Synopsis by section

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Introduction

Euler wrote of "oblique-angled diameters" and "orthogonal diameters" without explanation.

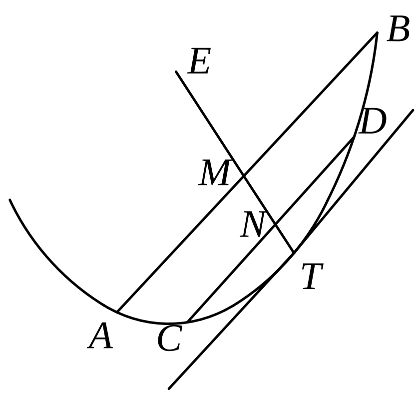

Figure 1:
ET is an oblique-angled diameter

Definition: Given a curve *ACTDB* shown in figure 1. The line *ET* , which intersects the curve at *T*, is called an *oblique-angled diameter* if it bisects all chords (such as *AB* and *CD*), that are parallel to the tangent line at *T*.

In other words $AM = MB$ *and* $CN = ND$ since the chords *AB* and *CD* are parallel to the tangent line at the point *T* where the *oblique-angled diameter ET* intersects the curve *ACTDB.*

Section 1. We will examine the concept of *oblique-angled diameters*, which applies to the conic sections, and try to find other curves with that property or a similar property.

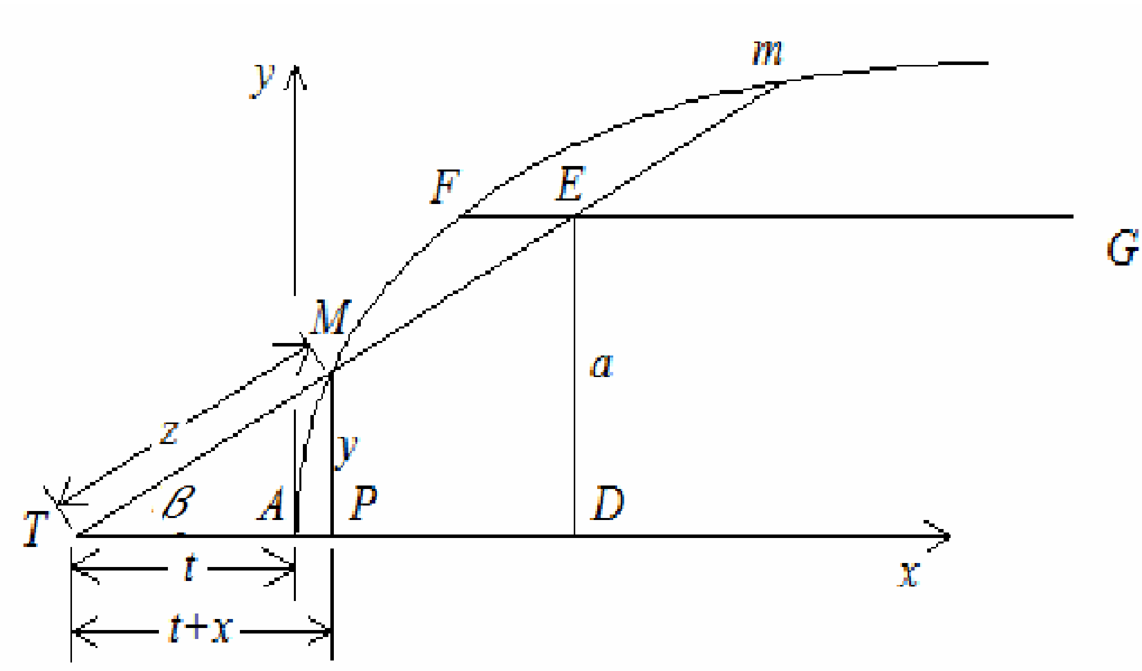

Figure 3.1

Section 2. Euler reminds us that any line parallel to the axis of a parabola is an oblique-angled diameter.

Section 3. We now study the following problem:

To describe a curve AMFm over the axis AD, which has the diameter FEG parallel to the axis AD at the distance given DE from the axis, and which bisects all chords Mm which make (*a fixed*) *angle* β *at T with the axis.*

With the notation shown in Figure 3.1, and with $n = \sin\beta$ and $m = \cos\beta$, Euler reasons that $z^2 = 2P(t)z - Q(t)$. Here $P(t)$ and $Q(t)$ are arbitrary functions of *t*. (Notice that the letters *m* and *P* have ambiguous meanings.)

