Triangles and parallelograms of equal area in an ellipse

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Introduction

In the paper [1], Euler looked at certain properties of the conic sections and tried to find other curves that shared these properties. In the eighteenth century, mathematicians were familiar with many properties of the parabola, ellipse and hyperbola that have been neglected in our modern education. This paper is about one such ignored property of the ellipse which we rediscovered in order to understand Euler's work. We will study an interesting family of parallelograms inscribed in the ellipse, all of which have the same area.

We begin by defining a few new terms, *diameters*, *reciprocal diameters* and *reciprocal points* in an ellipse.

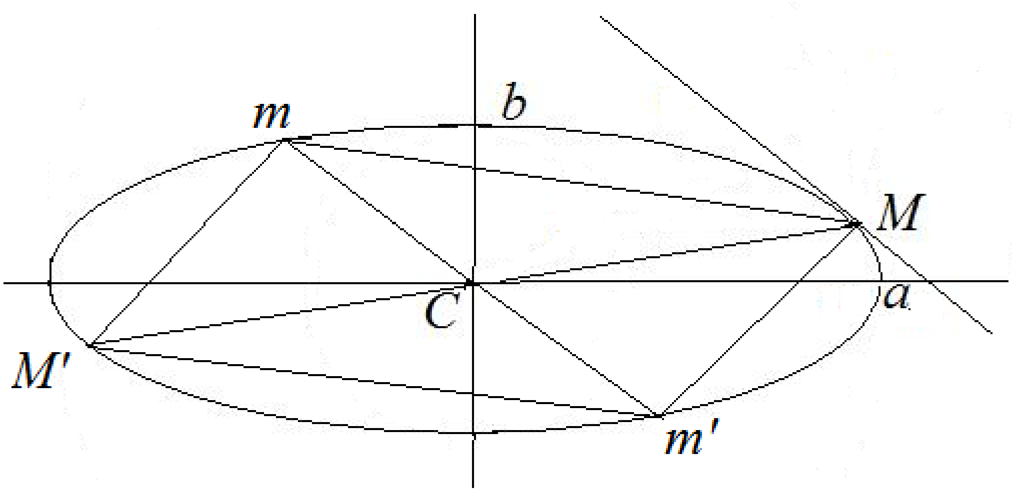

Figure 1

A *diameter* of an ellipse is any chord that passes through the center. In Figure 1 *MM'* and *mm'* are diameters. Now start with any diameter *MM'*. We say that the diameter *mm'* is *reciprocal* to diameter *MM'* if it is parallel to the tangent line to the ellipse at *M*. If we started with diameter *mm*', then *MM'* would be the reciprocal diameter. We say that the points *m* and *m'* are reciprocal to the point *M*. Euler assumes that his readers were familiar with reciprocal diameters and points. He also assumed that his readers would be aware that the area of the parallelogram *MmM'm'* is constant, regardless of the choice of the initial diameter, and is equal to 2*ab*. The area of the triangles *CMm* and *CMm'* are also constant and equal *ab*/2.

Before we derive our main result, we review parametric equations for the ellipse and their geometric consequences.

Parametric equations for the ellipse

These reciprocal diameters have an interesting relation to the parametric form of the equation of the ellipse given by the equations

(1) $x = a\cos\theta$ and $y = b\sin\theta$.

