APPROXIMATING THE PATH OF A CELESTIAL BODY WITH A CIRCULAR ORBIT FROM TWO CLOSE OBSERVATIONS

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Abstract

Data from at least three observations of an asteroid or planet is required to calculate the true elliptical orbit. Using ecliptic longitudes from only two close observations, we try to compute a circle that approximates the elliptical orbit. Our calculations result in not one, but two circular orbits. In some cases, we can use physical arguments to decide which of these two orbits is correct, otherwise, more observations may be necessary. The argument is elementary as only trigonometry and Kepler's laws of planetary motion are used. An example is given using data from a real asteroid. From such meager data, we can only expect a rough approximation.

1. Introduction

In January of 1801, Joseph Piazzi (see [1]) observed a new object with his telescope in Palermo. This new object appeared to move against the background stars. At this time, no asteroids were known, but the search for an unknown "planet" between the orbits of Mars and Jupiter was well under way. His observations came to an end in February, 1801 when the mysterious object became too dim to see as it approached the sun. Was this the undiscovered planet? In the 41 days Piazzi observed it, the object had only moved through an angle of 3 degrees. There was no known way to calculate the orbit from such close data. An elliptical orbit is specified by six parameters known as the “orbital elements”. Once these six numbers are known, the position of the planet can be calculated at any time. It was at this stage that the 24 year old Karl Friedrich Gauss. (see [2]) showed how to find the orbital elements from only three observations of the celestial latitude and longitude. His method is complicated, involving over 80 variables in three coordinate systems. By December, 1801, the mysterious object, now known as the asteroid Ceres, was rediscovered in the location predicted by Gauss. For this work, Laplace described Gauss as a “super-terrestrial spirit in a human body”, and the young mathematician’s fame was assured.

Suppose we have even less data, just two observations of the ecliptic longitude of an asteroid or planet on two different nights. (See the glossary at the end of this paper for astronomical terms that may be unfamiliar.) We have only two times, t_1 and t_2, and two corresponding angles θ_1 and θ_2. What can we conclude about the orbit of this object? Most asteroids and planets have orbits that are somewhat circular, having eccentricity e less than 0.2. (If $e = 0$, the orbit is a circle, and if e is near 1, the orbit is a very elongated ellipse, and the object is likely to be a comet.) Also, the plane of the orbit is usually inclined at a small angle i to the ecliptic plane. This angle is often less than 15 degrees. So we will make the following assumptions:

1. The orbit of the celestial object approximates a circle.
2. The orbit is on the ecliptic plane.
3. To simplify the calculations, we will also assume that the earth’s orbit is approximately circular. (The eccentricity of the earth’s orbit is $e = 0.0167$.)

With the above three assumptions, the derivations of the equations of an approximate circular orbit becomes relatively simple. The solution of these equations yields not one circular orbit, but three. One of the solutions is always the orbit of the earth, and can be ignored. Of the remaining two solutions, we can eliminate one, in some cases, by physical arguments, while in others, more observations may be required to determine which of the two is correct. The only mathematics needed is trigonometry, and the only physics used is Kepler's first and third laws of planetary motion. We will test our method using data from a real asteroid.

2. Deriving the Approximate Circular Orbit

The position of an asteroid or planet in a circular orbit can be described by

$$x = a\cos(\omega_A t + \gamma_A),\ \ y = a\sin(\omega_A t + \gamma_A)$$

in the ecliptic plane with the sun at the origin, and the x-axis pointing towards the vernal equinox. Here ω_A and γ_A are the angular velocity and initial longitude of the asteroid. The distance from the sun is a. We also assume that the earth is in a circular orbit given by

$$x = b\cos(\omega_E t + \gamma_E),\ \ y = b\sin(\omega_E t + \gamma_E)\ ,$$

with b, ω_E, and γ_E, the known radius, angular velocity and initial longitude of the earth respectively. Kepler's third law of planetary motion relates the unknowns a and ω_A by $\omega_A^2 = \frac{b^3\omega_E^2}{a^3}$. This means that only two of the three unknown parameters a, ω_A, and γ_A are independent, and need to be found from the observations of the asteroid's longitude.