Section 4. Call the z_M and z_m roots of the equation $z^2 = 2P(t)z - Q(t)$. Using $2P(t) = z_M + z_m$ and simple reasoning from the geometry of Figure 1, Euler concludes that the equation of the curve *AMFm* is $y^2 - 2ay = f(ny - mx)$, where the function *f* is arbitrary.

Section 5. We set $ny - mx = X$ and $y^2 - 2ay = Y$ and form the general equation between *X* and *Y*, by writing

$$0 = \alpha + \beta X + \gamma Y + \delta X^2 + \varepsilon XY + \zeta Y^2 + \eta X^3 + \theta X^2 Y + \cdots. \qquad (1)$$

In this general equation we find all possible relations between *X* and *Y* and thus we have *Y* = to any function of *X*, so that $y^2 - 2ay$ will be equal to any function of $ny - mx$ as required by our analysis.

Section 6. When the curve is a parabola, all lines parallel to the axis are oblique-angled diameters, while in the family of curves we found, only one line (at distance *a* from the axis) is an oblique-angled diameter.

Section 7. Now Euler considers curves that are symmetrical about the axis. He poses the problem:

Among all curves AMm, with the line AD as an axis of symmetry, in other words, to determine those that at a given distance on both sides of the axis AD have two oblique-angled diameters.

Because of the symmetry, the variable *y* cannot occur to an odd power in the equation of the curve. But because both $X = ny - mx$ and $Y = y^2 - 2ay$ feature *y* to the first power, Euler seeks new variables without odd powers. He selects $Z = Y + \frac{2aX}{n} = y^2 - \frac{2max}{n}$.

He notes that the preceding problem (without symmetry) is solved by the general equation between *Y* and *Z*, that is to say:

$$0 = \alpha + \beta Y + \gamma Z + \delta Y^2 + \varepsilon YZ + \varsigma Z^2 + \eta Y^3 + \theta Y^2 Z + \cdots . \qquad (2)$$

Section 8. Because the variable *Y* contains *y* to the first power, our second problem, with axis of symmetry, can only be satisfied if terms involving odd powers of *y* are eliminated. Euler argues that the removal of odd powers of *y* requires the use of infinitely many terms involving $Y^2, Y^3, Y^5, \ldots$. Thus if we restrict ourselves to algebraic equations, (finitely many terms), equation (2) must reduce to

$$0 = \alpha + \gamma Z + \varsigma Z^2 + \chi Z^3 + \cdots + \omega Z^q ,$$

that contains no *Y*. It follows that *Z* is a constant

$$yy - \frac{2max}{n} = C .$$

This is the parabola, and therefore all other algebraic curves are excluded.

Section 9. While the only algebraic equations with an axis of symmetry and oblique-angled diameters are parabolas, there may be many transcendental curves with this property.

Section 10. While Euler will not pursue transcendental curves further, he indicates a method for finding them. With $Y = yy - 2ay$ and $Z = y^2 - \frac{2max}{n}$ we seek a function of *Y* and *Z* with no odd powers of y, since the curve is symmetric about the *x* axis. Euler asks us, without explanation, to find a function $T(Y)$ satisfying the infinite differential equation

$$0 = \frac{dT}{dY} + \frac{4aYd^3T}{1.2.3.dY^3} + \frac{16a^4Y^2d^5T}{1.2.3.4.5dY^5} + \frac{64a^6Y^3d^7T}{1.2....7dY^7} + \cdots \quad . \qquad . \qquad (3)$$

He claims that this function T has no odd powers of y . With $W(Y,Z)$ an arbitrary function, the equation $W(T,Z)=0$ is a transcendental curve with axis of symmetry and oblique-angled diameter(s).

We give our derivation of (3). Using Taylor's theorem we have

$$T(Y)=T(y^2-2ay)=\sum_{n=0}^{\infty}\frac{d^nT(y^2)}{dY^n}\frac{(-2ay)^n}{n!}.$$

Separating even and odd powers we have

$$T(Y)=\sum_{n=0}^{\infty}\frac{d^{2n}T(y^2)}{dY^{2n}}\frac{(-2ay)^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{d^{2n+1}T(y^2)}{dY^{2n+1}}\frac{(-2ay)^{2n+1}}{(2n+1)!}.$$

The first series above is a function of only even powers of y , while the second series has both even and odd powers. Thus we set this second series equal to zero and after dividing by $2ay$ we get

$$0=\sum_{n=0}^{\infty}\frac{d^{2n+1}T(y^2)}{dY^{2n+1}}\frac{4^na^{2n}y^{2n}}{(2n+1)!}.$$

Since we are using this differential equation to solve for T , we can replace y^2 by the dummy variable Y and we have derived Euler's relation (3).