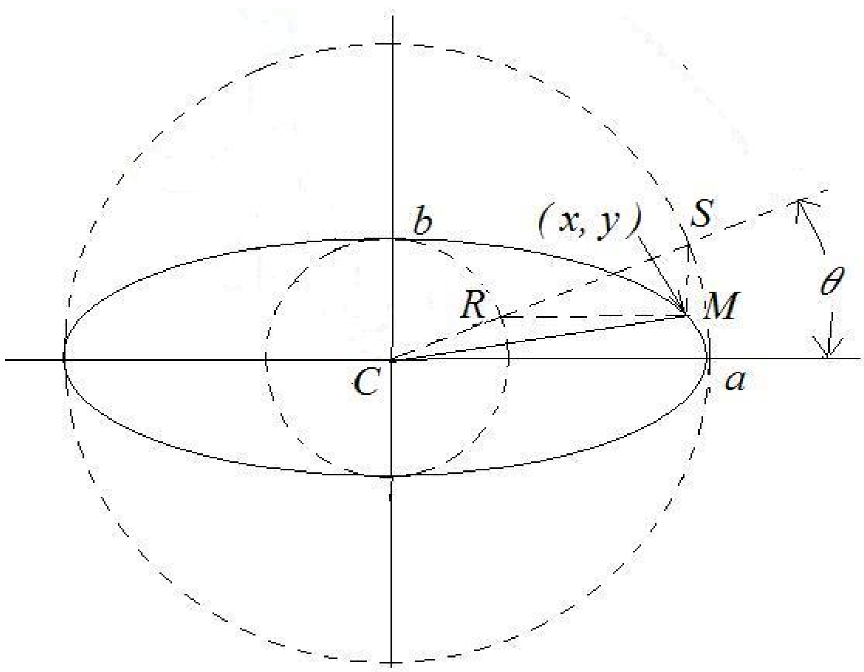

Figure 2

In Figure 2 we see two circles centered at point *C* with radii *b* and *a*. The ray *CRS* makes angle θ with the *x*-axis and intersects the smaller circle at *R* and the larger circle at *S*. From *R* extend a horizontal line and from *S* drop a vertical line. These two lines intersect at the point *M*. This point *M* is on the ellipse given by the parametric equations (1). As the angle θ varies between 0 and 2π, the point *M* generates the entire ellipse. Notice that the ray *CM* which identifies the point (x, y) on the perimeter of the ellipse differs from the ray *CS* that is made by the parameter θ. The angle θ is known historically as the *eccentric anomaly*. It is important to note that this is not the usual polar angle associate with the point (x, y).

