Discovering Bernoulli Number Identities via Euler-Maclaurin Summation

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Introduction

The Bernoulli numbers are one of the most fascinating number sequences in mathematics. Named after their discoverer, Jacob Bernoulli, they satisfy many interesting identities and appear in many important mathematical formulas. For example, the sums of powers formula first discovered by Bernoulli himself and published in his most famous work, Ars Conjectandi (1713), can be expressed as

$$\sum_{k=1}^{n} k^p = n^p + \sum_{k=0}^{p} \frac{p!}{k!(p-k+1)!} B_k n^{p+1-k}\,, \tag{1.1}$$

where B_n denote the n^{th} Bernoulli number. Another famous formula where Bernoulli numbers make their appearance is the Euler-Maclaurin Summation Formula, useful for approximating sums and integrals:

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}[f(n)+f(0)] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] \tag{1.2}$$

It is well known that sums of powers formula in (1.1) can be derived from the Euler Maclaurin Summation Formula in (1.2) by setting $f(x) = x^p$. Moreover, the classic identity

$$\sum_{k=0}^{p} \binom{p+1}{k} B_k = 0 \tag{1.3}$$

can be obtained by setting $n=1$ in (1.1). This begs the question as to whether other Bernoulli number identities can be obtained by a similar fashion such as the following quadratic identity discovered by Euler [E]:

$$\sum_{k=0}^{m+1} \binom{m+1}{k} B_k B_{m-k+1} = -(m+1)B_m - mB_{m+1}$$

The answer not surprisingly is yes. The surprise however is in the choice for $f(x)$. Before revealing the formula for $f(x)$, observe that $f(x)$ must involve the Bernoulli numbers. A natural candidate would be the Bernoulli polynomials $B_m(x)$ themselves, i.e. $f(x) = B_m(x)$; unfortunately, this choice yields the trivial identity. This is due to the fact that $B_n(1) = (-1)^n B_n(0)$ and $B_n'(x) = nB_{n-1}'(x)$ for all Bernoulli polynomials. The correct choice turns out to be $f(x) = (1-x)B_m(x)$ as we shall demonstrate later.

There are typically two equivalent approaches to defining Bernoulli polynomials. The first is in terms of the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and the second, which will be the approach that we take in this paper, is in terms of an Appell sequence:

$$\begin{aligned} &B_0(x) = 1, \\ &B_m^{'}(x) = mB_{m-1}(x), \\ &\int_0^1 B_m(x)dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} \end{aligned} \tag{1.4}$$

By definition, we set $B_k = B_k(0)$. Moreover, it can be shown that $B_k(1-x) = (-1)^k B_k(x)$ and thus $B_k(1) = (-1)^k B_k$.

Since we will often consider the Euler-Maclaurin Summation Formula in the case where $n = 1$, it will be useful for us to develop a version of it specific to this case. This simplified version can be derived quite easily as follows: Given a differentiable function *f*, we repeatedly integrate it by parts against the Bernoulli polynomials to obtain

$$\begin{aligned} \int_0^1 f(x)dx &= \int_0^1 B_0(x) f(x)dx \\ &= [B_1(x) f(x)]_0^1 - \int_0^1 B_1(x) f'(x)dx \end{aligned}$$

Continuing in this matter gives

$$\begin{aligned} \int_0^1 f(x)dx &= [B_1(1) f(1) - B_1(0) f(0)] - \left[\frac{B_2(x) f'(x)}{2!}\right]_0^1 + \frac{1}{2!}\int_0^1 B_2(x) f''(x)dx \\ &= -B_1[f(1) + f(0)] - [B_2(1) f'(1) - B_2(0) f'(0)] + \frac{1}{2!}\int_0^1 B_2(x) f''(x)dx \\ &= -B_1[f(1) + f(0)] - \frac{B_2}{2!}[f'(1) - f'(0)] + \left[\frac{B_3(x) f''(x)}{3!}\right]_0^1 - \frac{1}{3!}\int_0^1 B_3(x) f'''(x)dx \\ &= -B_1[f(1) + f(0)] - \frac{B_2}{2!}[f'(1) - f'(0)] + [B_3(1) f''(1) - B_3(0) f''(0)] - \frac{1}{3!}\int_0^1 B_3(x) f'''(x)dx \\ &\ldots \end{aligned}$$