Section 11. Euler gives a neat, simple geometric argument to prove the following theroem.

Theorem If a curve has two parallel oblique-angled diameters, separated by the distance *a*, then that curve has infinitely many parallel oblique-angled diameters, all separated by the same distance *a*.

Section 12. The figure shows a curve with oblique-angled diameters *Aa*, *Bb*, *Cc* and corresponding angles $\alpha,\ \beta,\ \gamma$. From the previous section we know that there are infinitely many diameters all equally spaced by the distance *a*. Referring to the figure we will use $\theta_1 = \alpha$, the angle made by the chord *Mm* with the first diameter *Aa*, $\theta_2 = \beta$, the angle made by the chord *MN* with the second diameter *Bb*, etc. Then Euler states without proof that

$$\cot\theta_3 = 2\cot\theta_2 - \cot\theta_1 .$$

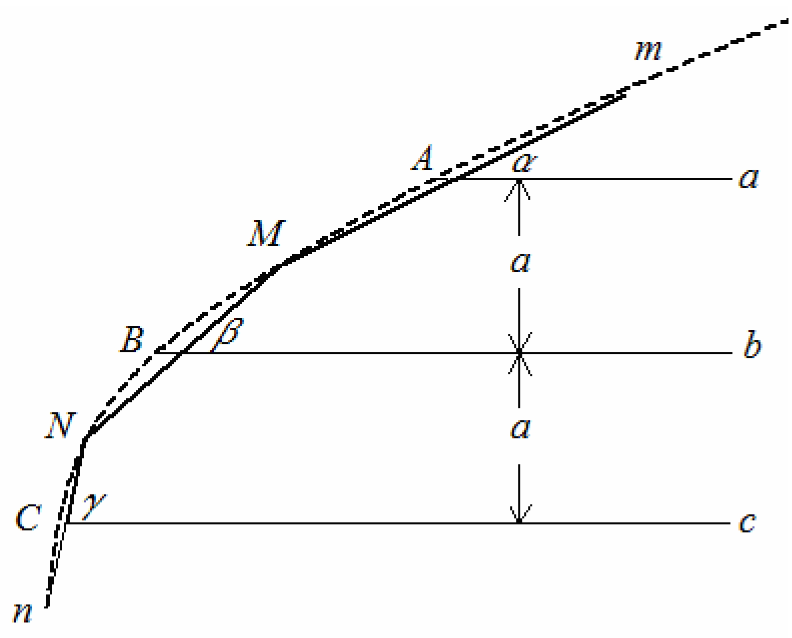

Figure 12.1

(We show why this is true in the notes that follow the translation.) It follows that.

$$\cot\theta_4 = 3\cot\theta_2 - 2\cot\theta_1 .$$

$$\cot\theta_5 = 4\cot\theta_2 - 3\cot\theta_1 ,$$

And in general

$$\cot\theta_n = (n-1)\cot\theta_2 - (n-2)\cot\theta_1 .$$

Section 13. Euler argues that the only curve in which all parallel lines are diameters is the parabola. He does this with the help of the remarkable differential equation of infinite order introduced in section 10.

Section 14. Having completed the discussion of parallel diameters like those found in parabolas, he turns to diameters that intersect at a common point, as with the ellipse and hyperbola. In the case of these curves, all straight lines that pass through their center are oblique angled diameters. However, are there other curves that have only one, two, three, etc. such diameters?

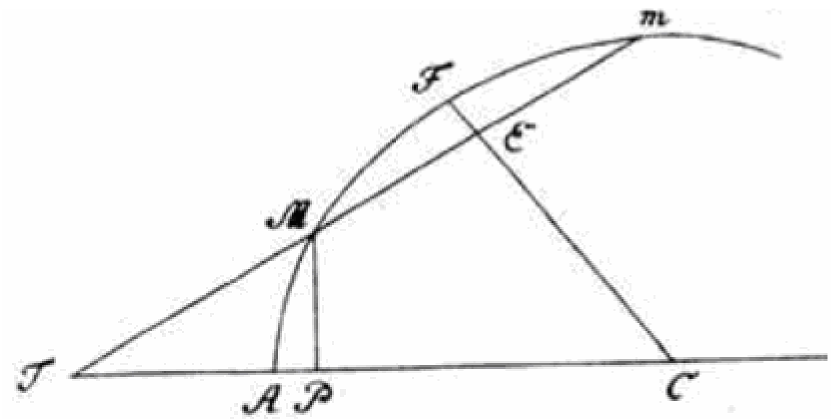

Figure 15.1

Section 15. Euler now presents the following problem:

Find all curves (Fig 15.1 above) *AMm constructed above an axis AC such that the line CF emerging from the point C, making angle ACF with the axis, bisects at E all chords Mm parallel to the tangent line at F.*

Euler now introduces the following notations:

$t = CT$ and $z = TM$ or Tm,

m = sine of angle ETC, and n = cosine of angle ETC,

p = sine of angle ECT, and q = cosine of angle ECT.