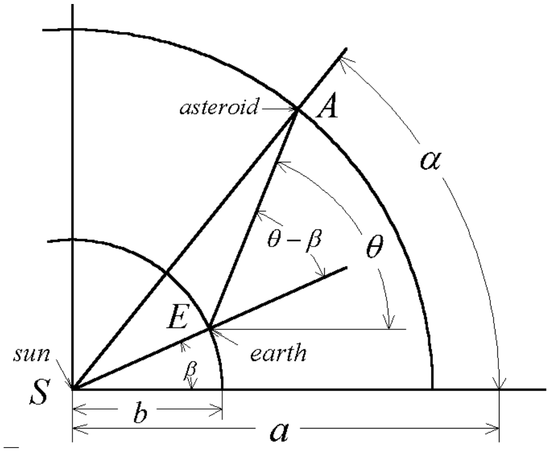


Figure 1A: Observing an asteroid or planet from the earth.

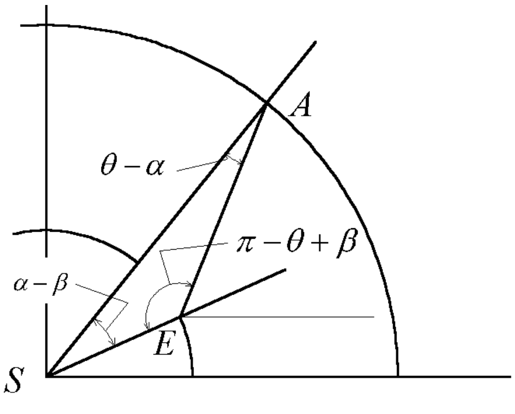


Figure 1B: Detail of the angles in triangle *SAE*.

In Figure 1, we imagine an observer on the earth E measuring the longitude θ of an asteroid at *A*. The sun is at point *S*. Using the law of sines for the triangle *SAE* we have

$$\frac{\sin(\theta-\alpha)}{b}=\frac{\sin(\pi-\theta+\beta)}{a}.$$

Since $\sin(\pi-\theta+\beta)=\sin(\theta-\beta)$, and $\sin(-x)=-\sin x$ we can rewrite this last expression as

$$\frac{\sin(\alpha-\theta)}{b}=\frac{\sin(\beta-\theta)}{a}.$$

Now $\beta=\omega_E t+\gamma_E$, and $\alpha=\omega_A t+\gamma_A$, so this last relation can be written as

$$\sin(\omega_A t+\gamma_A-\theta)=\frac{b\sin(\omega_E t+\gamma_E-\theta)}{a}. \tag{1}$$

This contains three unknowns, *a*, ω_A, and γ_A. Can we get a relation involving only one unknown? From Kepler's third law we know that $a^3\omega_A^2=b^3\omega_E^2$, so that

$$a = b\left(\frac{\omega_E}{\omega_A}\right)^{2/3}. \tag{2}$$

Removing a from (1) we get

$$\sin(\omega_A t + \gamma_A - \theta) = \frac{\omega_A^{2/3}\sin(\omega_E t + \gamma_E - \theta)}{\omega_E^{2/3}}. \tag{3}$$

Now, suppose we make two observations of the asteroid. At time t_1 we measure the longitude θ_1, and at time t_2 the longitude θ_2. If we substitute these measurements into (3) we notice that the expressions

$$c_1 = \frac{\sin(\omega_E t_1 + \gamma_E - \theta_1)}{\omega_E^{2/3}}, \text{ and } c_2 = \frac{\sin(\omega_E t_2 + \gamma_E - \theta_2)}{\omega_E^{2/3}}, \tag{4}$$

contain only known quantities. Therefore (3) yields two equations

$$\sin\left(\omega_A t_1 + \gamma_A - \theta_1\right) = c_1\omega_A^{2/3}, \tag{5}$$

and

$$\sin\left(\omega_A t_2 + \gamma_A - \theta_2\right) = c_2\omega_A^{2/3}. \tag{6}$$