Since $B_1 = -1/2$, this leads to the following formula, which can easily be proven by induction:

$$\int_0^1 f(x)dx = \frac{f(0) + f(1)}{2} - \sum_{k=2}^{p} \frac{B_k}{(k)!}[f^{(k-1)}(1) + (-1)^{k-1} f^{(k-1)}(0)] - \frac{1}{p!}\int_0^1 B_p(x) f^{(p)}(x)dx \tag{1.5}$$

where *p* is any positive integer. It is now clear that (1.5) is nothing more than a special version of the Euler-Maclaurin Summation Formula with $n = 1$ (recall that $B_{2k+1} = 0$ for $k > 0$).

Now, if in addition we assume *f* to be a polynomial of degree *q* and choose *p* so that $p = q$, then the integral on the right hand side of (1.5) vanishes. This reduces (1.5) to the form

$$\int_0^1 f(x)dx = \frac{f(0) + f(1)}{2} - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)] \tag{1.6}$$

We are now ready to hunt for Bernoulli number identities with the help of (1.6).

Example 1: Suppose $f(x) = x^{m-1}$ with $m > 1$. Then

$$\int_0^1 f(x)dx = \int_0^1 x^{m-1}dx = \frac{1}{m}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[1+0] = \frac{1}{2}$$

Set $p = m-1$. Then for $2 \le k \le m-1$, we have

$$f^{(k-1)}(1) - f^{(k-1)}(0) = \frac{(m-1)!}{(m-k)!}x^{m-k}\bigg|_{x=1} - \frac{(m-1)!}{(m-k)!}x^{m-k}\bigg|_{x=0} = \frac{(m-1)!}{(m-k)!}$$

Substituting the above results into the Euler-Maclaurin Summation Formula yields

$$\frac{1}{m} = \frac{1}{2} - \sum_{k=2}^{m-1} \frac{B_k}{k!}\left[\frac{(m-1)!}{(m-k)!}\right] \Rightarrow 1 - \frac{m}{2} + \sum_{k=2}^{m-1} \frac{m!}{k!(m-k)!} B_k = 0$$

This simplifies to the linear recurrence

$$\sum_{k=0}^{m-1} \binom{m}{k} B_k = 0 \qquad (1.7)$$

Of course this formula can also be derived by setting $x = 0$ in the formula below for Bernoulli polynomials:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}$$

Example 2: Suppose $f(x) = B_m(x)$ with $m > 1$. Then

$$\int_0^1 f(x)dx = \int_0^1 B_m(x)dx = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \ne 0 \end{cases}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[B_m(1) + B_m(0)] = \begin{cases} B_m & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases} = B_m$$

Set $p = m$. Then for $1 \le k \le m$,

$$\begin{aligned} f^{(k-1)}(1) - f^{(k-1)}(0) = B_m^{(k-1)}(1) - B_m^{(k-1)}(0) \\ &= \begin{cases} -2B_m^{(k-1)}(0) & \text{if } m+k-1 \text{ odd} \\ 0 & \text{if } m+k-1 \text{ even} \end{cases} \\ &= \begin{cases} -2(m)_{k-1}B_{m-k+1} & \text{if } m+k-1 \text{ odd} \\ 0 & \text{if } m+k-1 \text{ even} \end{cases} \\ &= \begin{cases} -2m!B_1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} m! & \text{if } k = m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

It follows that

$$\sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)] = \frac{B_m}{m!}(m!) = B_m$$

Thus, the Euler-Maclaurin Summation Formula reduces in this case to the trivial identity $B_m = B_m$ for $m > 1$.

Example 3: To yield a more interesting identity than that obtained in Example 2, let us instead define $f(x) = (1-x)B_m(x)$ where $m > 1$ Then

$$\int_0^1 f(x)dx = \int_0^1 (1-x)B_m(x)dx = -\frac{B_{m+1}}{m+1}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[0 + B_m(0)] = \frac{1}{2}B_m$$

To prove the integral formula above, we integrate by parts with to obtain

$$\begin{aligned} I &= \int_0^1 (1-x)B_m(x)dx \\ &= \left.\frac{(1-x)B_{m+1}(x)}{m+1}\right|_0^1 + \frac{1}{m+1}\int_0^1 B_{m+1}(x)dx \\ &= -\frac{B_{m+1}}{m+1} \end{aligned}$$

where we have used the integral condition in (1.4).

