EXACT VALUES OF THE HYPERBOLIC FUNCTIONS

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There are many special values of the trigonometric functions sine and cosine encountered in the study of trigonometry. Among the simplest are

(1) $$\sin\left(\frac{N\pi}{2}\right)=\begin{cases}0 & for\ N\ even\\ (-1)^{(N-1)/2} & for\ N\ odd,\end{cases}$$

and

(2) $$\cos\left(\frac{N\pi}{2}\right)=\begin{cases}(-1)^{N/2} & for\ N\ even\\ 0 & for\ N\ odd.\end{cases}$$

It is the purpose of this short article to show that similar relations exist for the hyperbolic functions sinh and cosh. The only exact values shown in most courses are $\sinh 0=0$ and $\cosh 0=1$. We will show that there are many more.

Suppose N is a positive integer, F_N and L_N are the Fibonacci and Lucas numbers, and $\phi=\frac{1+\sqrt{5}}{2}$ is the golden section. (The Fibonacci numbers are $F_1=1$, $F_2=1$, with the others determined by the recursion relation $F_n=F_{n-1}+F_{n-2}$, while the Lucas numbers are $L_1=1$, $L_2=3$ with the same recursion relation $L_n=L_{n-1}+L_{n-2}$.)

Notice that

$$\frac{2}{\sqrt{5}}\sinh\left(N\log\phi\right)=\frac{1}{\sqrt{5}}\left(e^{N\log\phi}-e^{-N\log\phi}\right)=\frac{1}{\sqrt{5}}\left(\phi^N-\left(\frac{1}{\phi}\right)^N\right)=F_N$$

for even *N*. (This last equality follows from Binet's formula [1, 2],

$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right)$, true for all positive *n*.) For odd *N* we have

$$2\sinh\left(N\log\phi\right) = e^{N\log\phi} - e^{-N\log\phi} = \phi^N - \left(\frac{1}{\phi}\right)^N = L_N \quad .$$

(This last equality follows from the Binet-like formula $L_n = \phi^n + \left(-\frac{1}{\phi}\right)^n$, which is true for all positive *n*.) Thus we have derived special values for the hyperbolic function sinh:

$$\text{(3)} \qquad \sinh\left(N\log\phi\right) = \begin{cases} \frac{\sqrt{5}}{2}F_N & \text{for } N \text{ even} \\ \frac{1}{2}L_N & \text{for } N \text{ odd.} \end{cases} .$$

In a similar way we can derive

$$\text{(4)} \qquad \cosh\left(N\log\phi\right) = \begin{cases} \frac{1}{2}L_N & \text{for } N \text{ even} \\ \frac{\sqrt{5}}{2}F_N & \text{for } N \text{ odd.} \end{cases} .$$

Comparing (1) and (2) with (3) and (4) we see in some ways the number $\log\phi$ acts with the hyperbolic functions as $\pi/2$ does with the trigonometric functions. Many more exact values of the hyperbolic functions of certain rational multiples of $\log\phi$ can be found by using familiar identities for these functions.

Reference

[1] Graham, R. L., Knuth, D. E., and Patashnik , O., *Concrete Mathematics*, Second Edition, Addison-Wesley, Reading, Massachusetts, 1994. ISBN 0-201-55802-5

[2] Weisstein, Eric W. *Binet's Fibonacci Number Formula*. From MathWorld--A Wolfram Web Resource.

http://mathworld.wolfram.com/BinetsFibonacciNumberFormula.html