Writing $\alpha_1 = \omega_A t_1 + \gamma_A - \theta_1$ and $\alpha_2 = \omega_A t_2 + \gamma_A - \theta_2$, we note that $\alpha_2 - \alpha_1 = \omega_A \Delta t - \Delta\theta$, (where we have used $\Delta t = t_2 - t_1$ and $\Delta\theta = \theta_2 - \theta_1$), does not contain γ_A. Using familiar trigonometric identities we have

$$\begin{aligned}\cos(\alpha_2 - \alpha_1) &= \cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2 \\ &= \pm\sqrt{1-\sin^2\alpha_1}\sqrt{1-\sin^2\alpha_2} + \sin\alpha_1\sin\alpha_2.\end{aligned}$$

From (5) and (6), we see that this is

$$\cos\left(\omega_A\Delta t - \Delta\theta\right) = \pm\sqrt{1-c_1^2\omega_A^{4/3}}\sqrt{1-c_2^2\omega_A^{4/3}} + c_1c_2\omega_A^{4/3}$$

Squaring to remove the radicals and simplifying we get

$$f(\omega_A) = \left[c_1^2 + c_2^2 - 2c_1c_2\cos\left(\omega_A\Delta t - \Delta\theta\right)\right]\omega_A^{4/3} - \sin^2\left(\omega_A\Delta t - \Delta\theta\right) = 0 \qquad (7)$$

Equation (7) contains only one unknown variable ω_A. We will see in the next section that equation (7) has three roots, and we will be required to decide which one fits our observations. We call (7) our *transcendental equation.* In (7) everything is known numerically except ω_A. So, in theory, we can solve (7) for the angular velocity ω_A of the asteroid. Using (2) we can find *a*, the radius of its orbit. It remains to find γ_A, the initial longitude of the asteroid. We can do this easily from (5) or (6). Solving (5) for γ_A we get

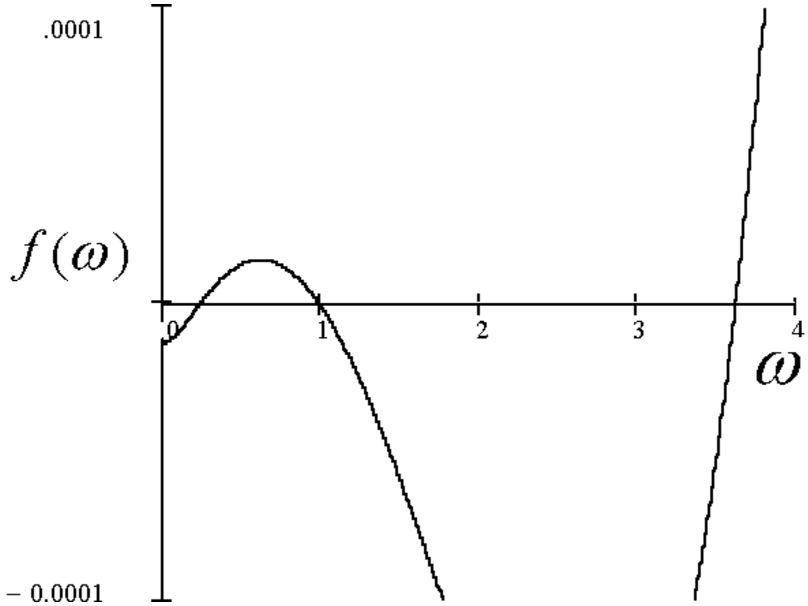
$$\gamma_A = \theta_1 - \omega_A t_1 + \sin^{-1}\left(c_1\omega_A^{2/3}\right). \qquad (8)$$

