ON SOME PROPERTIES OF CONIC SECTIONS THAT ARE SHARED WITH INFINITELY MANY OTHER CURVED LINES.

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Section 1

1.Conic sections have several properties that are shared only among themselves; but they also have several properties in common with infinitely many other curves. For instance, the axis that cuts the plane in two, with the origin placed at the vertex of the curve, is shared with infinitely many curves, both algebraic and transcendental. This is obvious to anyone who looks at the nature of curves. But geometry has shown other properties, which at first glance seem to be unique to conic sections, that are also shared with other curves. It is evident that the properties, by which conic sections are defined, are really their own, and that they cannot be shared with any other curve. However we encounter other properties beyond these, some of which are not easily seen to be unique, or not, to conic sections. To remedy this confusion, we must analytically find all curves that may have a certain property, and if we find that conic sections are the only curves satisfying that property, we will be certain that the property is unique to conic sections. Geometry has already given solutions to several questions **<u>of nature</u>** here and there; solutions that have considerably extended the art of mathematical analysis. We propose, thus, to add here some other similar questions, taken from the idea of oblique-angled diameters, that are shared principally with conic sections.

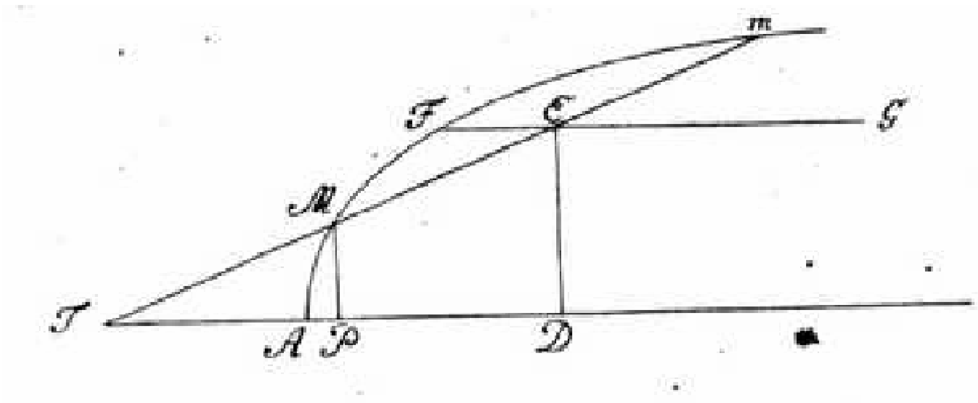


Fig. 1.

Section 2

2. We consider the following property of parabolas: any straight line parallel to the major axis is an oblique-angled diameter that divides all parallel lines drawn under a certain angle inside of the parabola in two equal parts. Indeed, if *AMFm* is a parabola with major axis *AD*, and if we draw *FG* parallel to the major axis with *F* on the parabola, we know that *FG* bisects at *E* any chord *Mm* parallel to the tangent of the curve at *F*. We can define the orientation of these chords as "parallel to the tangent at *F*", but this condition is true for all bisections. For, if in any curve whatsoever, the line FG meets all chords *Mm* making the given angle *GEm*, the tangent to the curve at *F* is always parallel to these chords. Consider that the lines very near the point *T* (*error, should be F*) will be parallel to the tangent. To see that this property is unique to the parabola, we turn to the following problem.

Section 3

3. *To describe a curve AMFm over the axis AD, which has the diameter FEG parallel to the axis AD at the distance given DE from the axis, and which bisects all chords Mm which make* (*a fixed*) *angle at T with the axis.*

Call the distance of the diameter to the axis *DE* = *a* and call the sine of the angle *MTA* *m*, the cosine will be called $n = \sqrt{1 - mm}$. If from any point *T* of the extended axis *AD*, we draw with the given angle the straight line *TMm*, it will cut the curve that we are looking for in two points *M* and *m*. This is why if we let *AT* = *t* and *TM* = *z*, the relation between *t* and *z* is given by an equation, with the property that for each value of *t* there are two values for *z*. Thus this equation will be a quadratic of the sort $zz = 2Pz - Q$, with *P* and *Q* any functions of *t*. Given any value of *t* we can determine two values of *z* which identify the double intersection of the straight line *Tm* and the curve.

Section 4

4. The line segments *TM* and *Tm* represent the two roots of z from the equation

$$zz - 2Pz + Q = 0 .$$

We will have

$$TM + Tm = 2P$$

and consequently

$$\frac{TM + Tm}{2} = P .$$

Because *E* is the midpoint of the line segment *Mm*, we have

$$\frac{TM + Tm}{2} = TE \text{ and thus } P = TE .$$

But because $DE = a$ and $\sin(DTE) = m$, it follows that $\frac{a}{TE} = m$ or $TE = P = \frac{a}{m}$. Thus the equation between *z* and *t* will be $zz = \frac{2az}{m} - Q$, where *Q* is any function of *t*. Suppose the equation for our curve *AMm* uses coordinates with the abscissa *AP* = *x* and corresponding ordinate *PM* = *y* Then $y : z$ will be = *m* and $\frac{t + x}{z} = n$, from which we get $z = \frac{y}{m}$ and $t = nz - x = \frac{ny}{m} - x$. The curve *AMm* will have the property previously required, if $\frac{yy - 2ay}{mm}$ is equal to any function whatsoever of $\frac{ny}{m} - x$.

Section 5

5. We set $ny - mx = X$ and $yy - 2ay = Y$ and then form the general equation between *X* and *Y*, that is to say

$$0 = \alpha + \beta X + \gamma Y + \delta X^2 + \varepsilon XY + \zeta Y^2 + \eta X^3 + \theta X^2 Y + \&c..$$

In this general equation we find all possible relations between *X* and *Y* and thus we have *Y* = to any function of *X*, so that $yy - 2ay$ will be = to any function of $ny - mx$ as required by our analysis. This is why to fully satisfy the proposed problem, we form any equation between the two variables *X* and *Y* and then we put $ny - mx$ in place of *X* and $yy - 2ay$ in place of Y. In this way we obtain an equation between *x* and *y* for the curve *AMm*. This equation has the property that the parallel *FG* to the axis *AD* which is the distance $DE = a$ from this axis, will be the oblique-angled diameter of the curve. This bisects all chords *Mm*, that make with it the angle *mEG*, whose sine is = *m*, the cosine = *n*.

Section 6

Thus there are an uncountable number of curves, which have the property that was written previously in the problem. In other words, at a given distance from the axis *AD*, the parallel diameter to the axis bisects all curves parallel to the tangent at *F*. In addition to this property, in the parabola, all straight lines parallel to the axis are at the same time the diameter. In the curve we have found, only a single straight line parallel to the axis has this property. We now ask if there are curves other than the parabola, in which two or more straight lines parallel to the axis are diameters. To simplify our search, we ask if among the curves found, besides the parabola, there is any other, in which the axis *Ad* is at least the orthogonal diameter. For this purpose the following problem is proposed.

