Surprising Connections between Partitions and Divisors

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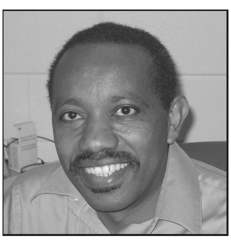

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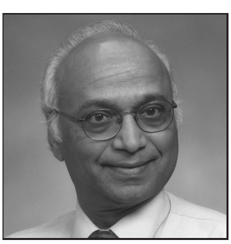

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Number theory, one of the oldest and best-loved areas of mathematics, has two main branches: the multiplicative and the additive. Multiplicative number theory has a rich heritage dating all the way back to Pythagoras, some 2600 years ago. Additive number theory, on the other hand, is much younger, going back only to Euler, less than 300 years ago. Although it is an area of intense research today, additive number theory is generally ignored in textbooks on elementary number theory. One exception is Andrews's text [**1**, pp. 149–200].

In multiplicative number theory we are concerned with how a positive integer is constructed as a product of prime numbers. One object of concern here is the "sum of divisors function" $\sigma(n)$, which is just that, the sum of the divisors of n. For example, the divisors of the number 6 are 1, 2, 3, and 6, and thus $\sigma(6) = 12$. Clearly, prime numbers are associated with this function in some way.

Additive number theory, on the other hand, is concerned with how a positive integer can be constructed as the sum of numbers from some set. One topic of interest is the "partitions" of a given number as sums of positive integers. For example, the seven partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. The "partition function" $p(n)$ is defined as the number of partitions of n. Thus, $p(5) = 7$. Prime numbers and divisors seem to be of little relevance here.

Some of the fascination with mathematics lies in unexpected results, including connections between concepts that appear to be unrelated. For the two functions, $p(n)$ and $\sigma(n)$, the first surprise is that they share a common recursive relation:

$$\begin{aligned} p(n) = {} & p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ & + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \cdots, \end{aligned}$$

and

$$\begin{aligned} \sigma(n) = {} & \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ & + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) + \cdots. \end{aligned}$$

Beyond this there is a really neat formula that combines the two functions:

$$np(n) = \sum_{k=0}^{n} \sigma(k) p(n-k).$$

We will discuss why these surprising relations should exist between two seemingly unrelated functions. Euler recognized this mystery as early as the eighteenth century, and it continues to both trouble and amaze us today.

The partition function

The partition function $p(n)$ is very important in number theory. It is the number of unrestricted partitions of the positive integer n, that is, the number of ways of writing n as a sum of positive integers without regard to order. In Table 1 we show all the partitions of the numbers up to 5, along with the value of $p(n)$.

Table 1. Partitions

n	Partitions of n	$p(n)$
1	1	1
2	2, $1+1$	2
3	3, $2+1$, $1+1+1$	3
4	4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$	5
5	5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$	7

While it is simple to determine $p(n)$ for very small numbers n by actually counting all the partitions, this soon becomes difficult as the numbers grow. Fortunately, there are easier ways to calculate $p(n)$. One way is the following remarkable recursion, where $p(0) = 1$ and $p(n) = 0$ if $n < 0$:

$$\begin{aligned} p(n) = {} & p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ & + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \cdots. \end{aligned} \tag{1}$$

Table 2. A brief table of the partition function

n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	11	56	21	792	31	6842	41	44583
2	2	12	77	22	1002	32	8349	42	53174
3	3	13	101	23	1255	33	10143	43	63261
4	5	14	135	24	1575	34	12310	44	75175
5	7	15	176	25	1958	35	14883	45	89134
6	11	16	231	26	2436	36	17977	46	105558
7	15	17	297	27	3010	37	21637	47	124754
8	22	18	385	28	3718	38	26015	48	147273
9	30	19	490	29	4565	39	31185	49	173525
10	42	20	627	30	5604	40	37338	50	204226

For example, using values from Table 2, we find that $p(10) = p(9) + p(8) - p(5) - p(3) = 42$.

This recursive formula was discovered by Euler himself; later we give a proof. The most mysterious feature of (1) is the appearance of the numbers 1, 2, 5, 7, 12, 15, ... These are related to the "pentagonal numbers" and will also be discussed later.

