An unusual proof that F_m divides F_{mn} for Fibonacci numbers using hyperbolic functions

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The Fibonacci numbers are given by $F_1 = 1$, $F_2 = 1$, and $F_{m+1} = F_m + F_{m-1}$. The fact that F_m always divides F_{mn}. is often proved [1, p. 33] by first establishing the relation $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$, then using induction. In this short note we give an unusual derivation using familiar identities for the hyperbolic functions. We will also need the Lucas numbers defined by $L_1 = 1$, $L_2 = 3$, and $L_{m+1} = L_m + L_{m-1}$. Our proof is based on the following unusual closed forms for these numbers shown to us by professor emeritus Richard Askey of the University of Wisconsin.

(1) $$F_m = \frac{2}{\sqrt{5}i^m}\sinh\left(m\log(i\phi)\right), \text{ and}$$

(2) $$L_m = \frac{2}{i^m}\cosh\left(m\log(i\phi)\right),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden section. To see that (1) is true we write

$$\frac{2}{\sqrt{5}i^m}\sinh\left(m\log(i\phi)\right) = \frac{2}{\sqrt{5}i^m}\frac{e^{m\log(i\phi)} - e^{-m\log(i\phi)}}{2}$$

$$= \frac{e^{\log(i^m\phi^m)} - e^{\log(i^{-m}\phi^{-m})}}{\sqrt{5}i^m}$$

$$=\frac{i^m\phi^m - i^{-m}\phi^{-m}}{\sqrt{5}i^m}$$

$$=\frac{\phi^m - i^{-2m}\phi^{-m}}{\sqrt{5}}$$

$$=\frac{\phi^m - (-\phi)^{-m}}{\sqrt{5}}.$$

This last relation is the well known Binet's formula $F_m = \frac{\phi^m - (-\phi)^{-m}}{\sqrt{5}}$, and thus (1) is verified. We verify (2) in a similar way and find that it reduces to the known formula $L_m = \phi^m + (-\phi)^{-m}$.

We will use the following:

Lemma 1: We have

(3) $$F_{m+n} = \frac{F_m L_n + L_m F_n}{2}.$$

Proof: Using the familiar identity $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ we have

$$\frac{2}{\sqrt{5}i^{m+n}}\sinh\left((m+n)\log(i\phi)\right) =$$
$$\frac{2}{\sqrt{5}i^m}\sinh\left(m\log(i\phi)\right)\frac{2}{2i^n}\cosh\left(n\log(i\phi)\right) + \frac{2}{2i^m}\cosh\left(m\log(i\phi)\right)\frac{2}{\sqrt{5}i^n}\sinh\left(n\log(i\phi)\right).$$

Using (1) and (2) this translates to (3) immediately and the lemma is proved.

We will also use the next lemma:

Lemma 2: Both F_{3m} and L_{3m} , for m = 1, 2, 3, …, are even numbers, while all other Fibonacci and Lucas numbers are odd. Also, $F_m L_n + L_m F_n$ is always even for all natural numbers m and n.

Proof: It follows immediately from (3) that $F_m L_n + L_m F_n$ must be even. Let o denote an odd number, and e denote an even number. Both Fibonacci and Lucas sequences begin with two odd numbers. Using the recursion relation defining these sequences we see at once that both sequences have parity $o,o,e,o,o,e,o,o,e,o,o,\cdots$, and the lemma is proved.

We are now read to prove our main result.

Theorem 1: For Fibonacci numbers, F_m divides F_{mn}.

Proof: We use induction on n. For n =1, we see at once that F_m divides F_m. Next assume that F_m divides F_{mN}, we must show that F_m also divides $F_{m(N+1)}$. From Lemma I we have $2F_{m(N+1)} = F_{mN}L_m + L_{mN}F_m$. Dividing by F_m we get

(4) $$2\frac{F_{m(N+1)}}{F_m} = \left(F_{mN} / F_m\right) L_m + L_{mN}\,.$$

By induction we know that F_{mN} / F_m is an integer, and thus the right side of (4) is an integer. If F_m is odd, it is clear at once from the left side of (4) that F_m divides $F_{m(N+1)}$. If F_m is even, then L_m , F_{mN} and L_{mn} are also even by Lemma 2 . Thus we see at once that the right side of (4) is even and therefore F_m divides F_{mn}. Thus the theorem is proved.

Another interesting theorem related to Theorem 1 is the following:

Theorem 2: We have

(5) $$F_{2^{n+1}m} = F_m L_m L_{2m} L_{4m} L_{8m} \cdots L_{2^n m}\,.$$

Proof: We begin with the familiar identity $\sinh 2x = 2\sinh x \cosh x$. If we set

$x = m\log(i\phi)$ and multiply by $\frac{2}{\sqrt{5}i^{2m}}$ we get

$$\frac{2}{\sqrt{5}i^{2m}}\sinh\left(2m\log(i\phi)\right) = \frac{2}{\sqrt{5}i^{m}}\sinh\left(m\log(i\phi)\right)\frac{2}{i^{m}}\cosh\left(m\log(i\phi)\right).$$

Using (1) and (2) we have $F_{2m} = F_m L_m$. Replacing *m* by 2*m* we have

$F_{4m} = F_{2m}L_{2m} = F_m L_m L_{2m}$. Again replacing *m* by 2*m* we get $F_{8m} = F_{4m}L_{4m} = F_m L_m L_{2m} L_{4m}$.

Continuing in this way we get (5) and the theorem is proved. The special case of (5) in which $m = 1$ is the interesting relation

$$F_{2^{n+1}} = L_1 L_2 L_4 L_8 \cdots L_{2^n}.$$

Reference

[1] N. N. Vorob'ev, *Fibonacci Numbers*, Pergamon Press, 1961.