Section 7

7. *Among all curves AMm, with the line AD as an axis of symmetry, in other words, it divides the curve into two similar and equal parts, to determine those that at a given distance on both sides of the axis AD have two oblique-angled diameters, like FG, that cut in two all chords Mm making the* (*fixed*) *angle with the axis AD.*

Because the axis *AD* divides the curve in two similar and equal parts, it is evident, if the straight line *FG* parallel to the axis *AD* is the diameter, that then in the other part of the curve at the same distance from the axis there must be a diameter parallel to the axis. But for the axis *AD* to be a similar orthogonal diameter, it is necessary that in the equation between *x* and *y* the variable *y* has only even powers and never an odd powers. We must thus exclude from the general equation found for the solution of the preceding problem all cases wherein the exponents of y are odd. But as X is $= ny - mx$ and $Y = yy - 2ay$, y is in both variables the exponent one , and consequently it is odd. We can form a new variable *Z* from the two variables *X* and *Y*, in which there will not be an odd power of *y*, and this selection is

$$Z = Y + \frac{2aX}{n} = yy - \frac{2m\,a\,x}{n}.$$

We will also satisfy the preceding problem by the general equation between *Y* and *Z*, which is:

$$0 = \alpha + \beta Y + \gamma Z + \delta Y^2 + \varepsilon YZ + \varsigma Z^2 + \eta Y^3 + \theta Y^2 Z + etc.$$

Setting

$$Y = yy - 2ay \text{ and } Z = yy - \frac{2m\,a\,x}{n},$$

this similarly comprises in itself all curves that fit there.

Section 8

Hence it appears that in all terms that do not contain Y, we do not find odd powers of *y* and that consequently these terms, α, γZ, ςZ^2, χZ^3 etc. must be distinguished because they are significant. But the term *Y* must be excluded, since it contains y^1, this power can't be subtracted by any of the following terms, and for the same reason we exclude the terms YZ, YZ^2, YZ^3 etc. What is more, if we allow the term Y^2, because in contains the power y^3, we will be obliged to use at the same time Y^3 so that we can subtract y^3. But Y^3 contains y^5 that we can not subtract without Y^4 . So it follows that any power of *Y* containing an odd power of *y*, that is not in the preceding terms, must be killed by the following terms, from which is born a progression ad infinitum. The same must be said of the terms $Y^2 Z$, $Y^3 Z^2$, etc. of which none are to be used, without allowing an infinitely many terms. We will thus satisfy the requirements only by the equation

$$0 = \alpha + \gamma Z + \varsigma Z^2 + \chi Z^3 + \text{etc}$$

that contains no Y. And for this equation Z will be = to a constant, that is to say

$$yy - \frac{2max}{n} = C\,,$$

This is a parabola, and all other (algebraic) curves are excluded.

Section 9.

Beyond the parabola of Appolonius, there are thus other curves (*which are transcendental*) with an axis of symmetry, that have at least one diameter parallel to the axis, so that this property does not come only to the parabola. But by virtue of the equation

$$yy - \frac{max}{n} = 0$$

(we can make the constant C equal to zero) it seems that not only at the distance given by *a*, but at all distances from the axis, we find a diameter parallel to the axis. If we set $\frac{2ma}{n} = c$, for the equation $yy - cx = 0$, that is the equation for any parabola, if at any distance = *a*, we construct a line parallel to the axis, it will be the diameter and it will bisect all the chords, that make with the axis an angle, whose tangent is = $\frac{m}{n} = \frac{c}{2a}$. With the exception of the axis of symmetry, there need not be any oblique-angled diameters, and thus no straight lines parallel to the axis are at the same time diameters. But this analysis must be restricted only to algebraic curves, for the transcendents are not excluded by this progression of terms Y, Y^2, Y^3, etc. ad infinitum.

We can produce many transcendent curves, that have many parallel diameters between them.

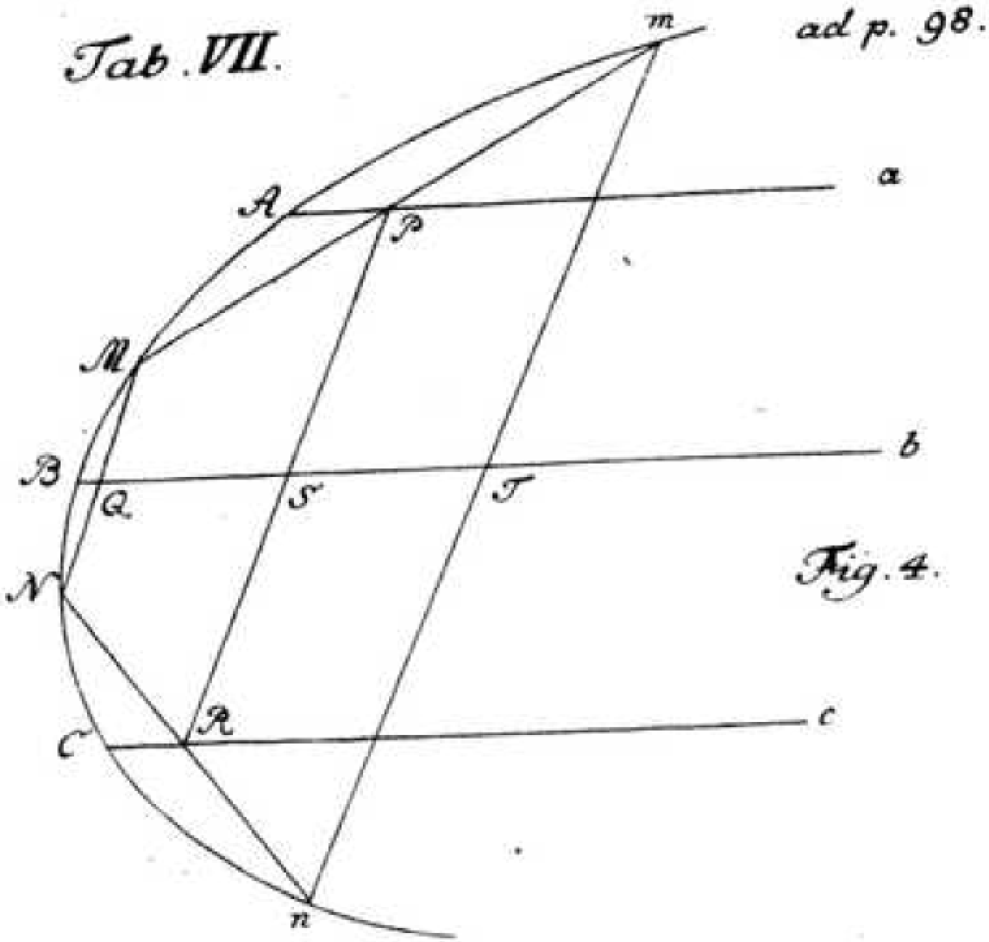
Section 10.

Our plan does not allow us to proceed here to examine these transcendental curves, since in this memoir we have in mind algebraic curves. However, until we discover that there actually exists similar transcendent curves, that satisfy the present question, we give a general equation, that incorporates in it all transcendent curves. Set $Y = yy - 2ay$, and we search for the value of the function T that satisfies this infinite differential equation, **<u>in setting the element dY constant</u>**:.

$$0 = \frac{dT}{dY} + \frac{4aYd^3Y}{1.2.3.dY^3} + \frac{16a^4Y^2d^5T}{1.2.3.4.5dY^5} + \frac{64a^6Y^3d^7T}{1.2....7dY^7} + \&c$$

Then T which is a function of Y is thereby a function of y, in which we find no odd dimensions of y. This is why, if we take W to be any function of T **<u>and</u>** $Z = y^2 - \frac{2ma\,x}{n}$, the equation W = 0 gives all curves that have the proposed property; that is to say, that beyond the axis *AD* which is an orthogonal diameter, we have on both sides at a given distance = a from the axis the oblique-angled diameters parallel to the axis.