(If we select this angle in the range $0 \le \gamma_A < 360$, then equation (8) yields two values for γ_A, because the inverse sine function is multiple valued. We will show how to select the appropriate values in an example.) We have now found parametric equations describing the circular orbit of the asteroid:

$$x = a\cos(\omega_A t + \gamma_A),\ \ y = a\sin(\omega_A t + \gamma_A).$$

3. Detailed Sample Calculations for the Asteroid Adeona

We now consider an example in which two observations of an unknown object are made within a day of each other. We show how the mathematical relations derived previously can be used to calculate a circular orbit that very roughly approximates the true elliptical orbit. Our data comes from the software *Starry Night Pro.*



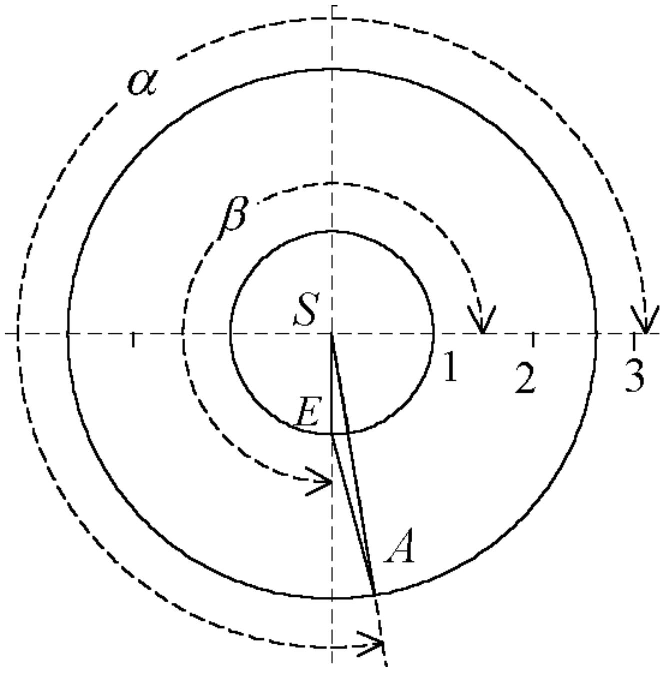
We observe the asteroid Adeona from Philadelphia at midnight on June 19 and 20 of 2002. On the first night we locate Adeona at 284.7277 ecliptic degrees longitude and on the second night 284.5216 degrees. Thus $\theta_1 = 284.7277$ and $\theta_2 = 284.5216$. Measuring time (in days) from midnight on January 1, 2000 to June 19, 2002, we calculate that 900 days have elapsed. Our time should be measured in Universal Time (UT) which is the time in Greenwich, England. Thus midnight in Philadelphia is 4 hours UT. So $t_1 = 900 + 4/24 = 900.1667$ and $t_2 = 901.1667$ days.

Next we calculate c_1 and c_2 using relations (4). Here we use $\omega_E = 0.98561$ degrees per day for the angular speed of the earth and $\gamma_E = 99.6794$ degrees as the longitude of the earth on Jan. 1, 2000. We obtain $c_1 = -0.309304$ and $c_2 = -0.289247$. We can now substitute these values into the transcendental equation (7). We used Mathcad to get a graph of $y = f(\omega)$, (Figure 2), where the roots are possible values for $\omega = \omega_A$, the angular velocity of the asteroid.

Figure 2: Exploring the three roots for the angular velocity ω_A of Adeona.

This graph shows the three possible solutions to equation (7). Only one root can represent the actual celestial object we are observing. Which one is it? Recall that the angular velocity of the earth is $\omega_E = 0.98561$. Thus values of ω_A less than 0.98561 represent objects orbiting beyond the orbit of the earth, and values of ω_A smaller than ω_E represent objects orbiting inside the orbit of the earth. Using the root finder from Mathcad, we find that the smallest root is $\omega_A = 0.237283$ degrees per day for the angular velocity of the asteroid. From (2) we can now estimate the radius of the orbit of our asteroid as $a = \left(\dfrac{\omega_E}{\omega_A}\right)^{2/3} = 2.5840$ AU.