Revealing *p*(*n*) through the generating function. We now examine some attractive features of the partition function $p(n)$. In places we give only heuristic explanations, but where this is done, we give references where systematic and rigorous treatments can be found.

Euler [**4**] began the mathematical theory of partitions in 1748 by discovering the "generating function"

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n. \tag{2}$$

The infinite product on the left side generates the values of $p(n)$ as the coefficients of the power series on the right side.

What follows is a glimpse at why (2) works; a full proof is given in Andrews [**1**, pp. 160–162]. If we expand the factors $1/(1 - x^n)$ as geometric series, we get the following:

$$\begin{aligned}
\frac{1}{1-x^1} &= 1 + x^{1\cdot 1} + x^{1\cdot 2} + x^{1\cdot 3} + x^{1\cdot 4} + x^{1\cdot 5} + \cdots \\
\frac{1}{1-x^2} &= 1 + x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + x^{2\cdot 5} + \cdots \\
\frac{1}{1-x^3} &= 1 + x^{3\cdot 1} + x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + x^{3\cdot 5} + \cdots \\
\frac{1}{1-x^4} &= 1 + x^{4\cdot 1} + x^{4\cdot 2} + x^{4\cdot 3} + x^{4\cdot 4} + x^{4\cdot 5} + \cdots \\
\frac{1}{1-x^5} &= 1 + x^{5\cdot 1} + x^{5\cdot 2} + x^{5\cdot 3} + x^{5\cdot 4} + x^{5\cdot 5} + \cdots \\
&\vdots
\end{aligned} \tag{3}$$

(Here we interpret the power in $x^{a\cdot b}$ to represent $a + a + \cdots + a$ with b terms). When we multiply the series on the right side of (3) and carefully observe what is taking place, we see that the partitions are being generated in the exponents. For example, if we collect the terms that generate x^5, we find

$$x^{1\cdot 5} + x^{1\cdot 3}x^{2\cdot 1} + x^{1\cdot 2}x^{3\cdot 1} + x^{1\cdot 1}x^{4\cdot 1} + x^{1\cdot 1}x^{2\cdot 2} + x^{2\cdot 1}x^{3\cdot 1} + x^{5\cdot 1} = 7x^5$$

Notice that the exponents are the seven partitions of 5 that we saw: $1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 2$, $1 + 1 + 3$, $1 + 4$, $1 + 2 + 2$, $2 + 3$, and 5. Thus 5 has seven partitions. This illustrates how the generating function (2) works.

Using a computer algebra system, (such as *Mathematica*), one can use this idea to find the partitions $p(n)$ for large n. However, this is not a good way to do this.

Two mysteries of $p(n)$: The pentagonal number theorem and a recursion relation. As we noted above, the values of the partition function can be obtained from (1). Observe that the series in (1) is finite in that all but a finite number of the terms are 0. We can also write (1) in the form

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(n - f(k)) + p(n - f(-k))\}, \tag{4}$$

where $f(k) = k(3k - 1)/2$. These are known as the *pentagonal numbers*, and a short list of them appears in Table 3.

Table 3. Pentagonal Numbers $f(k) = k(3k - 1)/2$

k	$f(k)$	$f(-k)$	k	$f(k)$	$f(-k)$
1	1	2	11	176	187
2	5	7	12	210	222
3	12	15	13	247	260
4	22	26	14	287	301
5	35	40	15	330	345
6	51	57	16	376	392
7	70	77	17	425	442
8	92	100	18	477	495
9	117	126	19	532	551
10	145	155	20	590	610

The full proof of the recursion relation (4) is beyond the scope of this paper, but can be found in Hardy and Wright [**6**, pp. 277–280]. However, since the proof itself is very interesting, we give here a brief outline of the main steps.

We begin with another discovery of Euler, his remarkable "pentagonal number theorem":

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2} = 1 + \sum_{n=1}^{\infty} (-1)^n \left(x^{n(3n-1)/2} + x^{n(3n+1)/2}\right). \tag{5}$$

Writing out the terms in (5) explicitly, we get

$$(1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots .$$

The reader can multiply out a few of the factors on the left side of this equation to verify that the terms having pentagonal numbers as exponents appear as given on the right side. Even Euler found it difficult to prove (5). A nice proof appears in Andrews [**2**, pp. 279–284].