Euler then argues that z, the intersection of all lines TE with the curve, satisfies a quadratic equation

$$z^2 = 2Pz - Q,$$

where *P* and *Q* are functions of *t*, and the angle *ECT* is constant.

Section 16. Setting $x = CP$ and $y = PM$ Euler gives an easy argument using simple trigonometry that $t = \frac{mx+ny}{m}$ and that the family of desired curves is given by the equation

$$y^2 + \frac{2mp}{np-mq}xy = f\left(\frac{mx+ny}{m}\right).$$

where f is an arbitrary function. If we let $X = x + \frac{ny}{m}$ and $Y = y^2 + \frac{2mp}{np-mq}xy$, Then it follows that this family of curves is given by $W(X,Y) = 0$, where *W* is an arbitrary function of X and Y

$$W(X,Y) = \alpha + \beta X + \gamma Y + \delta X^2 + \varepsilon XY + \zeta Y^2 + \eta X^3 + \cdots.$$

Section 17. This family of solutions has only one guaranteed oblique angled diameter passing through the point *C*. Euler now wishes to find the subset of the above curves that are symmetric about the axis *AC*. In this case AC is an orthogonal diameter. Now only even powers of y are allowed. Thus the coefficients $\alpha,\ \beta,\ \gamma, \cdots$, must be determined so that no odd powers of *y* remain in the equation.

Section 18. First we see that $\beta = 0$ because no following terms can remove by subtraction $\frac{ny}{m}$. However we can remove odd powers of y from the next two terms

$$\gamma Y + \delta X^2$$

by taking $\gamma = np - mq$ and $\delta = -\frac{m^2 p}{n}$, so that our expression becomes

$$\gamma Y + \delta X^2 = (np-mq)Y - \frac{m^2 p}{n}X^2 = -mqy^2 - \frac{m^2 px^2}{n}.$$

Thus Euler lets

$$Z = nqy^2 + mpx^2 = mpX^2 - \frac{n(np-mq)}{m}Y\ ,$$

which has only even powers of *y*.

Section 19. Now consider the arbitrary function of *X* and *Z*:

$$W(X,Z) = \alpha + \beta X + \gamma Z + \delta X^2 + \varepsilon XZ + \zeta Z^2 + \eta X^3 + \cdots .$$

Set $W(X,Z) = 0$, and this curve has an oblique angled diameter passing through the point *C* on the *x*- axis. If all the terms in which *X* appears vanish, then no odd powers of *y* occur and the *x*- axis is an axis of symmetry for our curve. In this case *Z* is a constant, and Euler writes $mpx^2 + nqy^2 = a^2$. These are all conic sections with center at the origin.

Let $b^2 = \frac{a^2}{nq}$ and get

$$y^2 = b^2 - kx^2\ ,$$

where $k = \frac{mp}{nq}$. This means that $k = \tan\sphericalangle ETC\ \tan\sphericalangle ECT$. Thus the angle ECT is arbitrary, and can be any value. Any line passing through the origin is an oblique angled diameter for these curves. If *k* = 1, the angle *TEC* is a right angle and the equation is therefore a circle.

Section 20. Euler now tries to remove odd powers of *y* from the equation

$$W(X,Z) = \alpha + \beta X + \gamma Z + \delta X^2 + \varepsilon XZ + \zeta Z^2 + \eta X^3 + \cdots = 0\ .$$