It is instructive to make an approximate drawing, (see Figure 3), showing the positions of the sun, earth and the Adeona in the ecliptic plane. Here the positive x-axis points in the direction with longitude zero. With the sun (*S*) at the origin, we draw the circular orbit of the earth with radius 1 and the orbit of Adeona with radius 2.6. On the date of our observations, the earth had ecliptic longitude $\beta = \omega_E t + \gamma_E \approx 267$ degrees. This allows us to locate the earth (point *E*) on the circle of radius 1. From the earth *E* , we observe Adeona *A* at angle $\theta \approx 286$ degrees, so we can draw the line segment *EA* at this angle. Now we can draw the line *SA* and measure the ecliptic longitude of Adeona to be



$\alpha \approx 276$ degrees.

Figure 3: Positions of Earth (E) and Adeona (A)

Finally we use (8) to find the initial longitude of our asteroid. The principal value for the inverse sine function, denoted by $Sin^{-1}\left(c\,\omega^{2/3}\right)$, is in the range $-90 < Sin^{-1}\left(c_1\omega^{2/3}\right) \leq 90$. It is this principal value that is computed by calculators and other software functions. Other possible values are

$$\sin^{-1}\left(c_1\omega^{2/3}\right) = Sin\left(c_1\omega^{2/3}\right) + 360n\text{, and} \tag{9}$$

$$\sin^{-1}\left(c_1\omega^{2/3}\right) = 180 - Sin\left(c_1\omega^{2/3}\right) + 360n, \tag{10}$$

where n is any integer. Using a calculator we find $Sin^{-1}\left(c_1\omega^{2/3}\right) = -6.81$ degrees. Recall that we select γ_A to be in the range $0 \leq \gamma_A < 360$. From (8) and (9) with $n = 0$ we calculate $\gamma_{A1} = 64.33$ degrees. The other solution is obtained using (8) and (10) with $n = 0$ and is $\gamma_{A2} = 258.01$ degrees. We need to determine which initial angle is correct, 64.33 or 258.01 degrees. A measurement from Figure 3 gives us the longitude of A as $\alpha \approx 276$ degrees. From $\alpha = \omega_A t + \gamma_A$ and $\omega_A t = 213.55$ degrees, we conclude that the correct value is $\gamma_A = 64.33$ degrees. The true orbit of the asteroid Adeona has an eccentricity of $e = 0.143$, a mean radius of 2.674 AU, and initial angle 55.6 degrees.

The second root obtained from (7) is $\omega_A = 0.985456$ degrees per day. This is really $\omega_A = \omega_E$, the angular velocity of the earth. In future problems, we can always ignore this root.

The largest root of (7) is $\omega_A = 3.619919$ degrees per day with orbital radius $a = 0.42001$ AU. At this speed, the object would be closer to the sun than Venus. This is impossible, because such an object would not be visible at midnight.

In Figure 4 we show the actual elliptical orbit of Adeona compared to our calculated circular orbit; the latter is not too bad an approximation.

