

Oct 20, 2006

## Euler's little summation formula and special values of the zeta function

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### 1. Introduction

In this note we present an elementary method of determining values of the zeta function  $\zeta(z)$  for  $z = 0, -1, -2, \dots$ . We use a little known summation formula due to Euler.

The zeta function, defined by the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \operatorname{Re}(z) > 1,$$

is one of the most important special functions in mathematics. The functional equation for the zeta function is

$$\zeta(1-z) = 2(2\pi)^{-z} \cos\left(\frac{\pi z}{2}\right) \Gamma(z) \zeta(z).$$

Euler found closed form expressions for  $\zeta(z)$  when  $z$  is an even natural number. He

showed that  $\zeta(2) = \frac{\pi^2}{6}$ ,  $\zeta(4) = \frac{\pi^4}{90}$ ,  $\zeta(6) = \frac{\pi^6}{945}$ , and in general

$$\zeta(2p) = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{(-1)^{p+1} 2^{2p-1} B_{2p}}{(2p)!} \pi^{2p}.$$

(See Knopp [3, page 237].) Here the numbers  $B_n$  are called Bernoulli's numbers, and they are all rational. The first few are

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \dots, \text{ and } B_3 = B_5 = B_7 = \dots = 0,$$

and their generating function is

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

These can all be calculated recursively by starting with  $B_0 = 1$ , and using

$$(1) \quad \binom{n}{0}B_0 + \binom{n}{1}B_1 + \binom{n}{2}B_2 + \cdots + \binom{n}{n-1}B_{n-1} = 0$$

for  $n = 2, 3, 4, \dots$ .

Using the functional equation and the special values of the zeta function at positive even integers shown above, we can calculate the values

$$\zeta(0) = B_1, \text{ and } \zeta(-n) = -\frac{B_{n+1}}{n+1}, \text{ for } n = 1, 2, 3, \dots$$

A second method [1, p. 12] of finding these values of the zeta function is to use the theory of residues applied to a contour integral representation of  $\zeta(z)$ .

In this paper we show that these values of the zeta function can be found without using either the functional equation or the special values of the zeta function at positive even integers. Nor do we need contour integrals and the theory of residues. We use only a rarely seen elementary summation formula due to Euler that we call ‘‘Euler’s little summation formula’’. This little summation formula [2] was found by Euler as an intermediate item in the derivation of his ‘‘big’’ result [3, pp. 518-535] that we call today the Euler-Maclaurin summation formula. In the next section we will show a simple derivation of the little summation formula and in section 3 we will use it to find the values of  $\zeta(z)$  for  $z = 0, -1, -2, \dots$ .

## 2. Euler’s little summation formula for infinite series

**Definition:** Let  $a$ ,  $b$  and  $h$  be constants and let

$${}_a S_{b,h} f(x) = f(a) + f(a+h) + f(a+2h) + \cdots + f(b),$$

where  $b = a + Nh$ , with  $N = 0, 1, 2, \dots$ . Here we read  ${}_a S_{b,h} f(x)$  as “summation of  $f(x)$  from  $a$  to  $b$  with increment  $h$ ”. Usually we sum with increment  $h = 1$ , and in this case we suppress the  $h$  and write  ${}_a S_{b,1} f(x) = {}_a S_b f(x)$ .

**Theorem (Euler’s little summation formula):** Let  $f(x)$  be a function with  $M + 1$  continuous derivatives over the interval  $a \leq x < \infty$ , and let us use the notation  $f(\infty) = \lim_{x \rightarrow \infty} f(x)$ . Then

$$(2) \quad {}_a S_{\infty} f'(x) = f(\infty) - f(a) - \sum_{n=2}^M \frac{{}_a S_{\infty} f^{(n)}(x)}{n!} - R_M,$$

where

$$R_M = \frac{1}{(M+1)!} \left( f^{(M+1)}(x_a^*) + f^{(M+1)}(x_{a+1}^*) + f^{(M+1)}(x_{a+2}^*) + \dots \right),$$

and  $x_c^*$  is in the interval  $c < x_c^* < c + 1$ .