It is convenient to introduce the numbers e_n given by

$$e_n = \begin{cases} 1 & n = 0 \\ (-1)^k & \text{if } n = \dfrac{k(3k \pm 1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, (5) can be rewritten as

$$g(x) = \prod_{n=1}^{\infty}(1 - x^n) = \sum_{n=0}^{\infty} e_n x^n. \tag{6}$$

The reader can verify that recursion relation (1) can be written as

$$p(n) = -\sum_{k=1}^{\infty} e_k p(n-k). \tag{7}$$

Note that the left side of (2) is the reciprocal of the left side of (7). From this it follows that

$$\left(\sum_{n=0}^{\infty} p(n)x^n\right)\left(\sum_{n=0}^{\infty} e_n x^n\right) = 1.$$

Using the standard Cauchy product of series we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} e_k p(n-k)\right)x^n = 1. \tag{8}$$

Since the right side is 1, it follows that $p(0) = 1$, and all the coefficients of the positive powers of x on the left side are zero. Thus $\sum_{k=0}^{n} e_k p(n-k) = 0$, for $n > 0$, from which our recursion relation (4) written in the form (8) follows. This completes our brief look at how this important recursion relation emerges. We now leave the partition function temporarily and get acquainted with the divisor function.

The divisor function

For positive integers n, the *divisor function* $\sigma(n)$ is defined as the sum of the positive divisors of n. Table 4 shows the divisors and the divisor function for $n \leq 20$.

Notice that while the partition function is clearly increasing, there is no apparent regularity in the divisor function. Whereas primes are intimately related to divisors of numbers, they do not appear to be related to partitions. On the other hand, we are not surprised that partitions satisfy a recursion relation, but we do not expect this of $\sigma(n)$. What do the divisors of n have to do with the divisors of $n-1, n-2, \ldots$? Yet Euler showed that $\sigma(n)$ satisfies the same recursion relation that $p(n)$ does, with only $\sigma(0)$ different from $p(0)$. We will see this later.

Table 4. Divisors of n and $\sigma(n)$

n	Divisors of n	$\sigma(n)$	n	Divisors of n	$\sigma(n)$
1	1	1	11	1, 11	12
2	1, 2	3	12	1, 2, 3, 4, 6, 12	28
3	1, 3	4	13	1, 13	14
4	1, 2, 4	7	14	1, 2, 7, 14	24
5	1, 5	6	15	1, 3, 5, 15	24
6	1, 2, 3, 6	12	16	1, 2, 4, 8, 16	31
7	1, 7	8	17	1, 17	18
8	1, 2, 4, 8	15	18	1, 2, 3, 6, 9, 18	39
9	1, 3, 9	13	19	1, 19	20
10	1, 2, 5, 10	18	20	1, 2, 4, 5, 10, 20	42

Getting to know $\sigma(n)$. We use the convention that $\sigma(0) = 0$. If p is prime, it is clear that $\sigma(p) = p + 1$. Also, since the only divisors of p^2 are 1, p, and p^2, $\sigma(p^2) = p^2 + p + 1$, which can be rewritten as $\frac{p^3-1}{p-1}$. Continuing in this way, we get the following result.

Lemma 1. *For any prime p and positive integer k,*

$$\sigma(p^k) = p^k + \cdots + p + 1 = \frac{p^{k+1} - 1}{p - 1}.$$

The following is a standard theorem in number theory textbooks (see, for example, [**1**, pp. 85–56]).

Lemma 2. *The function $\sigma(n)$ is multiplicative, that is, if m and n are relatively prime, then $\sigma(nm) = \sigma(n)\sigma(m)$.*

An immediate consequence of these lemmas is the following formula that can be used to evaluate $\sigma(n)$.

Lemma 3. *If the prime factorization of n is $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, then*

$$\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdots \frac{p_m^{k_m+1} - 1}{p_m - 1}.$$

Lemma 3 has a misleading simplicity. If n is a large number whose prime factors cannot easily be obtained, the formula is of no use. For this reason, a recurrence relation for $\sigma(n)$ would be of value and will be found later.

