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USING THEON'S LADDER TO FIND ROOTS OF QUADRATIC EQUATIONS

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1. Introduction

Theon of Smyrna (circa 140 A. D.) described a remarkably simple way to calculate rational approximations to $\sqrt{2}$. (See [2], [3], and [5].) It has become known as Theon's ladder and is shown below.

1	1
2	3
5	7
12	17
29	41
M	M

Each rung of the ladder contains two numbers. Call the left number on the *n* th rung x_n and the right number y_n. We see that $x_n = x_{n-1} + y_{n-1}$ and that $y_n = x_n + x_{n-1}$. So the next rung of the ladder is 29 + 41 = 70, and 70 + 29 = 99. The ratio of the two numbers on each rung, y_n / x_n gives us successively better approximations to $\sqrt{2}$.

1	1	$1/1 = 1.00000$L
2	3	$3/2 = 1.50000$L
5	7	$7/5 = 1.40000$L
12	17	$17/12 = 1.41666$L
29	41	$41/29 = 1.41379$L
70	99	$99/70 = 1.41428$L
169	239	$239/169 = 1.41420$L

Notice that the numbers are alternately above and below $\sqrt{2} = 1.41421\ldots$. The convergence of y_n / x_n to $\sqrt{2}$ is slow. From the above calculations, it appears that we gain an extra decimal digit in $\sqrt{2}$ after calculating another one or two rungs of the ladder.

In [1], Theon's ladder was generalized so that any square root could be calculated, and in [4] it was generalized to allow for the calculation of any cube, fourth, fifth, etc., root. In this paper we extend Theon's ladder so that we can find the roots of an arbitrary quadratic equation.

This paper is almost entirely precalculus mathematics. In the final section we discuss the details of the calculation of a limit, and this requires a more advanced background. If this last section is omitted, the remainder of the paper should be easily followed by precalculus students.

2. Theon's ladder for quadratic equations

Suppose we wish to find rational approximations to the roots of the quadratic equation

(1) $r^2 - br - c = 0$.

We will show that the recursion relations

(2) $x_n = (1-b)x_{n-1} + y_{n-1}$, and

(3) $y_n = cx_{n-1} + y_{n-1}$

achieve this end. (There will be restrictions on *b* and *c*, but we will find them later.)

For example, to find a root of $r^2 - r - 1 = 0$ we use $x_n = y_{n-1}$ and $y_n = x_{n-1} + y_{n-1}$. We always start with the first rung of 1 1. We get

1	1	$1/1 = 1.00000\ldots$
1	2	$2/1 = 2.00000\ldots$
2	3	$3/2 = 1.50000\ldots$
3	5	$5/3 = 1.66666\ldots$
5	8	$8/5 = 1.60000\ldots$
8	13	$13/8 = 1.62500\ldots$
13	21	$21/13 = 1.61538\ldots$

Notice that the Fibonacci numbers appear in this ladder, and the root being approximated is the golden section $\left(1+\sqrt{5}\right)/2$.

We now show why the recursion relations (2) and (3) lead to rational approximations of a root of (1). Our examination in this section is simple, but not rigorous, since we are required to assume that $\lim_{n\to\infty}\frac{y_n}{x_n}$ exists. Later, (section 3), independent of this section, we will prove that this limit exists.

Dividing (2) by (1) gives us

$$\frac{y_n}{x_n} = \frac{cx_{n-1} + y_{n-1}}{(1-b)x_{n-1} + y_{n-1}}$$

Dividing the numerator and the denominator on the right hand side by x_{n-1} we get

$$\frac{y_n}{x_n} = \frac{c + \frac{y_{n-1}}{x_{n-1}}}{(1-b) + \frac{y_{n-1}}{x_{n-1}}} \quad .$$

Assuming that the limit exists, we let $r = \lim \frac{y_n}{x_n}$. Then we have $r = \frac{c+r}{1-b+r}$, which reduces to $r^2 - br - c = 0.$ Thus we see that the ladder can be expected (at least in some cases), to gives rational approximations to one of the roots of the quadratic (1).

2. **Connection to $(1-r)^n$.**

We now show that the rungs of our ladder $x_n \quad y_n$ can be generated by powers of simple binomials

(4) $$y_n - r\,x_n = \left(1-r\right)^n .$$

We will prove (4) by induction. Notice that (4) is true when n = 1. Assume (4) is true for

$$\left(1-r\right)^{N+1} = \left(1-r\right)^N \left(1-r\right) = \left(y_N - x_N r\right)\left(1-r\right) = \left(x_N r^2 + y_N\right) - \left(x_N + y_N\right) r =$$
$$\left(x_N (br+c) + y_N\right) - \left(x_N + y_N\right) r = \left(cx_N + y_N\right) - \left((1-b)x_N + y_N\right) r.$$

But from the recursion relations (2) and (3), $x_{N+1} = (1-b)x_N + y_N$ and $y_{N+1} = cx_N + y_N$, we now have

(5) $$(1-r)^{N+1} = y_{N+1} - x_{N+1} r \ .$$

This completes our inductive proof.

3. A rigorous proof that the ladder converges to *r*

Earlier, in section 2, we gave a simple non rigorous demonstration that our extended ladder converges to r. Now we are able to give a rigorous demonstration. Suppose $b < 0$ and $0 < c$, then from (2) and (3), $x_{n+1} = (1-b)x_n + y_n$ and $y_{n+1} = cx_n + y_n$, we see that both x_n and y_n are strictly increasing series, and that both are always positive. Thus the positive root of the quadratic equation $r^2 - br - c = 0$ which is

(6) $$r = \left(\sqrt{b^2 + 4c} + b\right) / \, 2$$

is the only root that our ladder could hope to approach.

Dividing (5) by x_n we have

(7) $$\left| \frac{y_n}{x_n} - r \right| = \frac{\left|1-r\right|^n}{x_n} \ .$$

Since x_n increases to infinity, if we can show that

(8) $$|1-r| \leq 1,$$

then (7) will prove that our ladder converges. To this end, we seek restrictions on *b* and *c* that make (8) true. From (6) we have

$$|1-r| = \left| \frac{\sqrt{b^2+4c}+b}{2} - 1 \right|.$$

Thus if (8) is true we have $\left| \frac{\sqrt{b^2+4c}+b}{2} - 1 \right| \leq 1$ and so

$$-1 \leq \frac{\sqrt{b^2+4c}+b}{2} - 1 \leq 1.$$

Adding 1 to this inequality we get

$$0 \leq \frac{\sqrt{b^2+4c}+b}{2} \leq 2,$$

and so

$$\sqrt{b^2+4c} \leq 4-b.$$

Squaring and simplifying we get

(9) $$c+2b \leq 4.$$

Thus we have shown that under the restrictions

$$b<0\ ,\ 0<c \text{ and } c+2b \leq 4,$$

the inequality (8) is true and thus by (7) our ladder converges to a root of the quadratic equation. These restrictions are sufficient for convergence, but they are not necessary. The ladder does converge for a wider choice of *b* and *c*, and the reader might wish to explore this extension.

The recursion relation's (2) and (3) that generate the ladder are not the only pair that produce these numbers. We leave it to the reader to explore other possibilities.

References

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