He first observes that we must take $\beta = \delta = 0$, since the terms involving *y* and *xy* cannot be removed by following terms. Next he groups together terms that are homogeneous. He starts with X^3 and *XZ*, which are homogenous of degree 3, and states (without showing details), that if $\frac{mp}{nq} = 3$, then the terms involving $x^2 y$ and y^3 disappear. He also states that the terms X^5, $X^3 Z$ and XZ^2, which are homogeneous of degree 5, can be removed if we take

$$\frac{np}{mq} = 3 \text{ or } \frac{np}{mq} = 5 + 2\sqrt{5}.$$

Section 21 and 22. Euler gives algebraic details of his derivation from the homogenous curves of order 3 discussed in the previous section. He lets

$$\frac{p}{q} = \theta \quad \text{and} \quad \frac{m}{n} = \frac{1}{3}\theta,$$

and concludes that if we write

$$Z = yy + \frac{1}{3}\theta\theta xx \quad \text{and} \quad V = \frac{1}{3}\theta xyy - \frac{1}{27}\theta^3 x^3,$$

and if W denotes any function of Z and V, then the equation $W = 0$ has the desired properties. The equation is

$$0 = \alpha + \beta Z + \gamma V + \delta Z^2 + \varepsilon ZV + \zeta V^2 + \eta Z^3 + \cdots.$$

This equation has three diameters, the x-axis which is an orthogonal diameter, and the two lines from the origin making angles $\pm \tan^{-1}\theta$ with the negative x-axis.

Section 23 and 24. Now Euler considers the special case of the above equation

$$a^3 = bZ + V$$

which is

$$a^3 = by^2 + \frac{1}{3}\theta^2 bx^2 + \frac{1}{3}\theta xy^2 - \frac{1}{27}\theta^3 x^3, \text{ or}$$

$$y^2 = \frac{a^3 - \frac{1}{3}\theta\theta b x^2 + \frac{1}{27}\theta^3 x^3}{b + \frac{1}{3}\theta x}.$$

These curves are examples of what Euler calls the *redundant hyperbolas of Newton*,

$$yy = \frac{Av^3 + Bv^2 + Cv + D}{v}.$$

Here $v = b + \frac{1}{3}\theta x$ and $C = \frac{B^2}{4A}$. This curve shows oblique diameters *CE* and *CE'* and their reflections in the *v*- axis. The point *C* on the *v*-axis where these diameters meet has value $v = -\frac{B}{6A}$. Euler finds other values significant in the graph of this equation shown below in Figure 24.1

