# A CORRECTION TO LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES*

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**Abstract.** This paper calls attention to an error in the proofs of various extensions of the "Leibniz rule" for the fractional derivative of the product of two functions published previously by the author. The error occurs at a step where integration and summation must be interchanged, and justified. The justification requires that a new restriction be added to the functions involved. The new restriction, however, is a natural one, and in no way affects applications of the Leibniz rule previously published.

**1. Introduction.** Previously the author published papers which contained proofs and applications of generalized Leibniz rules for the fractional derivative of the product of two functions, $D^\alpha u(z)v(z)$ (and more generally $D^\alpha f(z, z)$), [1], [2], [5]. While exploring further generalizations, the author discovered an error in the proofs of Leibniz rule given in [1], [2], [5]. The error occurs at a stage where summation and integration are interchanged ($\int \sum = \sum \int$). This interchange does not seem to be valid unless an additional restriction is added to the hypotheses under consideration. It is the purpose of this note to state this added restriction, and to demonstrate that the interchange ($\int \sum = \sum \int$) is then valid. Fortunately the new restriction does not affect any of the applications of Leibniz rule to infinite series expansions given in [1], [2], [3], [5].

**2. The new restriction.** The new restriction is as follows.

*Restriction.* Let the singularity (if any) of $f(z, w)$ at $z = w = 0$ be such that $|f(z, w)| \leqq M|z|^p|w|^q$ for all $z$ and $w$ considered, where $M$ is constant and $p$, $q$, and $p + q$ are in the interval $(-1, \infty)$.

This Restriction should be added to the hypotheses of Theorems 4.1 and 5.1 of [1]; to the hypothesis of Theorem 1, p. 664 of [2] (in which $f(z, w)$ is $u(z)v(w)$); to the hypothesis of Theorem 4.1 of [5]; and to Theorem 1, p. 290 of [3] (in which $f(z, w)$ is $u(h^{-1}(z))v(h^{-1}(w))$.

In [1], [2], [5] we required the behavior of $f(z, w)$ near the origin to be such that $\oint_C f(z, z)\,dz$, $\oint_C f(z, w)\,dz$, and $\oint_C f(z, w)\,dw$ vanish over any closed path $C$ through the origin. Our new Restriction is stronger. Nevertheless, a function $f(z, w)$ having a singularity of the type $z^p w^q$ would still have the same restrictions placed on $p$ and $q$ by the integrals just mentioned as it would by our new Restriction. This is the reason why the series expansions considered in our applications of Leibniz rule are not affected by the new Restriction.

**3. The corrected proof.** The extended Leibniz rule given in [5] is more general than that given in [1], [2]. Thus we show where the error in reasoning occurs in [5] and how our new Restriction corrects it. The correction needed in [1], [2] is simply a special case of the one we are about to consider.

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Our restriction is needed to show that we can integrate the infinite series (4.2) of [5] term by term over the contour $C(z)$. The reason stated in [5] for this term by term integration is incorrect, as this series is not a Fourier series as the author previously assumed. According to [4, p. 44], the infinite series (4.2) of [5] can be written as a finite sum with two remainder terms in the form

$$\frac{\Gamma(\alpha+1)f(\xi,\xi)}{2\pi i(\xi-z)^{\alpha+1}} = \sum_{n=-N}^{N}\{\text{same terms as in (4.2) of [5]}\} + R_{\varepsilon}(N) + R_{-\delta}(N),$$

where

$$R_{\varepsilon}(N) = \frac{a\Gamma(\alpha+1)\theta(\xi;z)^{\gamma+a}}{-4\pi^2(\xi-z)^{\alpha+1}}\int_{C_{\varepsilon}}\frac{\theta(t;z)^{-\gamma-1}\{\theta(\xi;z)/\theta(t;z)\}^{aN}f(\xi,t)\theta_t(t;z)\,dt}{\theta(t;z)^a-\theta(\xi;z)^a},$$

and

$$R_{-\delta}(N) = \frac{a\Gamma(\alpha+1)\theta(\xi;z)^{\gamma+a}}{-4\pi^2(\xi-z)^{\alpha+1}}\int_{C_{-\delta}}\frac{\theta(t:z)^{a-\gamma-1}\{\theta(t;z)/\theta(\xi;z)\}^{aN}f(\xi,t)\theta_t(t;z)\,dt}{\theta(\xi;z)^a-\theta(t;z)^a}.$$

The contours of integration $C_x$ ($x = \varepsilon$ and $-\delta$) are shown in Fig. 1, and we note that each consists of three parts: $C_x = C_1(x) + C_2(x) + C_3(x)$, where

$C_1(x)$ starts at $t = 0$ and continues to the point where $|\theta(t;z)| = |\theta(0;z)| + x$ along the curve defined by $\arg\theta(t;z) = \arg\theta(0;z)$,

$C_2(x) = \{t \,|\, |\theta(t;z)| = |\theta(0;z)| + x\}$,

$C_3(x) = C_1(x)$ traversed in the opposite direction.

Notice that in the notation of [5], $C(z) = C_2(0)$.