**Proof:** From the hypothesis  $f(x)$  is a function of  $x$  with  $M+1$  continuous derivatives.

Thus (for  $a < x$ ), we can expand  $f(x)$  in a Taylor’s series with  $M+1$  terms and a

remainder: 
$$f(x+h) = \sum_{n=0}^M \frac{f^{(n)}(x)}{n!} h^n + r_M,$$

where  $r_M = \frac{f^{(M+1)}(x^*)}{(M+1)!} h^{M+1}$  and  $x^*$  is in the interval  $x < x^* < x + h$ . Summing from  $a$  to

$b = a + Nh$  we have

$$(3) \quad {}_a S_{b,h} f(x+h) = {}_a S_{b,h} f(x) + \sum_{n=1}^M \frac{{}_a S_{b,h} f^{(n)}(x)}{n!} h^n + R_M,$$

where  $R_M = {}_a S_{b,h} r_M$  is described by

$$R_M = \frac{h^{M+1}}{(M+1)!} \left( f^{(M+1)}(x_a^*) + f^{(M+1)}(x_{a+h}^*) + \cdots + f^{(M+1)}(x_b^*) \right).$$

and  $x_c^*$  is in the interval  $c < x_c^* < c+h$ . Notice that

$${}_a S_{b,h} f(x+h) = f(a+h) + f(a+2h) + \cdots + f(b) + f(b+h),$$

and

$${}_a S_{b,h} f(x) = f(a) + f(a+h) + f(a+2h) + \cdots + f(b),$$

so

$${}_a S_{b,h} f(x+h) - {}_a S_{b,h} f(x) = f(b+h) - f(a).$$

From this last relation and (3) we get

$$f(b+h) - f(a) = \sum_{n=1}^M \frac{{}_a S_{b,h} f^{(n)}(x)}{n!} h^n + R_M,$$

and isolating the first term in the sum and dividing by  $h$  we get

$$(4) \quad {}_a S_{b,h} f'(x) = \frac{1}{h} f(b+h) - \frac{1}{h} f(a) - \sum_{n=2}^M \frac{{}_a S_{b,h} f^{(n)}(x)}{n!} h^{n-1} - \frac{1}{h} R_M.$$

Now let  $h=1$ , and let  $b \rightarrow \infty$ , then this last relation becomes (2) and the theorem is proved.  $\square$

### 3. Euler's little summation formula for infinite series and the zeta function

We can use (2) to determine recursively the values of  $\zeta(z)$  for  $z = 0, -1, -2, -3, \dots$ . Let

$$f(x) = \frac{x^{-z+1}}{-z+1}.$$

The  $n$ th derivative is

$$f^{(n)}(x) = (-1)^{n+1} \frac{z(z+1)(z+2)\cdots(z+n-2)}{x^{z+n-1}},$$

If  $\operatorname{Re}\{z\} > 1$ , then  $\lim_{x \rightarrow \infty} f(x) = f(\infty) = 0$  and we have from (2)

$$(5) \quad {}_a S_{\infty} \frac{1}{x^z} = \frac{1}{(z-1)a^{z-1}} + \sum_{n=2}^M \frac{(-1)^n z(z+1)(z+2)\cdots(z+n-2)}{n!} {}_a S_{\infty} \frac{1}{x^{z+n-1}} + R_M,$$

where

$$R_M = (-1)^M \frac{z(z+1)(z+2)\cdots(z+M-1)}{(M+1)!} \left( \frac{1}{(x_a^*)^{z+M}} + \frac{1}{(x_{a+1}^*)^{z+M}} + \frac{1}{(x_{a+2}^*)^{z+M}} + \cdots \right)$$

and  $x < x_* < x+1$ . Notice that  $\frac{1}{(x_a^*)^{z+M}} + \frac{1}{(x_{a+1}^*)^{z+M}} + \frac{1}{(x_{a+2}^*)^{z+M}} + \cdots < \zeta(z+M)$ , and thus

we have

$$(6) \quad |R_M| \leq \frac{|z(z+1)(z+2)\cdots(z+M-1)| \zeta(z+M)}{(M+1)!}.$$

Let  $a = 1$  in (5) to get

$$(7) \quad \zeta(z) = \frac{1}{z-1} + \sum_{n=2}^M \frac{(-1)^n z(z+1)(z+2)\cdots(z+n-2)}{n!} \zeta(z+n-1) + R_M.$$

While (7) was proved with the restriction  $\operatorname{Re}\{z\} > 1$ , by the theory of analytic continuation, (6) is valid for all  $z$  except  $z = 1$ .

