notin S$$

$$\therefore \text{whether } x \in S \text{ or } x \notin S, Sx = xS$$

QED

Each of the following can easily be shown to be subgroups of their respective groups. Since they all have index 2, they are normal subgroups.

$$\{1, -1, i, -i\} \triangleleft Q_8$$

$$\{e, a, b^2, ab^2\} \triangleleft D_4$$

$$\{e, (123), (132)\} \triangleleft S_3$$

Theorem: Let G be a finite group. Let S be a subgroup of G . If S is the only subgroup of G of its order then S is normal.

Proof: Suppose $o(S) = n$. Let $S = \{s_1, s_2, \dots, s_n\}$ where each element is distinct. Let x be an arbitrary element of G . Form the set $Q = \{xs_1x^{-1}, xs_2x^{-1}, \dots, xs_nx^{-1}\}$. This set is a subgroup. To prove this all that is needed is to demonstrate closure.

Let xs_ix^{-1} and xs_jx^{-1} be elements of Q .

$$xs_ix^{-1} \cdot xs_jx^{-1} = xs_is_jx^{-1}$$

Since S is a subgroup, S is closed. $\therefore s_i \cdot s_j$ is an element of S . Let $s_i \cdot s_j = s_k$

$$\therefore xs_ix^{-1} \cdot xs_jx^{-1} = xs_kx^{-1} \in Q$$

$\therefore Q$ is closed

$\therefore Q$ is a subgroup

How many elements are in Q ? Clearly, at most n . Could there be less than n ? There are less than n elements in Q iff $xs_kx^{-1} = xs_lx^{-1}$ for some $k \neq l$. However, this would imply $s_k = s_l$ \implies \longleftarrow \implies $\therefore Q$ contains n elements.

Since S is the only subgroup that contains n elements, $Q = S$. $\therefore xsx^{-1} \in S$ for each s in S . Since x was arbitrary, $S \triangleleft G$.

QED

Let's examine an example of this theorem. It is rather easy to verify that $\{e, b^2, b^4\}$ is the only three element subgroup of D_6 . But our theorem, $\{e, b^2, b^4\} \triangleleft D_6$. This we had previously verified.

Theorem: For any group G , the trivial subgroups $\{e\}$ and G are always normal.

Proof: The fact that $G \triangleleft G$ is a consequence of the definition of a group. Since, $xe x^{-1} = e$ in any group, $\{e\}$ must also be normal.

QED

Consider the interesting group Q_8 . It's only subgroups are:

$$Q_8, \{1\}, \{1, -1\}, \{1, -1, i, -i\}, \{1, -1, j, -j\}, \text{ and } \{1, -1, k, -k\}.$$

Q_8 and $\{1\}$ are normal because they are the trivial subgroups. $\{1, -1\}$ is normal because it is the only subgroup of order 2. The last three are normal because they all have index 2.

Every subgroup of Q_8 is normal. Q_8 is a nonabelian group. We have a theorem that states that every subgroup of an abelian group is normal. It has just been demonstrated that the converse of this theorem is false.

Definition: Let G be a group. The center of G (denoted $C(G)$) is defined to be the set of all elements c in G such that $cx = xc$ for all x in G .

Clearly if G is an abelian group then $C(G) = G$. If G is nonabelian, there still will be some elements in G that commute with all other elements. e is an element of G that will always be in $C(G)$. Some examples from groups in our library:

$$\begin{aligned} C(D_4) &= \{e, b^2\} \\ C(Q_8) &= \{1, -1\} \\ C(\mathbb{Z}_6) &= \mathbb{Z}_6 \\ C(S_3) &= \{e\} \end{aligned}$$

Theorem: If G is a group, $C(G)$ is a normal subgroup of G .

Proof: We must first prove that $C(G)$ is a subgroup.

Closure

Let c_1 and c_2 be elements of $C(G)$. Let x be an arbitrary element of G .

Consider:

$$\begin{aligned} &(c_1 c_2)x \\ &= c_1(c_2 x) && \text{Associativity} \\ &= c_1(x c_2) && c_2 \in C(G) \\ &= (c_1 x)c_2 && \text{Associativity} \\ &= (x c_1)c_2 && c_1 \in C(G) \\ &= x(c_1 c_2) && \text{Associativity} \end{aligned}$$

$\therefore c_1 c_2$ commutes with any element of $G \therefore c_1 c_2 \in C(G)$

Inverse Closure

Let $c \in C(G)$ Let $x \in G$

$$\begin{aligned} cx &= xc \\ \Rightarrow x &= c^{-1}xc \\ \Rightarrow xc^{-1} &= c^{-1}x \\ \therefore c^{-1} &\in C(G) \end{aligned}$$

