EULER'S LITTLE SUMMATION FORMULA AND SUMS OF POWERS

Andrew Robertson and Thomas J. Osler
Mathematics Department
Rowan University
Glassboro NJ 06028

Osler@rowan.edu

1. Introduction

In this article we introduce an elementary summation formula due to Euler that we call "Euler's Little Summation Formula". This little summation formula [1] was found by Euler as an intermediate item in the derivation of his "big" result [2, pp. 518-535] that we call today the Euler-Maclaurin summation formula. We then apply this formula to the calculation of sums of powers. A recursive method is found to obtain closed forms for the sum $\sum_{n=1}^{N} n^p$, where p is a positive integer.

2. Derivation of Euler's little summation formula

We begin by introducing an unconventional notation for the sum of a series.

Definition: Let *a, b* and *h* be constants and let

$$ {}_aS_{b,h} f(x) = f(a) + f(a+h) + f(a+2h) + \cdots + f(b) \, , $$

where $b = a + Nh$, with N = 0, 1, 2, … . Here we read ${}_aS_{b,\,h} f(x)$ as "summation of $f(x)$ from a to b with increment h". Usually we sum with increment $h = 1$, and in this case we suppress the h and write ${}_aS_{b,1} f(x) = {}_aS_b f(x)$.

We use this for three reasons. Firstly, a similar notation was used by Euler in his original discussion of this work. Secondly, in the development below, several different summations occur, and having a special notation for the principal sum helps us to identify it in complex expressions. The third reason is that we sum over the increment *h* rather than the usual increment 1. There are applications to the summation of alternating series where other values of *h* are needed, but they will not be discussed here.

Theorem 1: Let $f(x)$ be a function with $M+1$ continuous derivatives for $x > a-1$. Then

(1) $$ {}_aS_b f'(x) = f(b) - f(a-1) + \sum_{n=2}^{M} \frac{(-1)^n\ {}_aS_b f^{(n)}(x)}{n!} + R_M , $$

where

$$ R_M = \frac{(-1)^{M+1}}{(M+1)!}\left(f^{(M+1)}(x_a^*) + f^{(M+1)}(x_{a+1}^*) + \cdots + f^{(M+1)}(x_b^*) \right)., $$

for some x_c^* in the interval $c-1 < x_c^* < c$.

Proof: From the hypothesis that $f(x)$ is a function of *x* with *M*+1 continuous derivatives, we can expand $f(x)$ in a Taylor's series and get

$$ f(x-h) = \sum_{n=0}^{M} \frac{(-1)^n f^{(n)}(x)}{n!} h^n + r_M , $$

where with *h*>0, $r_M = \frac{(-1)^{M+1} f^{(M+1)}(x_*)}{(M+1)!} h^{M+1}$ for some x_* in the interval $x-h < x_* < x$.

Summing from *a* to $b = a + Nh$ we have

(3) $$ {}_aS_{b,\,h} f(x-h) = {}_aS_{b,h} f(x) + \sum_{n=1}^{M} \frac{(-1)^n\ {}_aS_{b,h} f^{(n)}(x)}{n!} h^n + R_M \quad , $$

where R_M is described by

$$ R_M = \frac{(-1)^{M+1}}{(M+1)!} h^{M+1} \left(f^{(M+1)}(x_a^*) + f^{(M+1)}(x_{a+h}^*) + \cdots + f^{(M+1)}(x_b^*) \right) $$

for some x_c^* is in the interval $c-h<x_c^*<c$. Notice that

$$ {}_aS_{b,h}f(x-h)=f(a-h)+f(a)+f(a+h)+\cdots+f(b-h), $$

and

$$ {}_aS_{b,h}f(x)=f(a)+f(a+h)+f(a+2h)+\cdots+f(b) $$

so

$$ {}_aS_{b,h}f(x-h)-{}_aS_{b,h}f(x)=f(a-h)-f(b). $$

Now we have from (3)

$$ f(b)=f(a-h)-\sum_{n=1}^{M}\frac{(-1)^n\,{}_aS_{b,h}f^{(n)}(x)}{n!}h^n-R_M, $$

and isolating the first term in the sum we get

(4) $$ f(b)=f(a-h)+h\,{}_aS_{b,h}f'(x)-\sum_{n=2}^{M}\frac{(-1)^n\,{}_aS_{b,h}f^{(n)}(x)}{n!}h^n-R_M\quad. $$

from which

(5) $$ {}_aS_{b,h}f'(x)=\frac{1}{h}f(b)-\frac{1}{h}f(a-h)+\sum_{n=2}^{M}\frac{(-1)^n\,{}_aS_{b,h}f^{(n)}(x)}{n!}h^{n-1}+\frac{1}{h}R_M. $$

Now let $h=1$ and this last relation becomes (1), and the theorem is proved. □

We call (1) Euler's little summation formula.

