VIETA-LIKE PRODUCTS OF NESTED RADICALS WITH FIBONACCI AND LUCAS NUMBERS

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The beautiful infinite product of radicals

(1) $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

due to Vieta [1] in 1592, is one of the oldest noniterative analytical expressions for π. It is the purpose of this note to prove the following two Vieta-like products

(2) $$\frac{\sqrt{5}F_N}{2N\log\phi}=\sqrt{\frac{1}{2}+\frac{L_N}{4}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{L_N}{4}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{L_N}{4}}}}\cdots$$

for N even, and

(3) $$\frac{L_N}{2N\log\phi}=\sqrt{\frac{1}{2}+\frac{\sqrt{5}F_N}{4}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{5}F_N}{4}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\sqrt{5}F_N}{4}}}}\cdots$$

for N odd. Here N is a positive integer, F_N and L_N are the Fibonacci and Lucas numbers, and $\phi=\frac{1+\sqrt{5}}{2}$ is the golden section. (The Fibonacci numbers are $F_1=1$, $F_2=1$, with the recursion relation $F_n=F_{n-1}+F_{n-2}$, while the Lucas numbers are $L_1=1$, $L_2=3$ with the same recursion relation $L_n=L_{n-1}+L_{n-2}$.)

First we must explore a few exact values of the hyperbolic functions. Notice that

$\frac{2}{\sqrt{5}}\sinh\left(N\log\phi\right)=\frac{1}{\sqrt{5}}\left(e^{N\log\phi}-e^{-N\log\phi}\right)\;=\frac{1}{\sqrt{5}}\left(\phi^{N}-\left(\frac{1}{\phi}\right)^{N}\right)=F_{N}$ for even *N*. (This last equality follows from Binet's formula [2], $F_{n}=\frac{1}{\sqrt{5}}\left(\phi^{n}-\left(-\frac{1}{\phi}\right)^{n}\right)$, true for all positive *n*.) For odd *N* we have $2\sinh\left(N\log\phi\right)=e^{N\log\phi}-e^{-N\log\phi}=\phi^{N}-\left(\frac{1}{\phi}\right)^{N}=L_{N}$. (This last equality follows from the Binet-like formula $L_{n}=\phi^{n}+\left(-\frac{1}{\phi}\right)^{n}$, which is true for all positive *n*.) Thus we have derived

$$\sinh\left(N\log\phi\right)=\begin{cases}\frac{\sqrt{5}}{2}F_{N} & for\ N\ even\\ \frac{1}{2}L_{N} & for\ N\ odd\end{cases}. \tag{4}$$

In a similar way we can derive

$$\cosh\left(N\log\phi\right)=\begin{cases}\frac{1}{2}L_{N} & for\ N\ even\\ \frac{\sqrt{5}}{2}F_{N} & for\ N\ odd\end{cases}. \tag{5}$$

Notice that in some ways the number $\log\phi$ acts with the hyperbolic functions as π does with the trigonometric functions. The hyperbolic functions of certain rational multiples of $\log\phi$ can be expressed as exact values.

To derive (2) and (3) we start by applying the double angle formula for the hyperbolic sine function p times to obtain

$$\sinh x=2\cosh\frac{x}{2}\sinh\frac{x}{2}$$

$$= 2^2 \cosh\frac{x}{2}\cosh\frac{x}{2^2}\sinh\frac{x}{2^2}$$

$$= 2^3 \cosh\frac{x}{2}\cosh\frac{x}{2^2}\cosh\frac{x}{2^3}\sinh\frac{x}{2^3}$$

...

(6) $$\sinh x = 2^p \cosh\frac{x}{2}\cosh\frac{x}{2^2}\cosh\frac{x}{2^3}\cdots\cosh\frac{x}{2^p}\sinh\frac{x}{2^p}$$

We evaluate each of the hyperbolic cosine factors in (6) in terms of $\cosh x$ by repeated use of the half-angle formula for the hyperbolic cosine.

$$\cosh\frac{x}{2} = \sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}$$

$$\cosh\frac{x}{2^2} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}$$

...

(7) $$\cosh\frac{x}{2^p} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}}}$$

$$(p \ \ \mathit{radicals})$$

Combining (7) with (6) and dividing by x we obtain

$$\frac{\sinh x}{x} = \frac{2^p}{x}\sinh\left(\frac{x}{2^p}\right)\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}}}$$

If we let p tend to infinity we get (since $\lim_{\alpha\to 0}(\sinh\alpha)/\alpha = 0$),

(8) $$\frac{\sinh x}{x} = \prod_{n=1}^{\infty}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cosh x}}}} \quad .$$

$$(n \ \ \mathit{radicals})$$

Now let $x = N \log \phi$ in (8) and use (4) and (5) to obtain at once our desired products (2) and (3). This completes our proof.

It is interesting to notice that a common derivation of the original Vieta product (1) proceeds like our derivation of (8) with hyperbolic functions of x replaced by trigonometric functions of θ. In the final step where we set $x = N \log \phi$ in the hyperbolic functions to obtain (2) and (3), one sets $\theta = \pi / 2$ in the trigonometric functions to obtain (1).

This note was motivated by a discussion with Richard Askey in which he showed how the Fibonacci and Lucas numbers are related to the hyperbolic functions.

References

[1] L. Berggren, J. Borwein and P. Borwein, *Pi, A Source Book,* Springer, New York, 1997, pp. 53-67.

[2] N. N. Vorob'ev, *Fibonacci Numbers*, Pergamon Press, 1961, pp. 20-28.