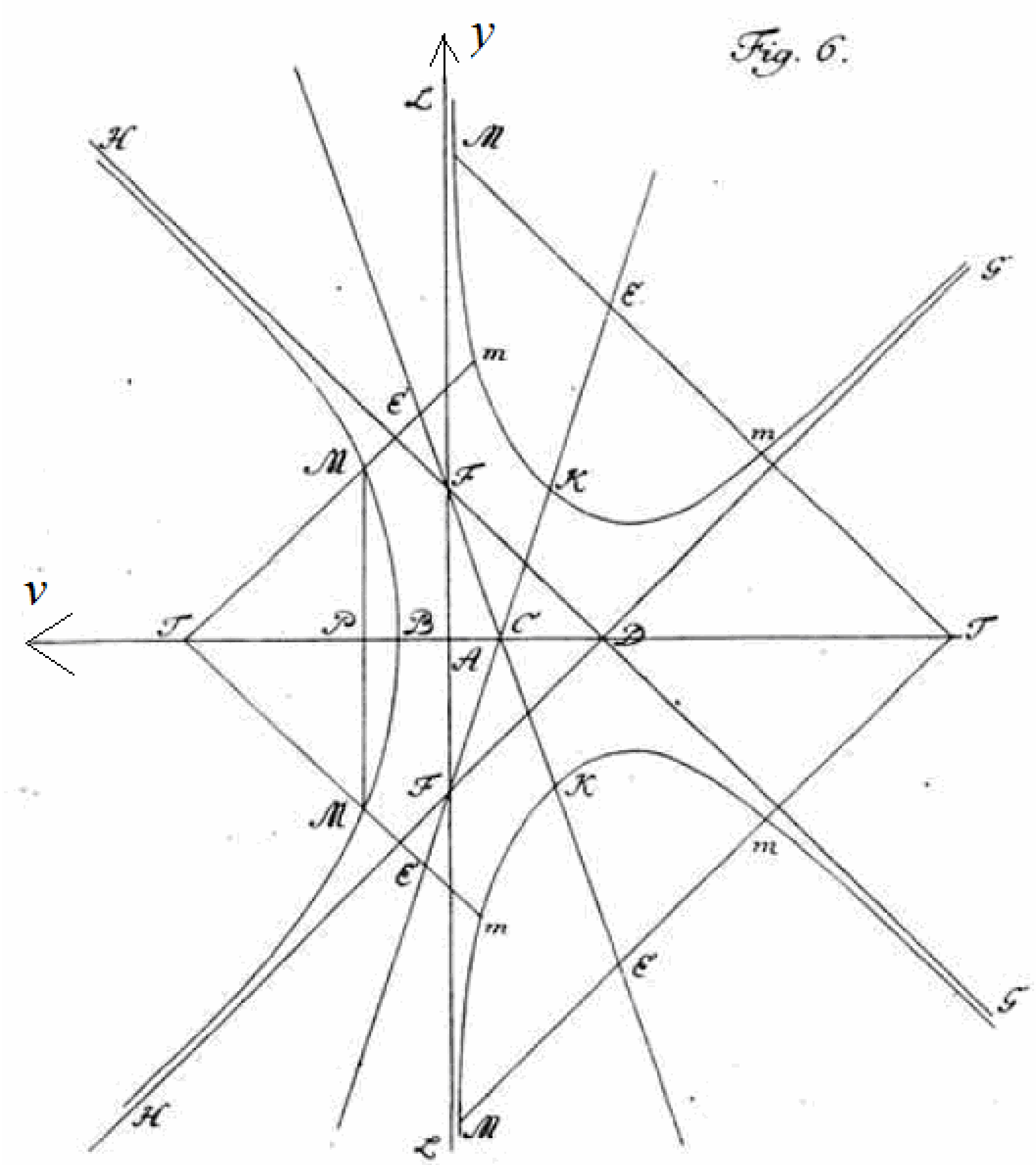

Figure 24.1

This is Euler's graph to which we have added the *v* and *y* axies. Notice that Euler takes the *v* axis directed to the left.

Section 25. Euler has examined curves obtained by starting with

$$X = x + \frac{ny}{m}, \quad Z = y^2 + \frac{1}{3}\theta^2 x^2$$

and studying the expression of third order $\alpha X^3 + \beta XZ$. He obtained finally the curves

$V=\frac{1}{3}\theta xy^2-\frac{1}{27}\theta^3x^3$, which satisfied his problem.

In this section he studies the expression of fourth order

$V=\alpha X^4+\beta X^2 Z$.

He finds that in this case we must take $\frac{m}{p}=\frac{p}{q}$ to remove odd powers of *y*. He calls

$\theta=\frac{m}{n}$ and gets $V=-\alpha\left(\theta^2xx-yy\right)^2$. Now any function $W(Z,V)=0$ will have the *x* axis as an orthogonal diameter and at least two oblique angled diameters. The simplest such curve is given by $\alpha^4=\theta^4x^4y^4$.

Section 26. Using the figure

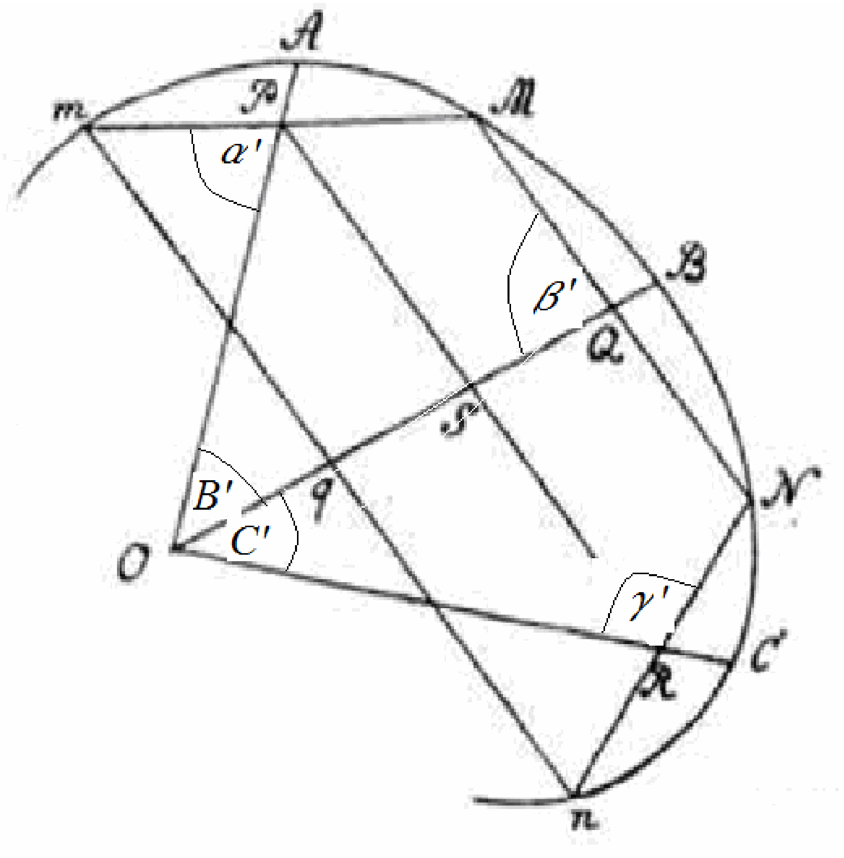

Figure 26.1

Euler proves that if we have two oblique angled diameters that meet at the point *O*, Then we have more diameters that meet at *O*. This is a remarkably simple geometric proof, much like the proof found in section 11. (We give a more detailed proof in the notes.)