3. Application of Euler's little summation formula to sums of powers

We now look for closed form expressions for sums of the form

$$ {}_1S_Nx^p=\sum_{n=1}^{N}n^p, $$

for p = 0, 1, 2, … . Using $f(x)=\dfrac{x^{p+1}}{p+1}$, and $a=1$ in (1) we get the finite series

(6) $$ {}_1S_b x^p = \frac{b^{p+1}}{p+1} - \frac{0^{p+1}}{p+1} + \sum_{n=2}^{p+1} \frac{(-1)^n}{n!} {}_1S_b \frac{p!x^{p-n+1}}{(p-n+1)!}. $$

(Note that the series is conveniently finite because the remaining derivatives all vanish.)

We start with $p = 0$ and easily see that

(7) $$ {}_1S_N x^0 = \sum_{n=1}^{N} 1 = N. $$

Next we let $p = 1$ and use (6) and (7) to get

(8) $$ {}_1S_N x = \frac{N^2}{2} + \frac{{}_1S_N 1}{2} = \frac{N^2 + N}{2}. $$

Let $p = 2$ and use the same idea to find ${}_1S_N x^2$.

$$ \begin{aligned} {}_1S_N x^2 &= \frac{N^3}{3} + \frac{{}_1S_N(2x)}{2} - \frac{{}_1S_N(2)}{6} \\ &= \frac{N^3}{3} + {}_1S_N(x) - \frac{{}_1S_N(1)}{3}. \end{aligned} $$

Using the previous sums (7) and (8) this last result becomes

$$ \begin{aligned} {}_1S_N x^2 &= \frac{N^3}{3} + \frac{N^2 + N}{2} - \frac{N}{3} \\ &= \frac{2N^3 + 3N^2 + N}{6}. \end{aligned} $$

It is now clear that, once we have found the sums ${}_1S_N x^0$, ${}_1S_N x^1$, …, ${}_1S_N x^{p-1}$, we can use Euler's little summation formula to obtain ${}_1S_N x^p$. We list the results for p from 1 to 10 below for reference:

$$ \sum_{n=1}^{N} n = \frac{N^2 - N}{2} $$

$$ \sum_{n=1}^{N} n^2 = \frac{2N^3 + 3N^2 + N}{6} $$

$$\sum_{n=1}^{N} n^3 = \frac{N^4 + 2N^3 + N^2}{4}$$

$$\sum_{n=1}^{N} n^4 = \frac{6N^5 + 15N^4 + 10N^3 - N}{30}$$

$$\sum_{n=1}^{N} n^5 = \frac{2N^6 + 6N^5 + 5N^4 - N^2}{12}$$

$$\sum_{n=1}^{N} n^6 = \frac{6N^7 + 21N^6 + 21N^5 - 7N^3 + N}{42}$$

$$\sum_{n=1}^{N} n^7 = \frac{3N^8 + 12N^7 + 14N^6 - 7N^4 + 2N^2}{24}$$

$$\sum_{n=1}^{N} n^8 = \frac{10N^9 + 45N^8 + 60N^7 - 42N^5 + 20N^3 - 3N}{90}$$

$$\sum_{n=1}^{N} n^9 = \frac{2N^{10} + 10N^9 + 15N^8 - 14N^6 + 10N^4 - 3N^2}{20}$$

$$\sum_{n=1}^{N} n^{10} = \frac{6N^{11} + 33N^{10} + 55N^9 - 66N^7 + 66N^5 - 33N^3 + 5N}{66}$$

For more information on sums of powers, see [3].

References

[1] Euler, L., *Excerpts on the Euler-Maclaurin summation formula*, from *Institutiones Calculi Differentialis* (translated by David Pengelley), at http://math.nmsu.edu/~davidp, New Mexico State University, 2000.

[2] Knopp, Konrad., *Theory and Application of Infinite Series*, Dover Publications, New York, 1990. (A translation by R. C. H. Young of the 4th German addition of 1947.) ISBN: 0486661652

[3] Weisstein, E. W. *Power Sum*, from Math World—A Woifram Web Resource. http://mathworld.wolfram.com/PowerSum.html