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A SIMPLE GOEMETRIC CONSTRUCTION OF THE HARMONIC MEAN OF *n* VARIABLES

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The harmonic mean of the *n* numbers $a_1, a_2, a_3, \cdots a_n$ is defined to be the number h such that $\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots\frac{1}{a_n}\right)=\frac{1}{h}$. The harmonic mean is the reciprocal of the mean of the reciprocals of the numbers $a_1, a_2, a_3, \cdots a_n$. In this short note we show a simple way to geometrically construct the harmonic mean of *n* positive numbers and give an application to finding the resistance of *n* resistors in parallel. In [2] a construction of the harmonic mean of two numbers was given, and in [1] a geometrical interpretation of several different means was presented.

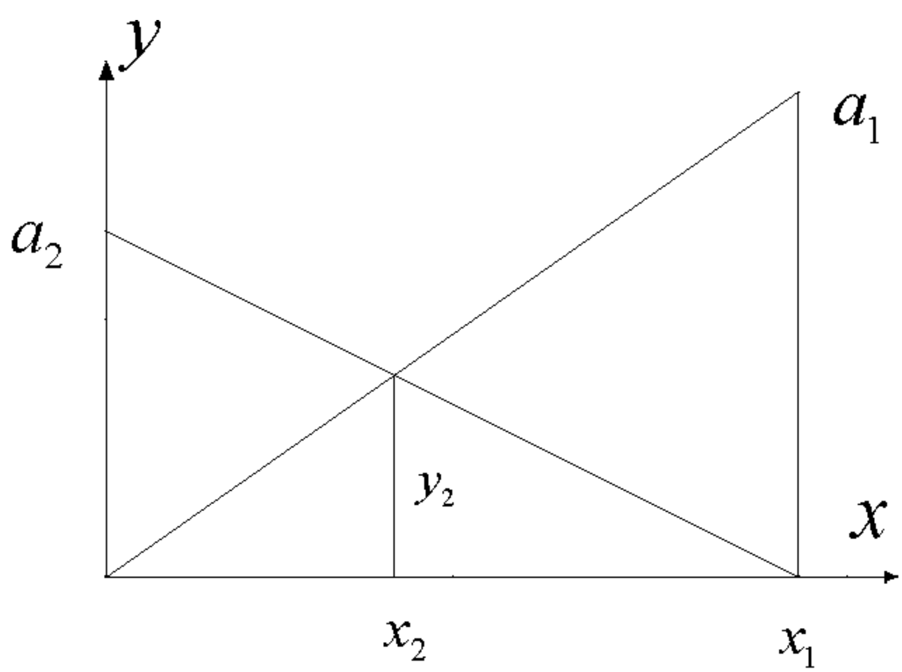

Figure 1

First we construct the line segment of length y_2 such that

(1) $$\frac{1}{a_1}+\frac{1}{a_2}=\frac{1}{y_2}.$$

Refer to Figure 1. Begin by locating the points $(0,a_2)$ and (x_1,a_1), where x_1 is any positive real number. Construct the line segment joining $(0,a_2)$ and $(x_1,0)$ and the line segment joining the origin and (x_1,a_1). Call the intersection of these two lines (x_2,y_2). Using similar triangles we see that

$$\frac{y_2}{a_1}=\frac{x_2}{x_1} \text{ and. } \frac{y_2}{a_2}=\frac{x_1-x_2}{x_1}.$$

Adding these two relations we get $\frac{y_2}{a_1}+\frac{y_2}{a_2}=1$, and so y_2 satisfies (1). Thus we have constructed the line segment whose length is the solution of (1).

Next we construct the lines segment of length y_3 that is the solution of the equation

(2) $$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}=\frac{1}{y_3} .$$

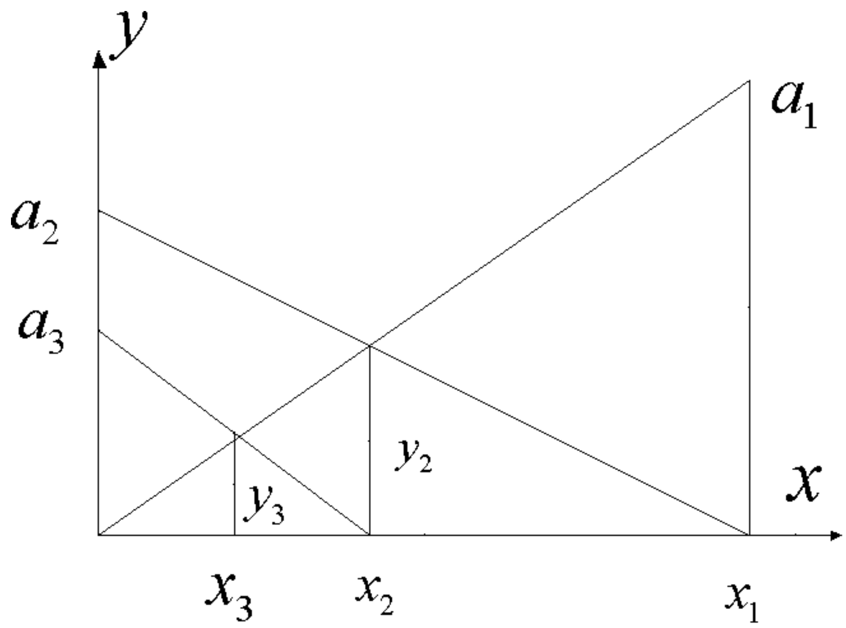

Figure 2

As before, we construct the line segment joining the points $(0, a_3)$ and $(x_2, 0)$. Let (x_3, y_3) be the intersection of this line with the previously drawn line from the origin to the point (x_1, a_1). Repeating the previous argument with the smaller triangles we get $\frac{1}{y_2} + \frac{1}{a_3} = \frac{1}{y_3}$. Replacing $\frac{1}{y_2}$ by (1) we have (2) at once. Thus we have constructed a line segment of length y_3 that solves equation (2). It is easy to see how this process can be repeated to construct a solution y_n of the general equation

(3) $$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} = \frac{1}{y_n} .$$

We will now show how to construct the harmonic mean of the set of numbers $a_1, a_2, a_3, \cdots a_n$. By the definition given at the beginning of this paper, the harmonic mean is the number $h = ny_n$. (Here y_n is the number given in (3)). A simple modification of the above construction will give us this number *h*. Recall that the number x_1 in our construction was arbitrary. Now suppose we took $x_1 = na_1$. Then we see that

$$\frac{1}{n} = \frac{a_1}{x_1} = \frac{y_n}{x_n} .$$

Thus the harmonic mean is simply the number $x_n = ny_n = h$, and we have constructed a line segment with this length.

As an application, we recall that when *n* resistors with resistances $R_1, R_2, \cdots, R_n$ are connected in parallel, the equivalent resistance *R* is given by

$$\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n} = \frac{1}{R}.$$

Thus we have given a graphic method of constructing the equivalent resistance of *n* resistors in parallel.

References

[1] Hoehn, Larry, *A geometrical interpretation of the weighted mean*, The College Mathematics Journal, 15(1984), pp. 135-139.

[2] Nataro, Dean C., *A proof without words: A construction of the harmonic mean of two positive numbers*, Mathematics and Computer Education, 35(2001), p. 233.