For  $z = -N$ , with  $N$  a nonnegative integer, the remainder in (7) is zero if  $M = N + 2$ . Notice also that

$$(z+n-2)\zeta(z+n-1) = (z+n-2) \left( \frac{1}{z+n-2} + A + B(z+n-2) + C(z+n-2)^2 + \cdots \right),$$

and thus

$$\lim_{z \rightarrow -(n-2)} (z+n-2)\zeta(z+n-1) = 1.$$

Returning to (7) we see that for  $N = 0, 1, 2, \dots$ , we have

$$\begin{aligned} \lim_{z \rightarrow -N} \zeta(z) &= \frac{1}{-N-1} + \sum_{n=2}^{N+1} \frac{(-1)^n (-N)(-N+1)(-N+2) \cdots (-N+n-2)}{n!} \zeta(-N+n-1) \\ &+ \frac{(-1)^{N+2} (-N)(-N+1)(-N+2) \cdots (-1)}{(N+2)!} \lim_{z \rightarrow -N} (z+(N+2)-2) \zeta(z+(N+2)-1) \end{aligned}$$

Thus we have

$$\begin{aligned} \zeta(-N) &= \frac{-1}{N+1} - \sum_{n=2}^{N+1} \frac{(N)(N-1)(N-2) \cdots (N-n+2)}{n!} \zeta(n-N-1) \\ &+ \frac{1}{(N+1)(N+2)}, \end{aligned}$$

which simplifies to

$$(8) \quad (N+2)(N+1)\zeta(-N) = -(N+1) - \sum_{n=1}^N \binom{N+2}{n} n \zeta(1-n).$$

With  $N=0$ , we can use (8) to find  $\zeta(0) = -\frac{1}{2}$ . Next using this value of  $\zeta(0)$  with

$N=1$  in (8) we calculate  $\zeta(-1) = -\frac{1}{12}$ . Continuing in this way we find successively

$$\zeta(-2) = 0, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-4) = 0, \quad \zeta(-5) = -\frac{1}{252}, \quad \zeta(-6) = 0, \quad \dots$$

We can also use (8) to prove the general formula

$$(9) \quad \zeta(0) = B_1, \quad \text{and} \quad \zeta(-n) = -\frac{B_{n+1}}{n+1}, \quad \text{for } n=1, 2, 3, \dots$$

.We have shown that  $\zeta(0) = B_1$ . By induction, assume (9) is true for  $n=1, 2, \dots, N-1$ .

Setting  $n\zeta(1-n) = -B_n$  in (8) for  $n=2, 3, \dots, N$  we get

$$(N+2)(N+1)\zeta(-N) = -(N+1) - \binom{N+2}{1} \zeta(0) + \sum_{n=2}^N \binom{N+2}{n} B_n.$$

Adding and subtracting 1 on the right we get

$$(N+2)(N+1)\zeta(-N) = 1 - \binom{N+2}{1} \{1 + \zeta(0)\} + \sum_{n=2}^N \binom{N+2}{n} B_n,$$

which we rewrite as

$$\binom{N+2}{N+1} (N+1)\zeta(-N) = \sum_{n=0}^N \binom{N+2}{n} B_n.$$

It follows at once from (1) that  $(N+1)\zeta(-N) = -B_{N+1}$ , and (9) is proved.

### References

- [1] Edwards, H. M., *Riemann's Zeta Function*, Dover Publications, New York, 1974.
- [2] Euler, L., *Excerpts on the Euler-Maclaurin summation formula*, from *Institutiones Calculi Differentialis* (transl. David Pengelley), at [math.nmsu.edu/~davidp](http://math.nmsu.edu/~davidp), New Mexico State University, 2000.
- [3] Knopp, Konrad., *Theory and Application of Infinite Series*, Dover Publications, New York, 1990. (A translation by R. C. H. Young of the 4<sup>th</sup> German addition of 1947.)