To show that we can integrate (4.2) of [5] term by term over $C(z)$ we must show that both

$$\int_{C(z)} R_{\varepsilon}(N)\,d\xi \quad\text{and}\quad \int_{C(z)} R_{-\delta}(N)\,d\xi$$

approach zero as $N$ approaches infinity. Both remainders are examined in the same way; thus we shall only examine

$$\left|\int_{C(z)} R_{-\delta}(N)\,d\xi\right| \leqq \left|\int_{C(z)}\int_{C_1(-\delta)+C_3(-\delta)}\right| + \left|\int_{C(z)}\int_{C_2(-\delta)}\right|.$$

The last term above can be made arbitrarily small by taking $N$ sufficiently large for fixed $\delta$ since

$$\left|\int_{C(z)}\int_{C_2(-\delta)}\right| \leqq \left|\frac{a\Gamma(\alpha+1)M\theta(0;z)^{\gamma}}{4\pi^2}\right| \max_{\xi\in C(z)}|(\xi-z)^{-\alpha-1}|\left|\frac{|\theta(0;z)|-\delta}{\theta(0;z)}\right|^{aN}$$
$$\cdot\int_{C(z)}\int_{C_2(-\delta)}\frac{|\theta(t;z)^{a-\gamma-1}\theta_t(t;z)|\,|\xi|^p|t|^q|dt|\,|d\xi|}{1-|(\theta(0;z)-\delta)/\theta(0;z)|^a}.$$

Thus we must show that

$$I = \left|\int_{C(z)}\int_{C_1(-\delta)+C_3(-\delta)}\right|$$

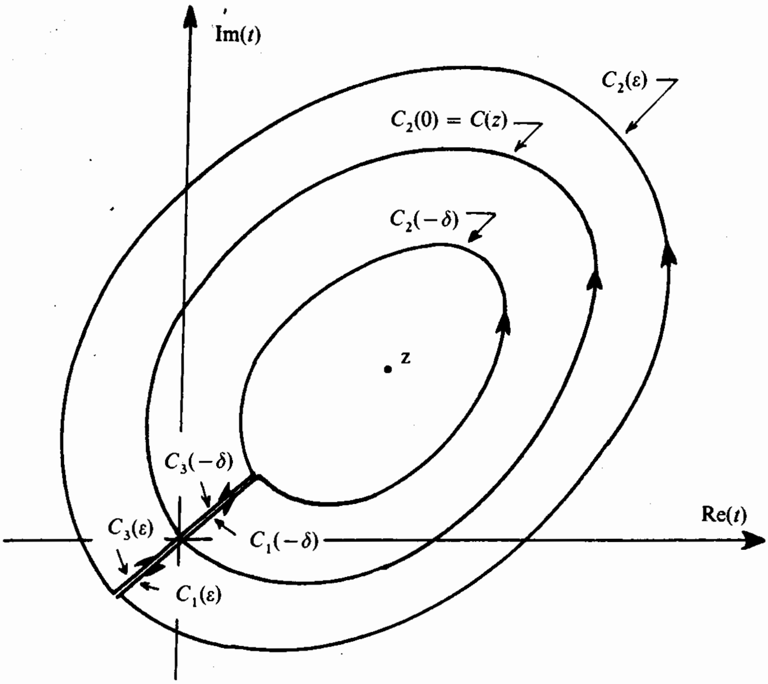


FIG. 1. *Contours of integration*

can be made arbitrarily small:

$$I \leqq \left| \frac{2aM\Gamma(\alpha+1)\theta(0;z)^{a+\gamma}}{4\pi^2} \right| \max_{\substack{t \in C_1(-\delta) \\ \xi \in C(z)}} \left| \frac{\theta(t;z)^{a-\gamma-1}\theta_t(t;z)}{(\xi - z)^{\alpha+1}} \right|$$

$$\cdot \int_{\xi \in C(z)} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt| \, |d\xi|}{|\theta(\xi;z)^a - \theta(t;z)^a|}.$$

If this last integral exists, we can make $I$ arbitrarily small by taking $\delta$ sufficiently small, that is, making the length of $C_1(-\delta)$ small. Note that $N$ does not appear in this integral.

$\theta(\xi;z)^a - \theta(t;z)^a$ has a simple zero at $\xi = t \neq z$ by the argument presented in [4, p. 43]. Thus, $\theta(\xi;z)^a - \theta(t;z)^a = G(\xi,t)(t-\xi)$, where $G(\xi,t)$ is not zero on the contours of integration. Thus we must show that

$$I' = \int_{\xi \in C(z)} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt| \, |d\xi|}{|t - \xi|}$$

exists. But

$$I' = \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{t \in C_1(-\delta)} + \int_{\substack{\xi \in C(z) \\ |\xi| \geqq \delta}} \int_{t \in C_1(-\delta)}$$

The second term on the right-hand side exists because $|t - \xi|$ does not approach zero. Thus our attention focuses finally on the critical integral,

$$I'' = \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt| \, |d\xi|}{|t - \xi|},$$

which must be shown to exist. The contours $C(z)$ and $C_1(-\delta)$ are smooth and intersect at right angles at the origin. (This is because $C_1(-\delta)$ defines constant argument and $C(z)$ constant modulus of $\theta$ at the origin.) Since the contours of integration for $I''$ are short and nearly straight line segments, we know that $\beta$ exists such that $0 < \beta \leqq |\arg(t) - \arg(\xi)| \leqq \beta + \pi/2$. Then it is clear that

$$|t - \xi| \geqq ||t| - |\xi| e^{i\beta}|,$$

and we have

$$I'' \leqq \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt| \, |d\xi|}{||t| - |\xi| e^{i\beta}|}.$$

Let $|t| = u|\xi|$. We obtain

$$I'' \leqq \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{u=0}^{\infty} \frac{u^q |\xi|^{p+q} \, du |d\xi|}{|u - e^{i\beta}|}$$

which exists. Thus we have shown that we can integrate (4.2) of [5] term by term provided we add the new Restriction.

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