Fig 4.



Section 11

In the following we are concerned with a general law, that all curves which have two parallel diameters, will have an infinity of similar diameters, equally distant from one another. Given that the curve *mABC* has two parallel diameters *Aa*, *Bb*, of which *A*a cuts in two all chords *Mm* parallel to the tangent at *A*, and *Bb* cuts likewise *MN*, *mn* parallel to the tangent at *B*. The terminal points *M* and *m* of any chord *Mm* are divided in two by the diameter *Aa*, and if we take the chords *MN* and *mn* parallel to the tangent at *B*, we will have *MQ* = *NQ* and *mT* = *nT*. Now we take the chord *Nn*, and it will make a (*certain*) angle with the diameters *Aa* or *Bb*, because (*it is one*) of all angles from the given quadrangle *MNnm*. If we take *PSR* parallel to *MN, mn,* then this new chord *Nn* will be divided in two at *R* and the point *R* will always lie on the straight line *Cc* parallel to *Aa* and *Bb* and its distance from the diameter *Bb* will be equal to the distance from the diameter *Bb* to the diameter *Aa*. Thus this line *Cc* will cut in two all chords *Nn* and will consequently be a diameter.

Section 12

Hence the angle *NRc* will be completely determined by the given angles *MQb* and *mPa*, since

$$\cot mPa + \cot NRc \text{ will be } = 2\cot MQb$$

and consequently

$$\cot NRc = 2\cot MQb - \cot mPa$$

The cotangents of the angles *mPa, MQb, NRc* constitute an arithmetic proportion. But as we have demonstrated that the **(*existence of*)**two diameters *Aa, Bb* the **(*implies the existence of the*)** third *C*c, likewise by any two adjacents we **will** (***He has already DONE THIS IN THE LAST SECTION)***demonstrate the following; that if a curve has two parallel diameters between them, it will have an infinity of distant diameters between them at equal intervals. That if the cotangent of the angle *mPa*, under which the first diameter cuts in two the chords, it is said = *p* and the cotangent of the angle *MQb*, for the second diameter *Bb* = *q* , the cotangent of the angle *NRc*, under which the third diameter cuts in two the chords, will be $= 2q - p$ and the cotangent of the angle, under which the fourth diameter in following cuts in two all chords, $= 3q - 2p$, the cotangent for the fifth diameter $= 4q - 3p$ and so it goes; so that the cotangents of all the angles, under which the diameters who follow by order cut the chords in two, constitute an arithemetic progression. In this way the transcendent curves, in which we find three diameters, in which the one in the middle is orthogonal, have at the same time an infinity of diameters.

Section 13

From this we can now demonstrate with full mathematical rigor, that the parabola is the only curve, in which all the lines without exception, that are parallel to the axis, are at the same time diameters. To attribute this property to a curve, it suffices that it has two diameters that approach infinitely; then, by the preceding demonstrations, it must be that all lines which are parallel to them are diameters. Hence we have posed here below (§10) the distance of two diameters who follow immediately = *a*, this is why this distance *a* must be vanishing. That being done, thr equation of (§10) becomes

$$0 = \frac{dT}{dY};$$

and consequently

T = to a constant.

If thus $W = 0$ shows the general equation for all curves, of which all straight lines parallel to the axis are diameters, W will be any function of T or of a constant quantity, and of

$$Z = yy - \frac{2ma\,x}{n}$$

We will take thus this equation Z = to a constant, and so

$$yy - \frac{2ma\,x}{n} = C$$

which equation contains in itself no other curve than the parabola.

Section 14

After having easily completed the discussion of parallel diameters, those which are generalizations of the parabola, we now examine the diameters that converge at a point, to understand more completely the nature of the ellipse and the hyperbola, curves in which all straight lines through their center are diameters. Reasoning in the same manner, we will determine, if this property is not found in any other curves. There is truly not any doubt, that it is an attribute of conic sections, that all straight lines without exception, that pass through the center, are at the same time diameters. However, there may exist other curves, that do not have an infinity of diameters which meet at the same point, but have two or three. To discover this, we propose to solve the following problem.

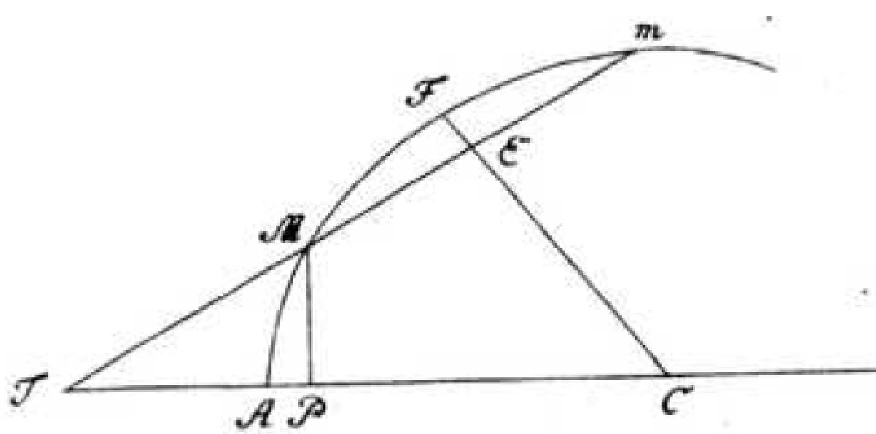


Fig. 2.

Section 15

Find all curves (Fig 3) **(Should be Fig 2.)** *AMm constructed above an axis AC with the following condition, that emerging from the point C the straight line CF, that makes the angle ACF with the axis, this line bisects at E all chords Mm parallel to the tangent at F.*

To begin it is apparent, that if all chords that the line *CF* cuts in two, are parallel to each other, the tangent to the point *F* must also be parallel to them. Thus the angle *ETC* is constant, so we put

the sine of the angle *ETC* = *m*,

the cosine $= n = \sqrt{1-mm}$,

and also

the sine of the angle *ACF* is = *p*,

the cosine $= q = \sqrt{1-pp}$.

The sine of the angle *CEm*, at which the diameter *CF* cuts in two the chords *Mm*, will be = *mq*+*np* and the cosine = *nq*-*mp*. Because the point *T* is variable, we set *CT* = *t* and when moving the line *TMm* keeping the constant angle *CTE*, this line will cut the curve in two points *M* and *m*. Thus the variable which is = *z*, describing the intersections of the line *TM*, will have two values, one for *TM*, another for *Tm*. This is why *z*, a function of *t*, will be determined by a quadratic equation, which is

$$zz = 2Pz - Q.$$

Here *P* and *Q* are functions of *t* and therefore *TM* will be $= P - \sqrt{PP - Q}$ and $Tm = P + \sqrt{PP - Q}$.