The union of partitions and divisors

It is now time for the promised union. The device that exposes this union is another series,

$$L(x) = \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n},$$

known as a Lambert series. Using the geometric series, we can expand each term of this series:

$$\begin{array}{rcccccccccccccl}
\frac{x}{1-x} & = x + & x^2 + & x^3 + & x^4 + & x^5 + & x^6 + & x^7 + & x^8 + & x^9 + & x^{10} + & x^{11} + & x^{12} + \cdots \\
\frac{2x^2}{1-x^2} & = & 2x^2 + & & 2x^4 + & & 2x^6 + & & 2x^8 + & & 2x^{10} + & & 2x^{12} + \cdots \\
\frac{3x^3}{1-x^3} & = & & 3x^3 + & & & 3x^6 + & & & 3x^9 + & & & 3x^{12} + \cdots \\
\frac{4x^4}{1-x^4} & = & & & 4x^4 + & & & & 4x^8 + & & & & 4x^{12} + \cdots \\
\frac{5x^5}{1-x^5} & = & & & & 5x^5 + & & & & & 5x^{10} + \cdots & & \\
\frac{6x^6}{1-x^6} & = & & & & & 6x^6 + & & & & & & 6x^{12} + \cdots \\
\frac{7x^7}{1-x^7} & = & & & & & & 7x^7 + \cdots & & & & &
\end{array}$$

Note that the coefficients of terms involving x^6 are 1, 2, 3, and 6. These are precisely the divisors of 6, so adding these terms, we get $\sigma(6)x^6$. This is true for every x^n, the coefficients of x^n are the divisors of n, so when these terms are added together we get $\sigma(n)x^n$. This establishes the following lemma.

Lemma 4. $\displaystyle\sum_{n=1}^{\infty} n\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n.$

We are now in a position to prove our first theorem connecting $\sigma(n)$ and $p(n)$.

Theorem 1.

$$\sigma(n) = -\sum_{k=0}^{n} ke_k p(n-k). \qquad (9)$$

Proof. Taking the logarithm of $g(x) = \prod_{n=1}^{\infty}(1-x^n)$, we get

$$\log g(x) = \sum_{n=1}^{\infty} \log(1-x^n).$$

Differentiating and then multiplying by x yields

$$-\frac{xg'(x)}{g(x)} = \sum_{n=1}^{\infty} n\frac{x^n}{1-x^n}.$$

On the other hand, we recall from (6) that $g(x) = \sum_{n=0}^{\infty} e_n x^n$, which implies that $xg'(x) = \sum_{n=0}^{\infty} ne_n x^n$. Also, from the generating function (2), we have

$$\frac{1}{g(x)} = \sum_{n=0}^{\infty} p(n)x^k.$$

Using Lemma 4 and substituting these last two series for $xg'(x)$ and $\frac{1}{g(x)}$, we find that

$$\left(\sum_{n=0}^{\infty} p(n)x^k\right) \cdot \left(\sum_{n=0}^{\infty} ne_n x^n\right) = -\sum_{n=1}^{\infty} n\frac{x^n}{1-x^n} = -\sum_{n=1}^{\infty} \sigma(n)x^n.$$

Expanding the product on the left, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} ke_k p(n-k)\right)x^n = -\sum_{n=1}^{\infty} \sigma(n)x^n,$$

and comparing coefficients completes the proof. ■

Theorem 2. *The divisor function satisfies the recurrence relation*

$$\sigma(n) = -ne_n - \sum_{k=0}^{n-1} e_{n-k}\sigma(k). \tag{10}$$

Proof. With $g(x)$ as above, we have noted that

$$-\frac{xg'(x)}{g(x)} = \sum_{n=1}^{\infty} \sigma(n)x^n.$$

Multiplying by $g(x)$ we get

$$-xg'(x) = g(x) \cdot \sum_{n=1}^{\infty} \sigma(n)x^n.$$

We use the facts that

$$g(x) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$xg'(x) = \sum_{n=0}^{\infty} ne_n x^n$$

to deduce

$$-\sum_{n=0}^{\infty} ne_n x^n = \left(\sum_{n=0}^{\infty} e_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} \sigma(n)x^n\right).$$