To compute the sum in (1.6), we begin with Leibniz' formula for the *n*th-derivative of *f*:

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}[1-x] \cdot B_m^{(n-k)}(x) \\ &= (1-x)B_m^{(n)}(x) - nB_m^{(n-1)}(x) \\ &= (1-x)\frac{m!}{(m-n)!}B_{m-n}(x) - n\frac{m!}{(m-n+1)!}B_{m-n+1}(x) \end{aligned}$$

Set $p = m+1$. Then for $1 \le k \le m+1$, we have (recall that $B_{2k+1} = 0$ for $k > 0$)

$$f^{(k-1)}(1) - f^{(k-1)}(0) = -\frac{m!}{(m-k+1)!}B_{m-k+1} + 2(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}\delta_{m-k+1}$$

where

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

It follows that

$$\begin{aligned} \sum_{k=2}^{m+1} \frac{B_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right] &= \sum_{k=2}^{m+1} \frac{B_k}{(k)!}\left(-\frac{m!}{(m-k+1)!}B_{m-k+1} + 2(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}\delta_{m-k+1}\right) \\ &= \frac{2m}{m+1}B_1 B_{m+1} - \frac{1}{(m+1)}\sum_{k=2}^{m+1}\binom{m+1}{k}B_k B_{m-k+1} \end{aligned}$$

Thus, formula (1.6) reduces to

$$-\frac{B_{m+1}}{m+1}=\frac{1}{2}B_m-\frac{2mB_1B_{m+1}}{m+1}+\frac{1}{(m+1)}\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

$$-B_{m+1}=\frac{(m+1)}{2}B_m+mB_{m+1}+\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

$$-mB_{m+1}=(m+1)B_m+\left[B_0B_{m+1}+(m+1)B_1B_m+\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}\right]$$

$$-mB_{m+1}=(m+1)B_m+\sum_{k=0}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

or equivalently,

$$\sum_{k=0}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}=-(m+1)B_m-mB_{m+1}$$

This identity was first established by Euler in 1755 in the form (see [E])

$$\sum_{k=0}^{n}\binom{2n}{2k}B_{2k}B_{2n-2k}=-(2n-1)B_{2n}$$

Example 4: We generalize Example 3 to hypergeometric Bernoulli polynomials by considering $f(x)=B_m(N,x)$, where $B_m(N,x)$ are polynomials defined as the Appell sequence (see [HN])

$$B_0(N,x)=1,$$
$$B_m^{'}(N,x)=mB_{m-1}(N,x),$$
$$\int_0^1(1-x)^{N-1}B_m(N,x)dx=\begin{cases}1/N & \text{if } n=0\\ 0 & \text{if } n\neq 0\end{cases}$$

For $N=1$, $B_m(1,x)=B_m(x)$. Consider now the case $N=2$. Since for $m>1$, we claim that

$$\int_0^1 f(x)dx=\int_0^1 B_m(2,x)dx=B_m(2)+\frac{1}{2}\delta_{m-1}$$

where again

$$\delta_k=\begin{cases}1 & \text{if } k=0\\ 0 & \text{if } k\neq 0\end{cases}$$

This follows from

$$\int_0^1(1-x)B_{m-1}(2,x)dx=\frac{1}{2}\delta_{m-1}$$

$$\Rightarrow\left.\frac{(1-x)B_m(2,x)}{m}\right|_0^1+\frac{1}{m}\int_0^1 B_m(2,x)dx=\frac{1}{2}\delta_{m-1}$$

$$\Rightarrow-\frac{B_m(2,0)}{m}+\frac{1}{m}\int_0^1 B_m(2,x)dx=\frac{1}{2}\delta_{m-1}$$

Moreover, we have for $m>2$,

$$\frac{1}{2}[f(1)+f(0)]=\frac{1}{2}[B_m(2,1)+B_m(2,0)]=B_m(2)+\frac{m}{2}B_{m-1}(2)$$

