Another geometric vision of the hyperbola

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Tom Osler is professor of mathematics at Rowan University. He received his Ph. D. from the Courant Institute at New York University in 1970 and is the author of 84 mathematical papers. In addition to teaching university mathematics for the past 46 years, Tom has a passion for long distance running. He has been competing for the past 53 consecutive years. Included in his over 1950 races are wins in three national championships in the late sixties at distances from 25 kilometers to 50 miles. He is the author of two running books.

There are two standard methods that are often used in defining the hyperbola.

(A) The hyperbola is the locus of all points P whose distance from a fixed point F (the focus) divided by its distance to a fixed line D (the directrix) is a constant e (the eccentricity) greater than one, i.e. $\frac{PF}{PD} = e > 1$.

(B) The hyperbola is the locus of all points P in which the difference of the distances from two fixed points (the foci F and F') is a constant, i.e. $PF - PF' = c$.

These definitions are related to the *conic sections* as described in [1].

In a recent article, [2], an unusual method of constructing the hyperbola was shown that is based on the asymptotes of the curve. In this short note we present another unusual geometric method of constructing the hyperbola

(1) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

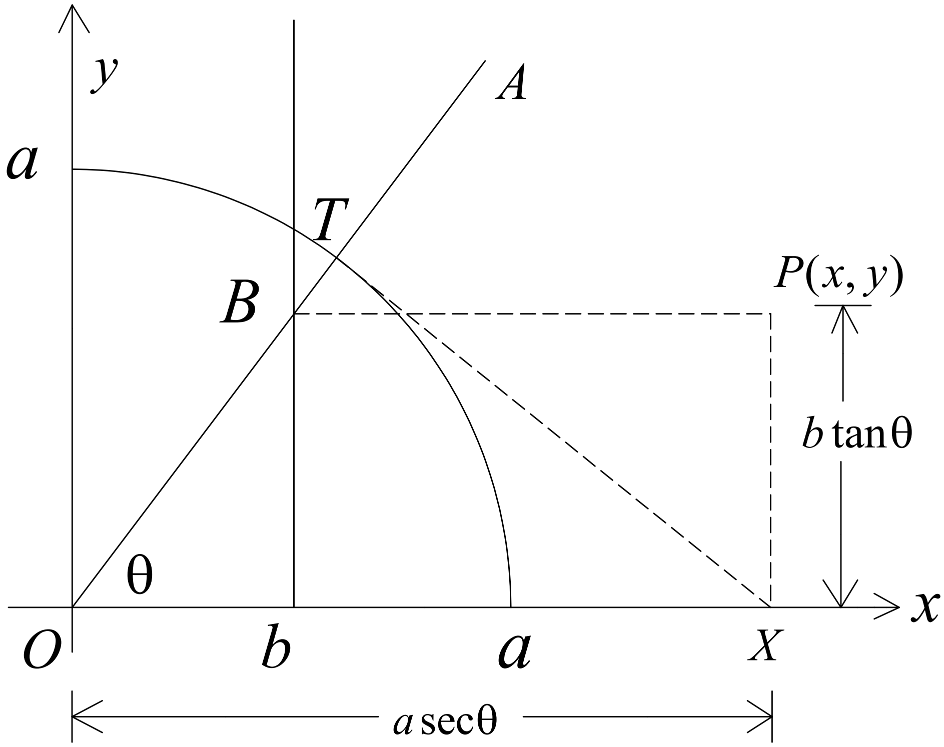

Figure 1

In Figure 1 we see a portion of the circle of radius *a* centered at the origin *O* of coordinates. The vertical line $x = b$ is also shown. Construct the ray OA making any angle θ with the *x* axis and intersecting the line $x = b$ at *B* and intersecting the circle at *T*. From the point *T*, construct a tangent to the circle intersecting the *x* axis at the point *X*. Construct a horizontal line through *B* and a vertical line through *X* meeting at *P*. We will show that the point *P* is on the hyperbola (1). By allowing θ to vary from 0 to 2π we generate the entire hyperbola described by the equation (1). In Figure 2 we see the portion of the hyperbola from the vertex at $(a, 0)$ to the point *P* being generated by this method.

It is easy to justify this construction. Examining Figure 1 we see that the coordinates of the point *P* are $x = a\sec\theta$ and $y = b\tan\theta$. From the identity

$$\sec^2\theta - \tan^2\theta = 1$$

we see at once that (1) is true.

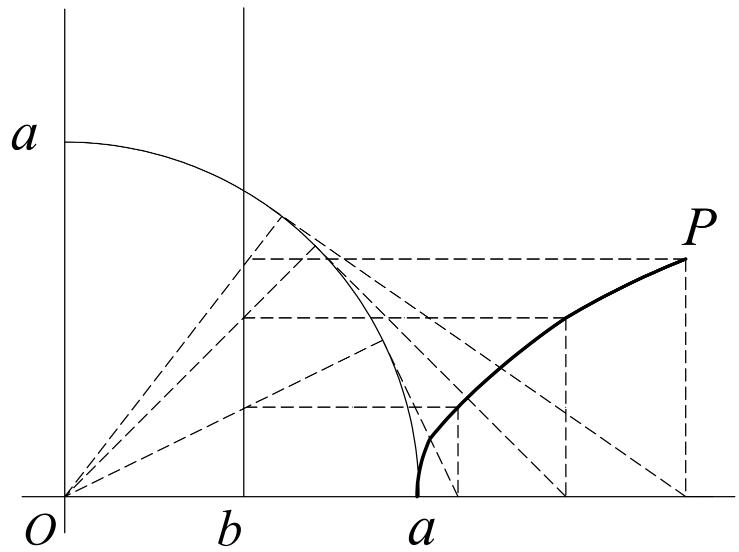

Figure 2

References

[1] Brannan, D. A., Esplen, M. F., and Gray, J. J., *Geometry*, Cambridge University Press, 1999, p. 5.

[2] Grochowski, D. and Osler, T. J., *An asymptotic approach to constructing the hyperbola*,. The Mathematical Spectrum 38(2005/2006), pp. 113-115.