Rewriting the product on the right of this last expression, we find that

$$-\sum_{n=0}^{\infty} ne_n x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} e_{n-k}\sigma(k)\right)x^n.$$

Equating the coefficients of x^n, we get

$$-ne_n = \sum_{k=0}^{n} e_{n-k}\sigma(k) = \sum_{k=0}^{n-1} e_{n-k}\sigma(k) + e_0\sigma(n).$$

Since $e_0 = 1$, the theorem is proved. ■

The union that we have discovered culminates in the following revelation.

Theorem 3. *The partition function and the divisor function are related by the formula*

$$np(n) = \sum_{k=0}^{n} \sigma(k)p(n-k). \tag{11}$$

Proof. Let $F(x) = \frac{1}{g(x)}$. Then

$$F'(x) = -\frac{g'(x)}{(g(x))^2} = -\frac{g'(x)}{g(x)} \cdot \frac{1}{g(x)} = -\frac{g'(x)}{g(x)} \cdot F(x). \tag{12}$$

On the other hand, $-\frac{g'(x)}{g(x)} = \sum_{n=1}^{\infty} \sigma(n)x^{n-1}$, and since $F(x) = \sum_{n=0}^{\infty} p(n)x^n$, $F'(x) = \sum_{n=0}^{\infty} n \cdot p(n)x^{n-1}$. Substituting these into (12), and multiplying by x we get

$$\sum_{n=0}^{\infty} n \cdot p(n)x^n = \left(\sum_{n=0}^{\infty} \sigma(n)x^n\right)\left(\sum_{n=0}^{\infty} p(n)x^n\right).$$

Multiplying the infinite series on the right and comparing coefficients completes the proof. ■

A final look at the union. We now take a last look at the surprising union of $p(n)$ and $\sigma(n)$. First, notice the (almost identical) recursion relations that may be used to calculate the functions. Seen before as (1) and (10), they are now written as

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ &\quad + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \cdots \end{aligned}$$

and

$$\begin{aligned} \sigma(n) &= \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ &\quad + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) + \cdots . \end{aligned}$$

There is one trick that is not obvious in this description. If $p(0)$ occurs in the calculation, it is replaced by 1; but if $\sigma(0)$ occurs, it must be replaced by n, and not the expected 0. Partitions and sums of differences, so different, yet so remarkably alike!

We now return to (9), which allows us to get $\sigma(n)$ in terms of $p(k)$, with $k < n$:

$$\sigma(n) = p(n-1) + 2p(n-2) - 5p(n-5) - 7p(n-7) \\ + 12p(n-12) + 15p(n-15) - 22p(n-22) - 26p(n-26) + \cdots .$$

It's quite remarkable that the pentagonal numbers appear again in this expansion.

Finally, we examine (11). Here $p(n)$ and $\sigma(n)$ join together in total equality to give $p(n)$ alone:

$$np(n) = \sigma(1)p(n-1) + \sigma(2)p(n-2) + \sigma(3)p(n-3)\cdots + \sigma(n)p(0).$$

This recursion relation is so simple; even the pentagonal numbers are not needed.

Our discussion is over, but we sense that the mystery is only partially revealed. How can the problem of expressing a number as a sum, be at all related to expressing that same number as a product? Euler was astonished at these results, and you can read a translation of his own words in Pólya [**9**, pp. 90–107] and Young [**10**, pp. 357–368]. Mathematics often guards its secrets jealously!

References

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Errata

(March 2007)

The Media Highlight "Some New Aspects of the Coupon Collector's Problem" should have been credited to Raymond N. Greenwell.

(May 2007 issue)

The opening of the Teaching Tip on page 184 should read as follows: "In explaining the concept of a function to their students, many instructors use the illustration 'a function is a machine.' It is a machine f that takes an element x from the domain A and produces $f(x)$ in the co-domain B . . . ."

In "An Oft-discovered Result" on page 220, the third line should read "a natural number is a sum of consecutive integers if and only if it is not a power of 2."

The Editor apologizes for these errors.