Section 16

Thus *TM* + *Tm* will be = 2*P* and because *E* is the mid-point of the chord *Mm*, *TE* will be = *P*. But from the angles given in the triangle *CTE*, we will have

$$CT : TE = \sin A.CET : \sin A.\,TCE,$$[1]

$$t : P = mq + np : p,$$

from which we get

$$P = \frac{pt}{mq + np}$$

and thus we have between z and t the equation:

$$zz = \frac{2ptz}{mq + np} - Q,$$

where Q is any function of t. Now to determine the curve, we set ($CP = x$)

$PM = y$, and we get

$$\frac{y}{z} = \frac{PM}{TM} = m,$$

and consequently

$$z = \frac{y}{m} \text{ and } PT = t - x = nz = \frac{ny}{m},$$

so that

$$t = \frac{mx + ny}{m}.$$ **<u>(Euler does not write the "*t*".)</u>**

From this the desired equation for the curve is

$$\frac{yy}{mm} = \frac{2p(mx + ny)y}{mm(mq + np)} - Q,$$

Q being any function of

$$t = \frac{mx - ny}{m}.$$ **<u>(The minus sign should be plus.)</u>**

This is why we will have

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[1] Here Euler writes sin *A.CET* to mean sine of the angle *CET*.

$$\frac{2pmxy+(np-mq)yy}{mm(mq+np)} \text{ or } yy+\frac{2mpxy}{np-mq}$$

as any function of $x-\frac{ny}{m}$. **(The minus sign should be plus.)**

Or if we set

$$x-\frac{ny}{m}=X$$ **(The minus sign should be plus.)**

$$\text{and } yy+\frac{2mpxy}{np-mq}=Y,$$

and let W be any function of X and Y , then the equation $W = 0$ will express the family of all curves, that we desire.

Section 17

This analysis is incomplete, for it contains only the curves, that have but one oblique-angled diameter. Here the intersection C is fully arbitrary, and depends on the position of the axis AC which is also arbitrary. We continue our search by selecting from these many curves, those that the axis AC divides in two similar and equal parts. In other words, look for curves for which the axis AC is an orthogonal diameter **(i.e. an axis of symmetry)**. We thus require that in the equation found below, the powers of y are everywhere even exponents. This can occur if the odd powers of y are cancelled by later terms from the series. Thus in the general equation sought, we put

$$X=x+\frac{ny}{m} \text{ and } Y=yy+\frac{2mpxy}{np-mq},$$

and get

$$0=\alpha+\beta X+\gamma Y+\delta X^2+\varepsilon XY+\zeta Y^2+\eta X^3+etc.,$$

where the coefficients must be determined so that the odd powers of y vanish.

Section 18

At once we see that β must be = 0, because the term $\frac{ny}{m}$ cannot be cancelled by any of the following terms. On the other hand, we see that γ and δ can be determined such that the terms $\alpha + \gamma Y + \delta X^2$ have no odd powers of *y*: if we put

$$\gamma = np - mq \quad \text{and} \quad \delta = -\frac{mmp}{n},$$

we get

$$(np - mq)Y - \frac{mmp}{n} X^2 = -mqyy - \frac{mmpxx}{n}.$$

Therefore we let

$$Z = nqyy + mpxx = mpX^2 - \frac{n(np - mq)}{m} Y,$$

and observe that *Z* is a function, in which y has only even powers. This is why, if *W* is any arbitrary function of

$$Z = nqyy + mpxx \quad \text{and} \quad X = mx + ny,$$

then the equation *W* = 0 will contain solutions to the previous problem, and beyond that we will want to throw out the odd powers of *y*.

Section 19

Therefore we set

$$X = mx + ny \text{ and } Z = mpxx + nqyy,$$

in the equation for the curves, in which the straight line *CF* is a diameter, and get

$$0 = \alpha + \beta X + \gamma Z + \delta X^2 + \varepsilon XZ + \zeta Z^2 + \eta X^3 + etc.$$

If all the terms, in which *X* appears vanish, then every appearance of the variable *y* has even powers and the resulting curve is simultaneously divided by the axis *AC i*n two similar and equal parts. Then *Z* will be = *C* **(Euler means "is a constant")** or *aa* = *mpxx*+*nqyy*. This equation contains conic sections with center *C* and with principal axis *AC*. Writing *bb* in place of $\frac{aa}{nq}$, we get

$$yy = bb - \frac{mp}{nq}xx \ .$$

At present we have the general equation for conic sections $yy = bb - kxx$ and it seems we should be able to determine more from this. Here lines *CF* passing through the center make an angle *FCA* with the axis, with tangent $= \frac{p}{q}$. This straight line bisects all chords *Mm*, that extended make the angle *MTC* with the axis *AC*, with tangent $\frac{m}{n} = k\frac{q}{p}$.

So the tangent of the angle *CEm*, under which the chords *Mm* are cut in two by the diameter *CF*, will be

$$= \frac{pp + kqq}{(1-k)pq}.$$

From this we see that the tangent $\frac{p}{q}$ is not a fixed number, but can be an arbitrary value, so that every straight line *CF* emerging from the center is a diameter. If *k* = *t* **(this should by *k* = 1)**, then all of these lines are orthogonal diameters and it is clear that the curve is a circle.

Section 20

We can find other curves, in which *AC* is the orthogonal diameter, if we determine the coefficients of the terms, in which *X* is found, such that no *y* term has an odd power. Initially we see that neither X nor X^2 can occur, because *y* and *xy* are not subtracted by any following terms. Otherwise, if *y* does not enter in by *X*, it must be that *n* is = 0 and so that the term *xy* is not in *XX*, *mn* should be = 0. But the terms X^3 and *XZ* generate homogeneous terms, from which the terms y^3 and *xxy* can be removed, if *np* is = 3*mq*. In a similar way, from the terms X^5, X^3Z and XZ^2, which are homogeneous, we can remove the odd terms, if we take

$$\frac{np}{mq} \text{ is } = 3 \text{ or } \frac{np}{mq} = 5 + 2\sqrt{5}.$$

In the same manner there must always be a certain relation between the tangents $\frac{m}{n}$ and $\frac{p}{q}$, so that the powers occur as required. If these relations are not found, it is impossible to produce other curves that are satisfied besides the conic sections.

Section 21

To investigate special cases, in which $\frac{m}{n}$ has a certain relation with $\frac{p}{q}$, we write

$$\frac{m}{n} = g \text{ and } \frac{mp}{nq} = k,$$

so that

$$X \text{ is } = gx + y \text{ and } Z = kxx + yy.$$

Taking homogeneous terms of order three, that is $\alpha X^3 + \beta XZ$; and after substitution we get

$$\begin{array}{cccc} +\alpha g^3 & +3\alpha g^2 & +3\alpha g & +\alpha \\ x^3 & x^2 y & xy^2 & y^3 \\ +\beta gk & +\beta k & +\beta g & +\beta \end{array}$$

in which the terms, that contain the odd dimensions of y, must vanish. Thus $\alpha+\beta$ will be = 0 and $3\alpha g^2+\beta k=0$. Therefore

$$\beta=-\alpha \text{ and } k = 3gg \text{ or } \frac{mp}{nq}=\frac{3mm}{nn},$$

and it follows that $\frac{p}{q}=\frac{3m}{n}$. Hence $\alpha X^3+\beta XZ$ changes into $2\alpha\left(gxyy-g^3x^3\right)$ or

$$2\alpha\left(\frac{m}{n}xyy-\frac{m^3x^3}{n^3}\right).$$

Section 22

Thus if the tangent of the angle $ACF = \frac{p}{q}$, is three times bigger than the tangent of the angle $CTE = \frac{m}{n}$, then we will be able to find infinitely many curves *AMm*, in which *AC* is the orthogonal diameter and *CF* is the oblique-angled diameter. Further, if we let the tangent of the angle $ACE = \theta$, we will have

$$\frac{p}{q}=\theta \text{ and } \frac{m}{n}=\frac{1}{3}\theta,$$

and the tangent of the angle *CEM* will be = $\frac{4\theta}{3-\theta\theta}$. Then write

$$Z = yy+\frac{1}{3}\theta\theta xx \text{ and } V=\frac{1}{3}\theta xyy-\frac{1}{27}\theta^3x^3,$$

and if W denotes any function of Z and V, then the equation $W = 0$ will express the curve, that possesses the aforementioned property. Hence it is obvious, with

