OBLIQUE-ANGLED DIAMETERS AND THE CONIC SECTIONS

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1. Introduction

Recently the authors were translating a paper of Leonhard Euler [1] from French into English. Euler wrote of "oblique-angled diameters" and "orthogonal diameters" without explanation. Apparently these ideas were familiar to mathematicians in his day, but have been ignored in the education of modern mathematicians. It is the purpose of this article to bring these simple ideas to the attention of today's readers.

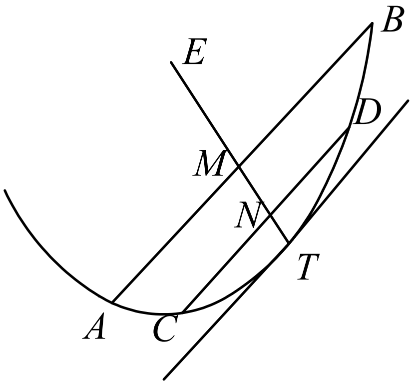

Figure 1:
ET is an oblique-angled diameter

Definition: Given a curve $ACTDB$ shown in figure 1. The line ET, which intersects the curve at T, is called an *oblique-angled diameter* if it bisects all chords (such as AB and CD), that are parallel to the tangent line at T.

In other words *AM = MB and CN = ND* since the chords *AB* and *CD* are parallel to the tangent line at the point *T* where the *oblique-angled diameter ET* intersects the curve *ACTDB.*

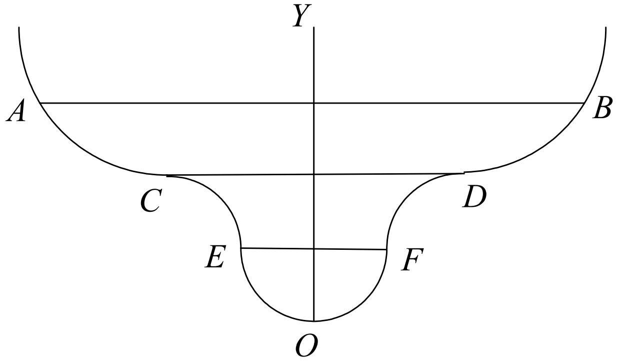

Figure 2: An orthogonal diameter *OY*

Many curves possess an oblique-angled diameter. For example, any curve that is symmetric about an axis has this property. In Figure 2 we see a curve that is symmetric about the line *OY*. Clearly this line bisects all horizontal chords. In this special case we call *OY* an *orthogonal diameter.* It is also clear that every diameter of a circle is an orthogonal diameter.

While many curves possess one oblique-angled diameter, in the case of the conic sections **all** appropriately defined diameters have this property. For the parabola, we define a “diameter” as any line parallel to the axis of the parabola. In this case, all diameters are oblique-angled diameters. We will prove this in section 2. For the ellipse and the hyperbola, we define a “diameter” as any line that passes through the center of the curve. In section 3 we will prove that **all** such diameters are oblique-angled diameters.

2. The parabola

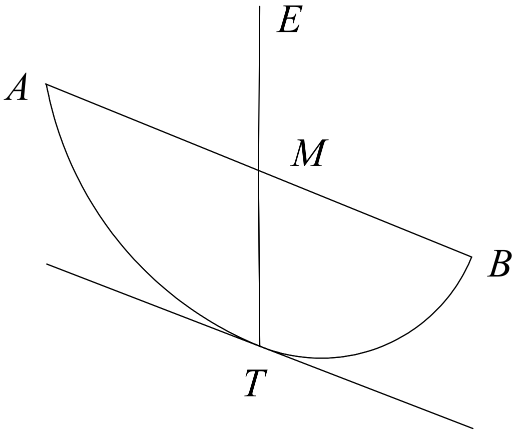

Figure 3: The parabola

Theorem 1. Let *ATB* be a parabola and let *ET* be any line parallel to the axis of the parabola. Let *AB* be any chord parallel to the tangent line at *T*. Then *ET* bisects the chord *AB* at the point of intersection *M*.

Proof Since all parabolas are similar, without loss of generality, we can call the equation of the parabola $y = x^2$. Call the coordinates of the point *T*, (x_T, y_T). The slope at this point is $\frac{dy}{dx} = m = 2x_T$. The line *AB* has the equation $y = mx + b$ and intersects the parabola at the points where $x^2 - mx - b = 0$. The solutions of this equation are $\frac{m}{2} \pm \frac{\sqrt{m^2 + 4b}}{2} = x_T \pm \sqrt{x_T^2 + b}$. Thus the *x* coordinate of the point *A* is $x_T - \sqrt{x_T^2 + b}$, and the *x* coordinate of the point *B* is $x_T + \sqrt{x_T^2 + b}$. Since the point *M* has *x* coordinate x_T it is clear from this result that the point *M* bisects the line segment *AB*.