Reciprocal diameters and the eccentric anomaly

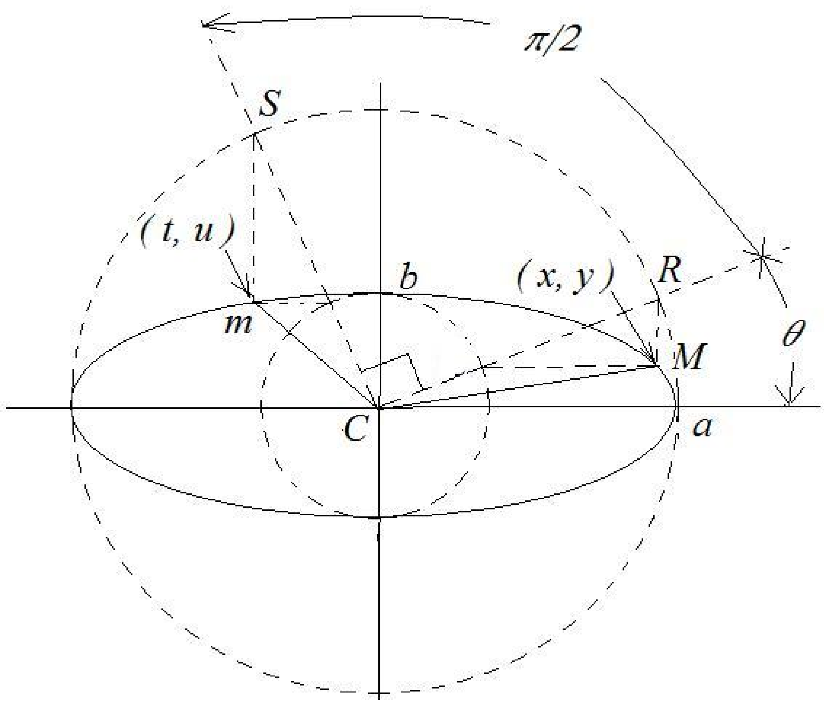

Figure 3

To see the relation between the reciprocal diameters and the eccentric anomaly, consider Figure 3. Start with the radius *CR* making eccentric anomaly θ, to identify the point on the ellipse *M*. Now increase the eccentric anomaly by $\pi/2$ to identify the radius ray *CS* and corresponding point on the ellipse *m*. We will show that this point *m* is reciprocal to *M*. Thus reciprocal points on the ellipse have their related eccentric anomalies separated by the angle $\pi / 2$.

To see that this is true, we use (1) to calculate the slope of the tangent at *M*. We get

$$\frac{dy}{dx} = \frac{b \cos\theta \, d\theta}{-a \sin\theta \, d\theta} = -\frac{b}{a}\cot\theta .$$

Therefore the slope of the reciprocal ray *Cm* is given by $-\frac{b}{a}\cot\theta$. (Notice that when the slope of the ray *CR* defined by the eccentric anomaly is $\tan\theta$, then the slope of the corresponding ray *CM* defined by the point on the ellipse is always given by $\frac{b}{a}\tan\theta$.) Thus the slope of the ray *CS* is given by $-\cot\theta$ But the identity $-\cot\theta = \tan(\theta + \pi/2)$ demonstrates the truth of the relation between *M* and *m* just stated. Thus the coordinates of the point *m* reciprocal to *M* are given by

(2) $t = a\cos(\theta + \pi/2) = -a\sin\theta$ and $u = b\sin(\theta + \pi/2) = b\cos\theta$.

The area of the triangle *CMm* is constant

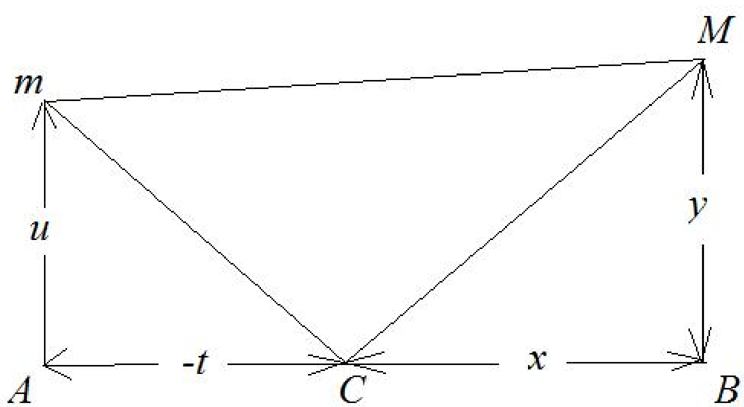

Figure 4

The area of the quadrilateral *ABMm* shown in Figure 4 is $\left(\frac{y+u}{2}\right)(x-t)$.

Subtracting the areas of triangles *ACm* and *CBM* we get the area of the triangle *CMm*

Area *CMm* = $\left(\frac{y+u}{2}\right)(x-t) - \frac{(-t)u}{2} - \frac{xy}{2}$.

This simplifies to $\frac{xu - ty}{2}$. Substituting the values of these variables in terms of θ from (1) and (2) we get

$$\text{Area } CMm = \frac{ab\cos^2\theta + ab\sin^2\theta}{2} = \frac{ab}{2}.$$

This proves that the area of the triangle *CMm* is constant. In the same way the area of triangle *CMm'* is $\frac{ab}{2}$ and thus the area of the parallelogram *MmM'm'* is 2*ab*.

This completes our study of the triangles and parallelograms of equal area in the ellipse.

Reference

[1] Greve, Edward and Osler, Thomas J., *Translation with notes of Euler's paper E83,* On some properties of conic sections that are shared with infinitely many other curved lines. On the web at the Euler Archive http://www.math.dartmouth.edu/~euler/.