AC being the orthogonal diameter, then the straight line from *C* passing under *AC*, will make with *AC* an angle below the axis, of which the tangent is = θ. This will be an oblique-angled diameter, of the same type as *CF* at the top. Therefore the curves with this property are contained in the general equation:

$$0 = \alpha + \beta Z + \gamma V + \delta Z^2 + \varepsilon ZV + \zeta V^2 + \eta Z^3 + etc.$$

Section 23

From the infinity of curves of this nature we select the curves of third order, that are contained in the equation:

$$a^3 = byy + \frac{1}{3}\theta\theta\zeta xx + \frac{1}{3}\theta xyy - \frac{1}{27}\theta^3 x^3$$

(In the second term on the right ζ should be *b*.)

or

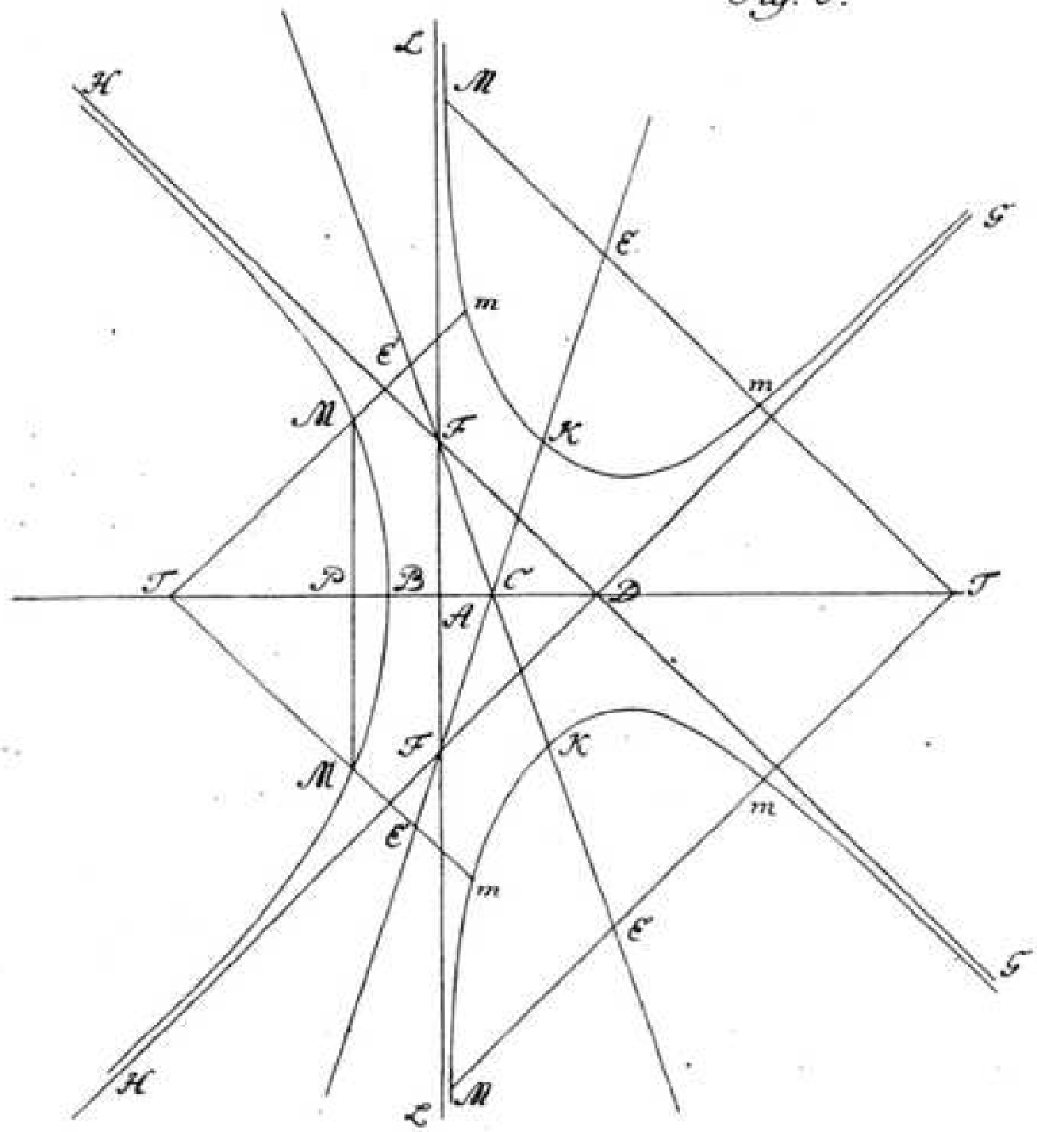
$$yy = \frac{a^3 - \frac{1}{3}\theta\theta bxx + \frac{1}{27}\theta^3 x^3}{b + \frac{1}{3}\theta x}.$$

These curves are examples of the redundant hyperbolas of Newton, that have only one orthogonal diameter. The general equation for these is

$$yy = \frac{Av^3 + Bv^2 + Cv + D}{v}.$$

Here the origin of the abscissa is the point of the axis, where the asymptote is parallel to the direction of *y*. **(This asymptote is the *y*-axis.)** Among these curves, those that will satisfy our requirements have *C* **(this is C from the above equation)** equal to $\frac{BB}{4A}$. So then, $v = -\frac{B}{6A}$, is the point *C*, **(on the axis)** from which the straight line *CF*, makes with the axis the angle *FCA*, for which the tangent is = $3\sqrt{A}$. This straight line will be the oblique-angled diameter, cutting in two the chords Mm, that make with the axis *CA* an angle, whose tangent = $\sqrt{A}$. The tangent of the angle, made where these chords meet the diameter is = $\frac{4\sqrt{A}}{1-3A}$.

Fig. 6.



