SOME LONG TELESCOPING SERIES

Thomas J. Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

osler@rowan.edu

In [1], Raymond Greenwell showed that

$$(1) \qquad \sum_{k=b+1}^{\infty} \frac{1}{k^2-b^2} = \frac{1}{2b}\sum_{k=1}^{2b}\frac{1}{k}.$$

Notice that the series on the left is an infinite sum, but the series on the right is finite and therefore is the sum viewed as a "closed form expression". This result is an unusual application of the technique from elementary calculus known as "telescoping series". In this short note we will show that (1) is a special case of a more general series.

Theorem: Let b be a positive integer and let the function $f(x)$ be such that $f(n) \neq 0$ for $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} f(n) = \infty$. Then

$$(2) \qquad \sum_{k=b+1}^{\infty} \frac{f(k+b)-f(k-b)}{f(k-b)f(k+b)} = \sum_{k=1}^{2b}\frac{1}{f(k)}.$$

It is easy to see that (1) is the special case of (2) in which $f(x) = x$.

To prove the theorem we first note that

$$\frac{f(k+b)-f(k-b)}{f(k-b)f(k+b)} = \frac{1}{f(k-b)} - \frac{1}{f(k+b)}.$$

Consider now the partial sum S_N of N terms for the series on the left side of (2)

$$S_N = \sum_{k=b+1}^{b+N} \frac{f(k+b)-f(k-b)}{f(k-b)f(k+b)} = \sum_{k=b+1}^{b+N} \frac{1}{f(k-b)} - \frac{1}{f(k+b)}.$$

Writing out the terms we have

$$
\begin{aligned}
S_N = {} & \frac{1}{f(1)} - \frac{1}{f(2b+1)} + \\
& \frac{1}{f(2)} - \frac{1}{f(2b+2)} + \\
& \frac{1}{f(3)} - \frac{1}{f(2b+3)} + \\
& \vdots \\
& \frac{1}{f(2b)} - \frac{1}{f(2b+2b)} + \\
& \frac{1}{f(2b+1)} - \frac{1}{f(2b+2b+1)} + \\
& \frac{1}{f(2b+2)} - \frac{1}{f(2b+2b+2)} + \\
& \vdots \\
& \frac{1}{f(N)} - \frac{1}{f(2b+N)}
\end{aligned}
$$

We see all the terms cancel except the first $2b$ terms of the first column and the last $2b$ terms of the second column. Thus we have

$$S_N = \sum_{k=1}^{2b} \frac{1}{f(k)} - \sum_{k=1}^{2b} \frac{1}{f(k+N)} .$$

Since $\lim_{n\to\infty} f(n) = \infty$, it is clear that the second sum above vanishes as N approaches infinity. Thus the theorem is proved.

We end by looking at a few special cases of (2). When $f(x) = x^p$, with $p > 0$ we get

$$\sum_{k=b+1}^{\infty} \frac{(k+b)^p - (k-b)^p}{(k^2-b^2)^p} = \sum_{k=1}^{2b} \frac{1}{k^p} \quad .$$

If we take $f(x) = \sinh ax$, with $a > 0$ we get after a little simplification

$$\sum_{k=b+1}^{\infty} \frac{\cosh ak}{\sinh a(k+b) \sinh a(k-b)} = \frac{1}{2\sinh ab} \sum_{k=1}^{2b} \frac{1}{\sinh ak} \quad .$$

With $f(x) = e^{ax}$, and $a > 0$ we get

$$\sum_{k=b+1}^{\infty} e^{-ak} = \frac{1}{2\sinh ab} \sum_{k=1}^{2b} e^{-ak} .$$

If $f(x) = \log ax$, with $a > 1$ we have

$$\sum_{k=b+1}^{\infty} \frac{\log \frac{k+b}{k-b}}{\log a(k+b) \log a(k-b)} = \sum_{k=1}^{2b} \frac{1}{\log ak} .$$

Reference

[1] Greenwell, R. N., *Donofrio's theorem*, The Mathematical Gazette, 89(2005), pp. 261-2.