NTC

Let $c \in C(G)$ and let $x \in G$

Consider:

$$\begin{aligned} &xcx^{-1} \\ &= cxx^{-1} \\ &= c \in C(G) \\ \therefore C(G) &\triangleleft G \end{aligned}$$

QED

Let A and B be subsets of a group G . We define the set product $A \cdot B$ to be the set of all products of the form $a \cdot b$ where $a \in A$ and $b \in B$. For example, let

$$G = D_6, A = \{a, b^2, ab\} \text{ and } B = \{b, ab^2, ab^3\}.$$

$$A \cdot B = \{a \cdot b, a \cdot ab^2, a \cdot ab^3, b^2 \cdot b, b^2 \cdot ab^2, b^2 \cdot ab^3, ab \cdot b, ab \cdot ab^2, ab \cdot ab^3\} \\ = \{ab, b^2, b^3, a, ab^2\}$$

Notice that duplication of results reduced the number of elements in $A \cdot B$ to five even though nine products had to be considered.

Suppose G is a group and H is a normal subgroup of G . We could use the definition in the previous paragraph to construct the product of two right cosets generated by H . Let's consider such a product. $Hx \cdot Hy$ can be computed alternatively as $H(xH)y$ by associativity. Since H is normal, $xH = Hx$. Therefore our product of cosets becomes $H(Hx)y$ or equivalently $(H \cdot H)x \cdot y$. Since H is closed and $e \in H$, $H \cdot H = H$. Therefore our product becomes Hxy . However, this last expression is simply the right coset generated by the element xy . This implies that the product of any two right cosets generated by a normal subgroup results in a right coset. We have proven:

Theorem: If $H \triangleleft G$, $Hx \cdot Hy = Hxy$ for any x and y in G .

Let's go further with this concept. Suppose we construct all of the distinct right cosets generated by H . Let's use set multiplication as an operator for this collection of sets. The fact that set multiplication is well defined is inherited from G , as is associativity. It is easy to see that He (which is equivalent to H) acts as an identity since:

$$He \cdot Hx = Hex = Hx \quad \text{and} \\ Hx \cdot He = Hxe = Hx$$

The coset Hx possesses Hx^{-1} as an inverse since:

$$Hx \cdot Hx^{-1} = Hxx^{-1} = He \quad \text{and} \\ Hx^{-1} \cdot Hx = Hx^{-1}x = He$$

Finally, our previous theorem that $Hx \cdot Hy = Hxy$ guarantees closure. Therefore, the collection of right (or left) cosets generated by H form a group. We call this type of group a "factor" group and denote it G/H .

Let's construct a factor group. Let $G = D_6$. Let $H = \{e, b^2, b^4\}$. H is normal in D_6 since it is the only subgroup of D_6 of order 3. However, normality can also be determined using the NTC or coset testing. Let's construct all right cosets generated by H :

$$He = \{e, b^2, b^4\} \\ Ha = \{a, ab^2, ab^4\} \\ Hb = \{b, b^3, b^5\} \\ Hab = \{ab, ab^3, ab^5\}$$

Any other right coset is equivalent to one of these cosets.

$$\begin{aligned} \text{Consider } Ha \cdot Hab &= \{a, ab^2ab^4\} \cdot \{ab, ab^3, ab^5\} \\ &= \{a \cdot ab, a \cdot ab^3, a \cdot ab^5, ab^2 \cdot ab, ab^2 \cdot ab^3, ab^2 \cdot ab^5, ab^4 \cdot ab, ab^4 \cdot ab^3, ab^4 \cdot ab^5\} \\ &= \{b, b^3b^5\}. \end{aligned}$$

This should not surprise us since our theorem guarantees us that $Ha \cdot Hab = Ha \cdot ab = Hb$ which is precisely $\{b, b^3, b^5\}$.

Let's make a table for D_6/H :

D_6/H	He	Ha	Hb	Hab
He	He	Ha	Hb	Hab
Ha	Ha	He	Hab	Hb
Hb	Hb	Hab	He	Ha
Hab	Hab	Hb	Ha	He

Some of the results in this table are more subtle than they may appear. For example, $Hb \cdot Ha = Hba = Hab^5$. However, the coset Hab^5 is identical to the coset Hab . Again, we call the table above " D_6 factored by $\{e, b^2, b^4\}$ ".

The major result from this discussion bears repeating. If a group is factored by one of its normal subgroups, the result is a group. Factoring is the process of:

- a) Finding a complete collection of distinct cosets
- b) Using set multiplication as an operator for these cosets.

This process can only be accomplished with normal subgroups. Indeed, the product of two right cosets generated by a non-normal subgroup is not generally a right coset. Therefore subgroups that are not normal will wane in importance in this text. You should begin the process of determining which subgroups are normal for the groups in your library.