Section 24

Therefore, (Fig. 6) *MMmm* is a similar redundant hyperbola, having the axis *AP*, that is at the same time an orthogonal diameter, such that in taking the abscissa *AP* = *v* and in letting the ordinate *PM* = *y*, *yy* is

$$= \frac{Av^3 + Bv^2 + Cv + D}{v} = \frac{(2Av + B)^2}{4A} + \frac{D}{v}.$$

Here *C* is = $\frac{B^2}{4A}$. The straight line *LAL* normal to the axis will be an asymptote of the curve and the two other asymptotes *HDG* will cross at the point *D* of the axis, such that *AD* is = $\frac{B}{2}$. The tangent of the angle *HDA* will be = $\sqrt{A}$ and the whole curve will be composed of three parts in hyperbola like form *MBM*, *mKm*, and *mKm*. Now take AC = $\frac{B}{6A}$ and construct above and below the lines *CF*, *CF*, such that the tangent of the angle *FCA* is $= 3\sqrt{A}$, these two diameters will bisect all chords *Mm*, that were extend to make with the axis the angle *MTA*, with tangent = $\sqrt{A}$. Those bisected straight lines *Mm* will therefore be parallel to one of these diameters. As for the rest this curve, it can take many different shapes, depending on the value of *B*, seeing that *A* is an affirmative **(positive)** number. The one intersection of the axis at *B* shown in the figure is not the only possible case. It can occur that the curve cuts the axis in three points and this happens if we let *AB* = *a*, and we take

$$v = \frac{-a}{2} - \frac{B}{24} \pm \sqrt{\left(-\frac{Ba}{2A} - \frac{3aa}{4}\right)}.$$

Section 25

The same values (§22)

$$Z = yy + \frac{1}{3}\theta\theta xx \quad \text{and} \quad V = \frac{1}{3}\theta xyy - \frac{1}{27}\theta^3 x^3,$$

can be made to find infinitely many curves of higher order, that in addition to the orthogonal diameter, have two or more oblique-angled diameters. But like the formula V found by $\alpha X^3 + \beta XZ$ (§21) when making the odd powers of y vanish, seemingly we can do the same thing with greater powers. For example, let

$$V = \alpha X^4 + \beta X^2 Z$$

and setting

$$\frac{m}{n} = g \quad \text{and} \quad \frac{mp}{nq} = k,$$

we get

$$\begin{array}{lllll}
+\alpha g^4 & +4\alpha g^3 & +\zeta\alpha g^2 & +4\alpha g & +\alpha \\
\quad x^4 & \quad x^3 y & & \quad xy^3 & \\
+\beta g^2 k & +2\beta gk & +\beta k\, x^2 y^2 & +2\beta g & +\zeta\, y^4 \\
 & & +\beta gg & &
\end{array}$$

Thus $2\alpha g^2 + \beta k$ must be = 0 and $2\alpha + \beta = 0$: from which we get

$$\beta = -2\alpha \text{ and } k = gg \text{ or } \frac{mp}{nq} = \frac{mm}{nn}.$$

Thus $\frac{m}{n} = \frac{p}{q}$ and if we set as before $\frac{p}{q} = \theta$, this will make

$$\frac{m}{n} = \theta\,,\; g = \theta\,,\; k = \theta\theta\,;$$

and like this *V* will be

$$= -\alpha\theta^4 x^4 + 2\alpha\theta^2 x^2 y^2 - \alpha y^4 \text{ or } V = -\alpha\left(\theta^2 xx - yy\right)^2 .$$

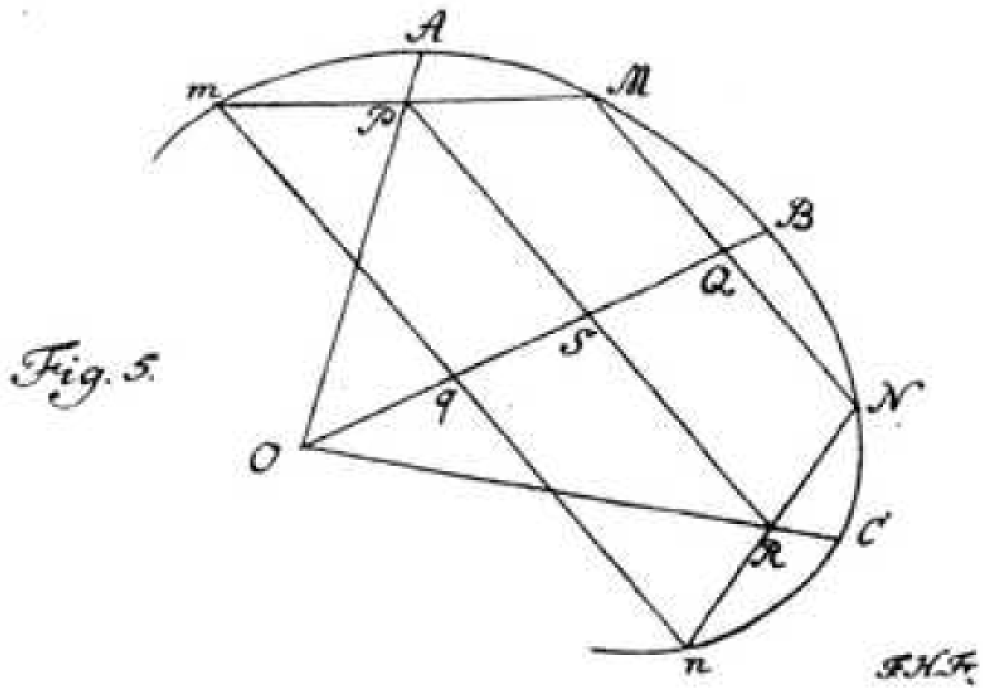
This is why if *W* is taken as any function of

$$Z = \theta^2 xx - yy \text{ and } V = \left(\theta^2 xx - yy\right)^2$$

and we set *W* = 0, the curve, besides the orthogonal diameter *CA*, will have oblique-angled diameters emerging from *C*, that make with *CA* an angle, whose tangent $= \theta$, and these diameters will cut in two the chords *Mm* inclined at the axis *CA* under the angle, of which the tangent $= \theta$. Among these curves, the simplest is that which is given by the equation

$$\alpha^4 = \theta^4 x^4 y^4 .$$

In addion to all these curves, besides the orthogonal diameter *CA*, many also have an orthogonal diameter that is vertical at *C*; for these the equations have not only *y*, but also *x*, with only even powers.



Section 26

Using the same method we can go further by studying homogeneous expressions of higher powers to eliminate odd powers that must be thrown away. We find other functions for V, that require other relations between

$$\frac{m}{n} \text{ and } \frac{p}{q}.$$

We will not stop there however, but we bring back a property of very grand importance regarding diameters, that can be accommodated by all the curves that we find. Here is the question. If the curve *ABC* (Fig. 5) has two diameters *AO, BO,* that intersect at *O*, this same curve will have many more diameters, that meet at the point *O*. Sometimes an infinite number occur, if new diameters do not coincide with previous ones. To see this, consider that the curve has two diameters *AO* and *BO*, where *AO* bisects all chords *Mm* making angle *mPO* and *BO* intersects likewise the chord *MN* with the angle *MQO*. From points *M* and *m* of any chord *Mm* bisected by the diameter

AO, we extend cords through the other diameter *BO*, that are *MN* and *Mn*. These are bisected by the diameter *BO* at *Q* and *q*. After connecting the chord *Nn*, all the angles will be known in the quadrilateral *MNnm.* If from *P* we extend *PR* parallel to *MN*, *mn*, this line will intersect the chord *Nn* at *R*. Now construct *ORC*, the angles *BOC* and *NRC* will also be defined, from which it follows that the line *OC* will be again a diameter, that bisects the chord *Nn* making given angle *NRO*.