This follows from

$$B_m(2,1) - B_m(2,0) = m\int_0^1 B_{m-1}(2,x)dx$$
$$= m[B_{m-1}(2) + \frac{1}{2}\delta_{m-2}] \tag{1.8}$$

Thus,

$$f^{(k-1)}(1) - f^{(k-1)}(0) = B_m^{(k-1)}(2,1) - B_m^{(k-1)}(2,0)$$
$$= \frac{m!}{(m-k+1)!}(m)_{k-1} \;\; B_{m-k+1}(2,1) - B_{m-k+1}(2,0)$$
$$= \frac{m!}{(m-k+1)!}(m-k+1)[B_{m-k}(2) + \frac{1}{2}\delta_{m-2}]$$
$$= \frac{m!}{(m-k)!}[B_{m-k}(2) + \frac{1}{2}\delta_{m-k-1}]$$

It follows that

$$\sum_{k=2}^{m}\frac{B_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right] = \sum_{k=2}^{m}\frac{B_k}{k!}\left[\frac{m!}{(m-k)!}[B_{m-k}(2) + \frac{1}{2}\delta_{m-k-1}]\right]$$
$$= \sum_{k=2}^{m}\binom{m}{k}B_k[B_{m-k}(2) + \frac{1}{2}\delta_{m-k-1}]$$
$$= \frac{m!}{(m-1)!}\cdot\frac{B_{m-1}}{2} + \sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2)$$
$$= \frac{mB_{m-1}}{2} + \sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2)$$

Thus, for $m > 1$ formula (1.6) reduces to

$$B_m(2) = B_m(2) + \frac{m}{2}B_{m-1}(2) - \frac{mB_{m-1}}{2} - \sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2),$$

or equivalently,

$$\sum_{k=0}^{m}\binom{m}{k}B_kB_{m-k}(2) = B_m(2) - \frac{mB_{m-1}}{2} \tag{1.9}$$

(Or

$$B_{m-1}(2) = B_{m-1} + \frac{m}{2}\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2)$$

)

Observe that this formula allows us to calculate the hypergeometric Bernoulli numbers $B_m(2)$ in terms of the Bernoulli numbers B_k. This is useful since the odd Bernoulli numbers are known to vanish except for B_1 and thus (1.9) essentially separates the calculation of the even coefficients $B_{2k}(2)$ and the odd coefficients $B_{2m+1}(2)$.

Example 4.5: Consider now the case $f(x) = B_m(3,x)$. For $m > 2$, we claim that

$$\int_0^1 f(x)dx = \int_0^1 B_m(3,x)dx = \frac{m(m-1)}{2}\left[\frac{B_{m-1}(3)}{m-1} + \frac{2B_m(3)}{m(m-1)} + \frac{1}{3}\delta_{m-2}\right]$$

This follows from

$$\begin{aligned}\frac{1}{3}\delta_{m-2} &= \int_0^1 (1-x)^2 B_{m-2}(3,x)dx \\ &= \frac{(1-x)^2 B_{m-1}(3,x)}{m-1}\bigg|_0^1 + \frac{2}{m-1}\int_0^1 (1-x)B_{m-1}(3,x)dx \\ &= -\frac{B_{m-1}(3)}{m-1} + \frac{2(1-x)B_m(3,x)}{m(m-1)}\bigg|_0^1 + \frac{2}{m(m-1)}\int_0^1 B_m(3,x)dx \\ &= -\frac{B_{m-1}(3)}{m-1} - \frac{2B_m(3)}{m(m-1)} + \frac{2}{m(m-1)}\int_0^1 B_m(3,x)dx\end{aligned}$$

Moreover, we have for $m > 3$,

$$\frac{1}{2}[f(1)+f(0)] = \frac{1}{2}[B_m(3,1)+B_m(3,0)] = B_m(3) + m\left[\frac{(m-1)}{2}B_{m-2}(3) + B_{m-1}(3)\right]$$