Thus any line parallel to the axis of a parabola is an oblique-angled diameter. Next we consider the ellipse and the hyperbola.

3. The ellipse and the hyperbola

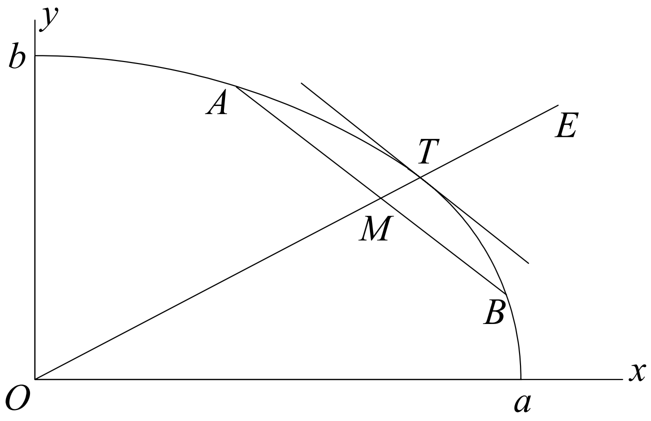

Figure 4: The ellipse

Theorem 2: Every diameter of an ellipse is an oblique-angled diameter.

Proof: Recall that a diameter is any line that passes through the center of the ellipse. In Figure 4 we see a section of the ellipse given by the equation

(1) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The line *OE* is by definition a *diameter* of the ellipse since it passes through the centre of the ellipse. This diameter intersects the ellipse at the point *T* which has coordinates (x_T, y_T). Differentiating (1) we get $y'=-\frac{b^2x}{a^2y}$. Thus the slope of the tangent line at the point *T* is given by

(2) $$m=-\frac{b^2x_T}{a^2y_T}.$$

The equation of the chord AB, which is parallel to the tangent at *T* is

(3) $$y=mx+c,$$

for some number *c*. The coordinates of the points *A* and *B* are found by solving (1) and (3) simultaneously. We are lead to consider

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

which simplifies to the quadratic equation

$$\left(b^2+a^2m^2\right)x^2+2a^2mcx+a^2c^2-a^2b^2=0.$$

The roots of this quadratic are

(4) $$x=-\frac{a^2cm}{b^2+a^2m^2}\pm\Delta,$$

where Δ is an expression involving a, b, c and m whose exact value will not concern us.

The x-coordinates of the points A and B are thus given by

$$x_A=-\frac{a^2cm}{b^2+a^2m^2}-\Delta \text{ and } x_B=-\frac{a^2cm}{b^2+a^2m^2}+\Delta.$$

Thus it is clear that the x-coordinate of the midpoint M of AB is

(5) $$x_M=-\frac{a^2cm}{b^2+a^2m^2}.$$

We must show that the diameter, which passes through T also bisects the chord AB and thus passes through the point M. This diameter has the equation

(6) $$y=\frac{y_T}{x_T}x.$$

Thus we must solve the equations (3) and (6) simultaneously and show that the solution

is identical to (5). Thus we have $\frac{y_T}{x_T}x=mx+c$. and solving for x we get

(7) $$x=\frac{cx_T}{y_T-mx_T}.$$

Using (2) to eliminate m, after some manipulation, we see that (5) and (7) both become

$$x = \frac{a^2 c\, x_T y_T}{a^2 y_T^2 + b^2 x_T^2}$$

and are thus identical and the theorem is proved.

Theorem 3: Every diameter of a hyperbola is an oblique-angled diameter.

Proof: The equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Repeating the above argument with this equation produces terms which differ only by an occasional sign. We leave it to to the reader to complete the details.

Reference

[1] Euler, L., *Sur quelques proprietes des Sections coniques qui conviennent a un infinite d'autres lignes courbes*, (On some properties shared between conic sections and infinitely many other curves) Originally published in Memoires de l'academie des sciences de Berlin 1, 1746, pp. 71-98. Available in *Opera Omnia*: Series 1, Volume 27, Birkhauser, 1989, pp. 51 – 73, ISBN-10: 3-7643-1475-3. Also available on the web site *The Euler Archive*, http://www.math.dartmouth.edu/~euler/.