Section 27

To understand more fully the above phenomenon, let the tangent of the angle *mPO* = α, the tangent of the angle *AOB* = *B* and the tangent of the angle MQO = β. It follows from this that the cotangent of the angle *BOC*

$$=\frac{1}{B}+\frac{2}{\beta},$$

and consequently the tangent of the angle *BOC,* that is = *C* = $\frac{\beta B}{\beta+2B}$. If we let the tangent of the angle *NRO* = γ , γ will be

$$=\frac{\alpha\beta^2(1+BB)}{2\alpha\beta+4\alpha B-2\alpha\beta B^2-\beta^2-4\beta B-4BB-\beta\beta BB}.$$

Now it follows that if the tangents of the next angles in the same order are called *D* and δ,

$$\text{D will be } = \frac{\gamma C}{\gamma+2C}$$

and

$$\delta=\frac{\beta\gamma^2(1+CC)}{2\beta\gamma+2\zeta C-2\zeta\,\gamma C^2-\gamma^2-4\gamma C-4CC-\gamma\gamma CC}.$$

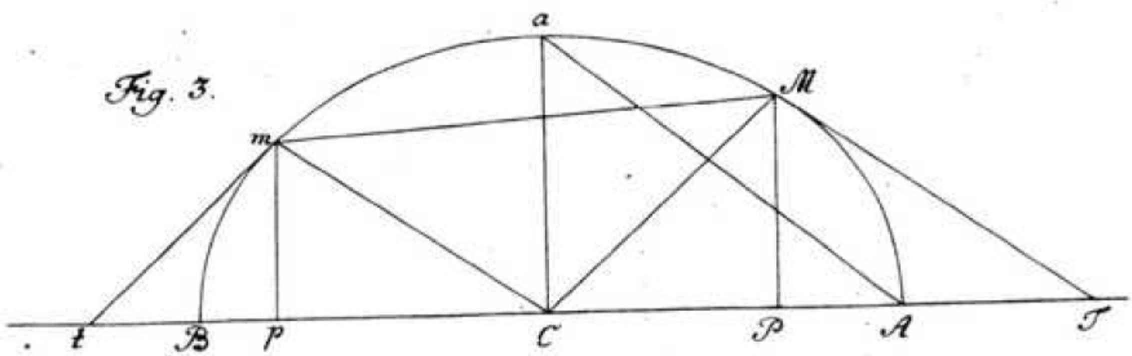
<u>(In the numerator the original has $\beta\gamma(1+CC)$. Also the second term in the denominator should be $4\zeta C$.))</u>

In some cases we can continue to infinity, at least for those **<u>(diameters)</u>** that do not coincide exactly with the first.

Section 28

To these problems on parallel diameters, or those that intersect at a given point, I will add another that is closely related and which the skilled Mr. Clairaut mentions in one of the letters he has done me the honor of writing. The origin of this problem comes from the property of the ellipse, in which the inscribed parallelograms around the two conjugate diameters, encompass everywhere the same area. Since all the lines in the other curves intersecting at a fixed point, as from a center, are not diameters, we will not pay attention in this research to the conditions, that define diameters and we propose only the following problem.

Fig. 3.



We seek a curve AMamB that has two orthogonal diameters ACB and aC that are *perpendicular to each other. Thus the center of this curve is at C. Like the ellipse, it* *must have the following property: that extending from the cente*r *C any ray CM and at the same time another ray Cm parallel to the tangent MT at the point M*, the area of the triangle *MCm is always constant , and is equal to the area of the triangle ACa*.

Section 29

To solve this problem, let, after having dropped a perpendicular *MP* from the point *M* on the axis *AC*, we let the abscissa *CP = x* and the ordinate *PM = y*. First the equation for the curve, that is *W* = 0, will necessarily be such that *x* and *y* have on both sides even power. Letting *x* or *y* or both be negative, the equation remains always the same. Thus *W* will be any function of *xx* and *yy*. This condition follows from the above property, in virtue of which, the lines *AC* and *aC* must be orthogonal diameters of the curve. Now from the point *m* we costruct on the axis *AB* the perpendicular *mp.* We say that *Cp = t* and *pm = u*. It follows from the continuity of the curve that the same equation is found between *t* and *u*, that is between *x* and *y*. If in the equation *W* = 0 in place of *xx* we put *tt*, the value of *yy* changes into *uu.*

Section 30

We introduce a new variable **(parameter)** *z*, from which we can determines the values *xx* and *yy*, such that when *z* is eliminated, we arrive at the equation for the curve *W* = 0. We conceives some quantity *z*, which when made negative, *xx* changes into *tt* and *yy* to *uu*. Thus it is obvious that in eliminating *z*, the equation between *tt* and *uu* must be the same as that between *xx* and *yy*, as the law of continuity requires. Suppose we are given *P* and *R* as even functions of *z*, that remain the same in setting $-z$ in the place of +*z*, and given *Q* and *S* as functions of odd powers of *z*, that change into their negatives, if we set -z in the place of +z. Thus if we set $xx = P + Q$ and $yy = R + S$, then when making z negative, we will have $tt = P - Q$ and $uu = R - S$. By these manipulations we arrive at a curve that continuous connects the parts *AMa* and *amB* and where both *AC* and *aC* are orthogonal diameters.

Section 31

What is more, because the straight line *Cm* must be parallel to the tangent *MT*, the undertangent $PT = -\frac{ydx}{dy}$. This will make $PT : PM = Cp : pm$ or

$-dx : dy = t : u$, from which we get

$$udx + tdy = 0 .$$

Finally since the area of the triangle *MCm* must be constant, we calculate this area and get

$$= \frac{1}{2} CM \cdot Cm \cdot \sin A \cdot MCm .$$

<u>(Recall that Euler writes "sin *A.MCm*" to mean sine of angle *MCm*.)</u>

But

$$\sin A \cdot MCm = \sin A \cdot (MCP + mCp) = \frac{PM \cdot Cp + CP \cdot pm}{CM \cdot Cm},$$

from which we find that the area of the triangle *MCm* is $= \frac{ty + ux}{2}$. Consequently the value of $\frac{ty + ux}{2}$ must be constant and so its differential is equal to zero. Therefore

$$ydt + tdy + udx + xdu = 0 .$$

Since $udx + tdy$ is = 0, this makes

$$ydt + xdu = 0 .$$

From this equation we read that the tangent *mt* is parallel to the ray *CM*

and that the rays *CM* and *Cm* are reciprocally parallel to their tangents *MT* and *mt*.