This follows from

$$\begin{aligned}B_m(3,1) - B_m(3,0) &= m\int_0^1 B_{m-1}(3,x)dx \\ &= m\frac{(m-1)(m-2)}{2}\left[\frac{B_{m-2}(3)}{m-2} + \frac{2B_{m-1}(3)}{(m-1)(m-2)} + \frac{1}{3}\delta_{m-3}\right]\end{aligned} \tag{1.10}$$

Thus, we have

$$\begin{aligned}f^{(k-1)}(1) - f^{(k-1)}(0) &= B_m^{(k-1)}(3,1) - B_m^{(k-1)}(3,0) \\ &= \frac{m!}{(m-k+1)!}\ B_{m-k+1}(3,1) - B_{m-k+1}(3,0) \\ &= \frac{m!}{(m-k+1)!}\frac{(m-k+1)(m-k)(m-k-1)}{2}\left[\frac{B_{m-k-1}(3)}{m-k-1} + \frac{2B_{m-k}(3)}{(m-k)(m-k-1)} + \frac{1}{3}\delta_{m-k-2}\right] \\ &= \frac{m!}{2(m-k-2)!}\left[\frac{B_{m-k-1}(3)}{m-k-1} + \frac{2B_{m-k}(3)}{(m-k)(m-k-1)} + \frac{1}{3}\delta_{m-k-2}\right]\end{aligned}$$

where again

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

It follows that

$$\sum_{k=2}^{m}\frac{B_k}{k!}\left[f^{(k-1)}(1)-f^{(k-1)}(0)\right]=\sum_{k=2}^{m}\frac{B_k}{k!}\left[\frac{m!}{2(m-k-2)!}\left[\frac{B_{m-k-1}(3)}{m-k-1}+\frac{2B_{m-k}(3)}{(m-k)(m-k-1)}+\frac{1}{3}\delta_{m-k-2}\right]\right]$$
$$=\sum_{k=2}^{m}\binom{m}{k}B_k\left[\frac{(m-k)B_{m-k-1}(3)}{2}+B_{m-k}(3)+\frac{(m-k)(m-k-1)}{2\cdot 3}\delta_{m-k-2}\right]$$
$$=\frac{m(m-1)}{2\cdot 3}B_{m-2}+\sum_{k=2}^{m}\binom{m}{k}\frac{(m-k)}{2}B_kB_{m-k-1}(3)+\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(3)$$

Thus, for $m>2$ formula (1.6) reduces to

$$\frac{m(m-1)}{2}\left[\frac{B_{m-1}(3)}{m-1}+\frac{2B_m(3)}{m(m-1)}\right]=B_m(3)+m\left[\frac{(m-1)}{2}B_{m-2}(3)+B_{m-1}(3)\right]$$
$$-\frac{m(m-1)}{2\cdot 3}B_{m-2}-\sum_{k=2}^{m}\binom{m}{k}\frac{(m-k)}{2}B_kB_{m-k-1}(3)-\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(3)$$

which can be rewritten as

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(m-k)}{2}B_kB_{m-k-1}(3)+\sum_{k=0}^{m}\binom{m}{k}B_kB_{m-k}(3)=\cancel{\frac{m}{2}B_{m-1}}(3)+\frac{m(m-1)}{2}B_{m-2}(3)-\frac{m(m-1)}{2\cdot 3}B_{m-2}$$
$$+\binom{m}{0}\frac{(m-0)}{2}B_0B_{m-1}(3)+\binom{m}{1}\frac{(m-1)}{2}B_1B_{m-2}(3)+\binom{m}{0}B_0B_m(3)+\cancel{\binom{m}{1}B_1B_{m-1}}(3) \tag{1.11}$$

or equivalently,

$$\sum_{k=0}^{m}\binom{m}{k}B_k\left[\frac{(m-k)}{2}B_{m-k-1}(3)+B_{m-k}(3)\right]$$
$$=\frac{m}{2}B_0B_{m-1}(3)-\frac{m(m-1)}{2}B_1B_{m-2}(3)+B_m(3)-\frac{m(m-1)}{2\cdot 3}B_{m-2}$$

References

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[HN] A. Hassen and H. Nguyen, Hypergeometric Bernoulli Polynomials and Appell Sequences, to appear in Intern. J. Number Theory.