Section 27. Euler gives formulas for important angles in the above figure. The calculations are left out, and our detailed derivations of these can be found in the notes. Using the notation that all angles are "primed" and $\tan x' = x$ he gets the relations

$$\frac{1}{C} = \frac{1}{\mathrm{B}} + \frac{2}{\beta}, \quad \text{and} \quad \frac{2}{\beta} = \frac{1+\alpha B}{\alpha - B} + \frac{1-\gamma C}{\gamma + C}.$$

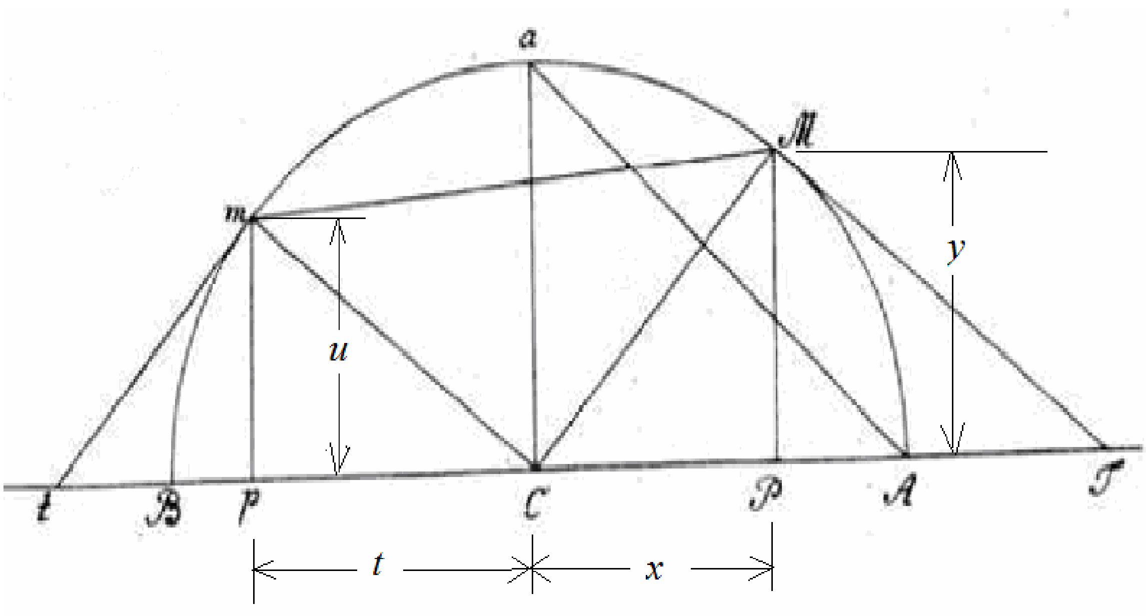

Figure 28.1

Section 28. Euler now begins a new problem:

We seek a curve AMamB that has two orthogonal diameters ACB and aC that are *perpendicular to each other. Thus the center of this curve is at C. Like the ellipse, it must have the following property: that extending from the cente*r *C any ray CM and at the same time another ray Cm parallel to the tangent MT at the point M*, the area of the triangle *MCm is always constant , and is equal to the area of the triangle ACa*.

Section 29. Euler notes that the equation we seek $W(x, y) = 0$, has, in terms of the variables *x* and *y*, only even powers of both variables. This follows from the fact that the curve is symmetric about both the *x* and *y* axies. In addition, if we replace *x* by *t*, then *y* is replaced by *u*.