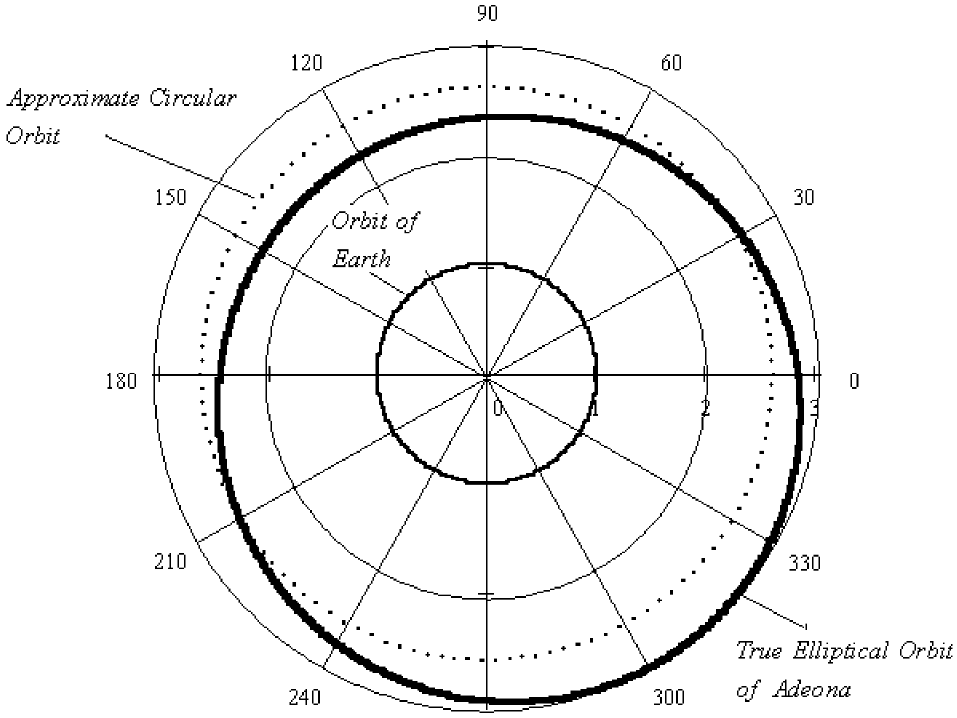


Figure 4: The true elliptical orbit of Adeona compared to Our calculated circular orbit.

Glossary of Astronomical Terms

Astronomical unit (AU). A unit of length approximately equal to the mean distance between the earth and the sun. The currently accepted value of the AU is 149 597 870 691 ± 30 meters (about 150 million kilometers or 93 million miles).

Celestial equator. The great circle on the celestial sphere which lies directly above the equator of the earth.

Celestial sphere. An imaginary sphere, on which we plot the apparent position of the sun, moon, planets and stars as viewed from the earth.

Ecliptic. The path of the sun against the background stars as seen from the earth over the course of a full year. It is a great circle on the celestial sphere that intersects the celestial equator at an angle of 23.4 degrees, (obliquity of the ecliptic). This is due to the axis of the earth being tilted in relation to the plane containing the earth's orbit. The points where the ecliptic crosses the celestial equator are called equinoxes. The planets, most asteroids and the moon have orbits which lie close to the ecliptic.

Ecliptic latitude and longitude. An orthogonal coordinate system for locating the position of a celestial object against the background of the stars. The measurements of these angles are often made with the earth or the sun as the center of observation. The ecliptic latitude gives the angular position of a celestial body north or south of the ecliptic. It is measured in degrees from 0 at the ecliptic to 90 at the ecliptic pole. The ecliptic longitude gives the angular position of a celestial body around the plane of the ecliptic. It is measured in degrees from 0 to 360 along the ecliptic, starting at the vernal equinox.

Ecliptic plane. The plane in which the earth's orbit revolves about the sun.

Perihelion. The point on an elliptical orbit that is nearest to the sun's center.

Retrograde motion. The movement of a body from east to west on the celestial sphere. Normally planets and asteroids appear to move from west to east along the ecliptic. When the sun, the earth and the object are nearly in a straight line in space, the object appears to move backward.

Universal time. The time at the longitude of Greenwich, England. Universal time (UT) is used widely as the standard time in astronomical publications.

Vernal equinox. The point on the celestial sphere at which the sun passes from south to north of the celestial equator.

References

[1] Dunnington, G. Waldo, *Carl Friedrich Gauss: Titan of Science*, Exposition Press, New York, N.Y., 1955, pp. 49-66.

[2] Gauss, Karl Friedrich, *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections,* (a translation of *Theoria Motus* (published in 1809) by Henry Davis), Dover Pub., 1963.