Section 32

We let *ty + ux = 2cc* and since

x is $=\sqrt{P+Q}$, $y=\sqrt{R+S}$, $t=\sqrt{P-Q}$ and $u=\sqrt{R-S}$,

we get after making substitutions

$$\sqrt{(P+Q)(R-S)}+\sqrt{(P-Q)(R+S)}=2cc.$$

Suppose *V* denotes any odd functions of *z* and that we let

$$\sqrt{(P+Q)(R-S)}=cc+V.$$

If we make *z* negative, $\sqrt{(P+Q)(R-S)}$ changes to $\sqrt{(P-Q)(R+S)}$ so that

$$\sqrt{(P-Q)(R+S)} \text{ is } = cc - V,$$

as it is required by the nature of the given conditions. One can thus infer that

$$R+S=\frac{(cc-V)^2}{P-Q} \text{ and } R-S=\frac{(cc+V)^2}{P+Q}.$$

So

$$x \text{ is } =\sqrt{P+Q},\ t=\sqrt{P-Q},\ y=\frac{cc-V}{\sqrt{P-Q}} \text{ and } u=\frac{cc+V}{\sqrt{P+Q}}.$$

From these we have

$$dx=\frac{dP+dQ}{2\sqrt{P+Q}} \text{ and } dy=-\frac{dV}{\sqrt{P-Q}}-\frac{(cc-V)(dP-dQ)}{2(P-Q)\sqrt{P-Q}},$$

and thus $udx+tdy=\frac{(cc+V)(dP+dQ)}{2(P+Q)}-dV-\frac{(cc-V)(dP-dQ)}{2(P+Q}.$

(This last expression should be

$$udx + tdy = \frac{(cc+V)(dP+dQ)}{2(P+Q)} - dV - \frac{(cc-V)(dP-dQ)}{2(P-Q)}$$ **.)**

Since *udx* + *tdy* must be = 0,

0 will be = $(PP - QQ)\, dV - V\,(PdP - QdQ) - cc\,(PdQ - QdP)$.

Section 33

We divide this equation by $(PP-QQ)^{3/2}$ and we get

$$\frac{dV}{\sqrt{PP-QQ}} - \frac{V(PdP-QdQ)}{(P^2-Q^2)^{3/2}} = \frac{cc(PdQ-QdP)}{(P^2-Q^2)^{3/2}}.$$

After integration have

$$\frac{V}{\sqrt{PP-QQ}} = \int \frac{cc(PdQ-QdP)}{(PP-QQ)^{3/2}}.$$

Now let $Q = Pz$, and because P is an even function, Pz is an odd function, as is Q . Because $dQ = Pdz + zdP$, we have

$$\frac{V}{P\sqrt{1-zz}} = \int \frac{ccdz}{P(1-zz)^{3/2}}.$$

This must be an integrable formula, if we want to discover algebraic curves. We write

$$\int \frac{ccdz}{P(1-zz)^{3/2}} = \frac{Z}{\sqrt{1-zz}},$$

so that *V* becomes = *PZ*. We see that *Z* is an odd function of *z*, because *V* was defined as an odd function. From this we get

$$\frac{ccdz}{P} = (1-zz)dZ + Zzdz \text{ and } P = \frac{ccdz}{(1-zz)dZ + Zzdz}.$$

Section 34

Now let *Z* be an arbitrary odd function of *z*, then

$$P = \frac{ccdz}{(1-zz)dZ + Zzdz},$$

is an even function. Thus

$$Q = Pz = \frac{cczdz}{(1-zz)dZ + Zzdz} \text{ and } V = PZ = \frac{ccZdz}{(1-zz)dZ + Zzdz}.$$

Thus we arrive at the complete solution of the problem by this method in the following way. Let *x* and *y* be determined by taking Z as an arbitrary odd function of *z*, and get **(the parametric equations)**

$$xx \text{ is } = \frac{cc(1+z)dz}{(1-zz)dZ + Zzdz} \quad \text{and} \quad yy = \frac{cc(1-z)((1+z)dZ - Zdz)^2 dz}{((1-zz)dZ + Zzdz)dz}.$$

Hence virtue of the previous argument, by making *z* and *Z* negative we get *tt* and *uu*

$$tt = \frac{cc(1-z)dz}{(1-zz)dZ + Zzdz} \text{ and } uu = \frac{cc(1+z)((1-z)dZ + Zdz)^2 dz}{((1-zz)dZ + Zzdz)dz}.$$

Thus we get from these an infinite number of curves endowed with the proposed properties, First they have symmetry around the principal axes *aC* and *AC.* Next, extending from the center C the two rays CM and *Cm*, to the tangents of the curve at *M* and m reciprocally parallel, we get for the area of the triangle $MCm = cc$.

Section 35

Thus we find the equation for the curve in *x* and *y*, if we eliminate the variable *Z* from the two equations :

$$xx = \frac{cc(1+z)dz}{(1-zz)dZ + Zzdz}$$

and

$$yy = \frac{cc(1-z)((1+z)dZ - Zdz)^2 dz}{((1-zz)dZ + Zzdz)dz}.$$

Dividing the one by the other we get

$$\frac{yy}{xx} = \frac{(1-z)((1+z)dZ - Zdz)^2 dz}{(1+z)dz^2}$$

and

$$\frac{y}{x} = \frac{((1+z)dZ - Zdz)\sqrt{1-zz}}{(1+z)dz},$$

and the product is

$$yx = \frac{cc((1+z)dZ - Zdz)\sqrt{1-zz}}{(1-zz)dZ + Zzdz}.$$

But if we do not desire the equation between *x* and *y*, the same discovered formulas give a convenient construction. Taking any value for *Z*, by which *z* is at the same time determined, we find the values for *xx* and *yy* and they determine one point of the curve. We are also able to create a geometric construction, if select a curve, with the coordinates z and Z that has this property, that in making *z* negative, the other *Z* becomes negative as well. The relation between *dZ* and *dz* is defined by the tangent of this curve.

Section 36

Because *Z* must be some function of z that changes to -*Z*, when using -*z* in place of *z*, we take the simplest case $Z = \alpha z$ and get

$$xx = \frac{cc(1+z)}{\alpha} \text{ and } yy = \alpha cc(1-z);$$

from which we have

$$1+z = \frac{\alpha xx}{cc} \text{ and } 1-z = 2 - \frac{\alpha xx}{cc}.$$

Thus

$$yy = 2\alpha cc - \alpha\alpha xx,$$

the equation for the ellipse, which obviously satisfies our problem.

Section 37

Now we let $Z = \alpha z^n$, where n is an odd number, so that Z becomes an odd function of z. Also

$$\frac{dZ}{dz} \text{ will be } = \alpha n z^{n-1}$$

and

$$(1+z)dZ - Zdz = \alpha z^{n-1} dz(n+(n-1)z)$$

and

$$(1-zz)dZ + Zzdz = \left(\alpha n z^{n-1} - \alpha(n-1)z^{n+1}\right)dz = \alpha n z^{n-1} dz(n-(n-1)zz)$$

from which we find

$$xx = \frac{cc(1+z)}{\alpha z^{n-1}(n-(n-1)zz)},$$

and

$$yy = \frac{\alpha cc(1-z)z^{n-1}(n+(n-1)z)^2}{n-(n-1)zz}.$$

Thus

$$xxyy = c^4(1-zz)\frac{(n+(n-1)z)^2}{(n-(n-1)zz)^2};$$

from which by eliminating *z* there results an algebraic equation of higher degree.

If we let $Z = \frac{\alpha z}{1-zz}$,

$$xx \text{ is } = \frac{cc(1+z)(1-zz)}{\alpha(1+2zz)} \text{ and } yy = \frac{\alpha cc(1-z+2zz)^2}{(1+2zz)(1-zz)(1-z)}.$$

We can in the same manner substitute infinitely many functions of *z* in place of *Z*, that will always generate equations for the curves, that satisfy that our requirements. I have found no selections among them, that lead to a simpler equation between *x* and *y*, however they are all easy to construct.