Section 30. Euler will try to find the solution in parametric parametric equations

$$x^2 = f(z) \text{ and } y^2 = g(z)$$

where *z* is the parameter. He imposes the condition that changing *z* to –*z* changes *x* to *t* and changes *y* to *u*. Thus

$$t^2 = f(-z) \text{ and } u^2 = g(-z).$$

Euler uses *P* to denote the even part of *f* and *Q* to denote the odd part. In the same way *R* and *S* denote the even and odd parts of *g*. Thus we have

$$x^2 = P(z) + Q(z) \text{ and } y^2 = R(z) + S(z),$$

and replacing *z* by –*z* we get

$$t^2 = P(z) - Q(z) \text{ and } u^2 = R(z) - S(z).$$

Section 31. From the fact that the ray *Cm* has the same slope as the tangent line at *M* we get $\frac{u}{t} = -\frac{dy}{dx}$, and therefore

$$udx + tdy = 0 .$$

An easy calculation shows that area of the triangle *MCm* is $= \frac{ty + ux}{2}$. Since the area is constant, its differential is zero and we get

$$ydt + tdy + udx + xdu = 0 .$$

Since $udx + tdy$ is = 0, this makes

$$ydt + xdu = 0 .$$

Section 32. Euler denotes the area of triangle *MCm* by c^2 so that we have *ty* + *ux* = *2cc*. Substituting

$$x = \sqrt{P+Q}\,,\ y = \sqrt{R+S}\,,\ t = \sqrt{P-Q} \text{ and } u = \sqrt{R-S}\,,$$

we get

$$\sqrt{(P+Q)(R-S)} + \sqrt{(P-Q)(R+S)} = 2cc\,.$$

Euler now introduces *V* as an odd function of *z* such that $\sqrt{(P+Q)(R-S)} = cc + V$. An easy argument now leads to

$$x = \sqrt{P+Q}\,,\ t = \sqrt{P-Q}\ ,\ y = \frac{cc-V}{\sqrt{P-Q}} \text{ and } u = \frac{cc+V}{\sqrt{P+Q}}\,.$$

From these we have after a some manipulations

$$udx + tdy = \frac{(cc+V)(dP+dQ)}{2(P+Q)} - dV - \frac{(cc-V)(dP-dQ)}{2(P-Q)},$$

and since this is zero we get

$$(P^2 - Q^2)dV - V(PdP - QdQ) - cc\,(PdQ - QdP) = 0\,.$$

Section 33. Dividing the above equation by $(P^2 - Q^2)^{3/2}$ and integrating we get

$$\frac{V}{\sqrt{P^2 - Q^2}} = \int \frac{cc(PdQ - QdP)}{(P^2 - Q^2)^{3/2}}\,.$$

So far, we have arbitrary even functions *P* and *Q* and odd function *R, S* and *V*. Euler now begins to specify some of these by trying *Q* = *Pz*. Now the above equation is

$$\frac{V}{P\sqrt{1-z^2}} = \int \frac{c^2 dz}{P(1-z^2)^{3/2}}\,.$$

To get an algebraic solution, we must be able to integrate. Euler writes $Z = \frac{V}{P}$, and by differentiating the above equation he gets

$$P = \frac{c^2 dz}{(1-z^2)dZ + Zzdz}.$$

Sections 34 and 35. Euler now gets expressions for the parametric equations

$$x^2 = \frac{c^2(1+z)dz}{(1-z^2)dZ + Zzdz} \quad \text{and} \quad y^2 = \frac{c^2(1-z)((1+z)dZ - Zdz)^2 dz}{((1-z^2)dZ + Zzdz)dz},$$

also

$$t^2 = \frac{c^2(1-z)dz}{(1-z^2)dZ + Zzdz} \text{ and } u^2 = \frac{c^2(1+z)((1-z)dZ + Zdz)^2 dz}{((1-z^2)dZ + Zzdz)dz}.$$

Here *Z* is an arbitrary odd function of *z*.

Section 36. Euler now considers the simplest case $Z = \alpha z$ and gets

$$x^2 = \frac{c^2(1+z)}{\alpha} \text{ and } y^2 = \alpha c^2(1-z);$$

from which we get $y^2 = 2\alpha c^2 - \alpha^2 x^2$, which is an ellipse.

Section 37. Now we let $Z = \alpha z^n$, where *n* is an odd number, from which we find

$$x^2 = \frac{cc(1+z)}{\alpha z^{n-1}(n-(n-1)z^2)},$$

and

$$y^2 = \frac{\alpha c^2(1-z)z^{n-1}(n+(n-1)z)^2}{n-(n-1)z^2}.$$

If we let $Z = \dfrac{\alpha z}{1-zz}$, we get

$$x^2 = \frac{c^2(1+z)(1-z^2)}{\alpha(1+2z^2)} \text{ and } y^2 = \frac{\alpha c^2(1-z+2z^2)^2}{(1+2z^2)(1-z^2)(1-z)}.$$