02/22/05

FINDING $\zeta(2n)$ FROM A RECURSION RELATION FOR BERNOULLI NUMBERS

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The zeta function ([4] and [9]) is defined by $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$, where $1<\mathrm{Re}(z)$.

Euler [5] found that $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, $\zeta(6)=\frac{\pi^6}{945}$, and in general

$$(1)\qquad \zeta(2n)=\frac{(-1)^{n+1}2^{2n-1}B_{2n}}{(2n)!}\pi^{2n}.$$

Here n is a positive integer and the numbers B_{2n} are Bernoulli numbers defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n x^n}{n!}.$$

The first few are

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42},\ldots, \text{ and } \quad B_3=B_5=B_7=\ldots=0.$$

These can all be calculated recursively by starting with $B_0=1$, and using

$$\binom{n}{0}B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-1}B_{n-1}=0$$

for $n=2,3,4,\cdots$. (See Knopp [6], page 237.) Several additional methods of deriving (1) have been given since Euler, some of which are found in [1], [3], [4], [6], [7] and [8]. In this paper we show how to derive (1) using another recursion relation for the Bernoulli numbers

(2) $$\sum_{k=1}^{n}\binom{2n+1}{2k}2^{2k-1}B_{2k}=n\text{, for } n = 1, 2, 3, \ldots .$$

It is also instructive to write (2) in the matrix form

(2a) $$\begin{pmatrix}\binom{3}{2} & 0 & 0 & \cdots \\ \binom{5}{2} & \binom{5}{4} & 0 & \cdots \\ \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \cdots \\ \cdots & & & \end{pmatrix}\begin{pmatrix}2B_2 \\ 2^3B_4 \\ 2^5B_6 \\ \cdots\end{pmatrix}=\begin{pmatrix}1 \\ 2 \\ 3 \\ \cdots\end{pmatrix}.$$

We will prove (2) at the end of this paper.

We begin with the function $f(x)=\frac{\sin\sqrt{x}}{\sqrt{x}}$ and expand it in a product and in a Taylor's series

(3) $$f(x)=\prod_{n=1}^{\infty}\left(1-\frac{x}{\pi^2n^2}\right)\text{, and}$$

(4) $$f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^n .$$

Next we take the logarithmic derivative of (3) and get

$$f'(x)=-f(x)\sum_{n=1}^{\infty}\frac{\frac{1}{\pi^2n^2}}{1-\frac{1}{\pi^2n^2}x}.$$

Expanding in the geometric series we find

$$f'(x)=-f(x)\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{x^k}{\pi^{2k+2}n^{2k+2}},$$

and interchanging summations we get

$$f'(x) = -f(x)\sum_{k=0}^{\infty}\frac{1}{\pi^{2k+2}}\zeta(2k+2)\,x^k\,.$$

Using (4) we can express this last relation as

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(2n+3)!}x^n = \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n\right)\sum_{n=0}^{\infty}\frac{1}{\pi^{2n+2}}\zeta(2n+2)x^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{n-k}\zeta(2k+2)}{\pi^{2k+2}(2n-2k-1)!}\right)x^n\,.$$

Equating coefficients of x^n we get

$$\sum_{k=0}^{n}\frac{(-1)^k\zeta(2k+2)}{\pi^{2k+2}(2n-2k-1)!} = \frac{n+1}{(2n+3)!}\,.$$

Replacing n by $n-1$ and changing the index we have

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\zeta(2k)}{\pi^{2k}(2n-2k+1)!} = \frac{n}{(2n+1)!}\,.$$

We can multiply by $(2n+1)!$ and rearrange items to get

(5) $$\sum_{k=1}^{n}(-1)^{k+1}\binom{2n+1}{2k}\frac{(2k)!\zeta(2k)}{\pi^{2k}} = n\,,$$ for n = 1, 2, 3, … .

A matrix version of (5) is

(5a) $$\begin{pmatrix} \binom{3}{2} & 0 & 0 & \cdots \\ \binom{5}{2} & \binom{5}{4} & 0 & \cdots \\ \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & \cdots \\ \cdots & & & \end{pmatrix}\begin{pmatrix} \frac{2!\zeta(2)}{\pi^2} \\ -\frac{4!\zeta(4)}{\pi^4} \\ \frac{6!\zeta(6)}{\pi^6} \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ \cdots \end{pmatrix}.$$

Comparing (2) and (5) or the matrix versions (2a) and (5a) we get (1) at once. Note that Euler was aware of the equations (5), although he found them in a different way. The novelty of our presentation, is to relate (5) to (2).

We can derive (2) by starting with $z\cot z = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}B_{2k}}{(2k)!}z^{2k}$ (see Knopp [6], p. 204). Multiply by $\frac{\sin z}{z}$ and get $\cos z = \left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k}\right)\sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}B_{2k}}{(2k)!}z^{2k}$.

Multiplying the series we have $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n} = \sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{2^{2k}B_{2k}}{(2n-2k+1)!(2k)!}z^{2n}$. Equating coefficients of z^{2n} we get (2) after a little manipulation.

The paper by Ayoub [2] gives a very readable account of how Euler evolved several derivations of (1) and similar results for related series. The derivation of (1) using the theory of residues and the functional equation for the zeta function can be found in [4], pages 11-14 and [9], pages 19 –20.

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