AN ASYMPTOTIC APPROACH TO CONSTRUCTING THE HYPERBOLA

Revised

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There are two standard methods that are often used in defining the hyperbola.

(A) The hyperbola is the locus of all points P whose distance from a fixed point F (the focus) divided by its distance to a fixed line D (the directrix) is a constant e (the eccentricity) greater than one, i.e. $\frac{PF}{PD} = e > 1$.

(B) The hyperbola is the locus of all points P in which the difference of the distances from two fixed points (the foci F and F') is a constant, i.e. $PF - PF' = c$.

These definitions are related to the *conic sections* as described in [1].

In this short note we present another method of defining the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

by visualizing simple constructions based on the asymptote

$$y = \frac{b}{a}x . \tag{2}$$

In Figure 1 we see the asymptote (2) drawn as the line $\overline{OA}$. An auxiliary line $\overline{O'A'}$ is constructed parallel to line $\overline{OA}$ with distance a separating them. Now consider any point P on the asymptote. From P draw a line $\overline{PH}$ parallel to the x-axis and a line

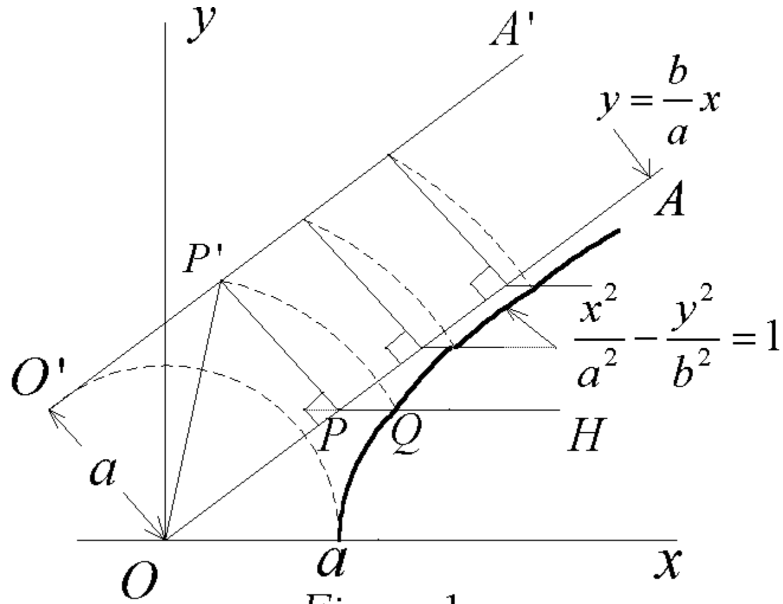

Figure 1

$\overline{PP'}$ perpendicular to $\overline{OA}$ and meeting $\overline{O'A'}$ at *P'*. Using *O* as center and radius the length *OP'*, construct an arc meeting line $\overline{PH}$ at the point *Q*. We will show that this point *Q* is on our hyperbola (1). By letting *P* move from *O* to infinity on the asymptote $\overline{OA}$, the corresponding points *Q* constructed as described will trace out the branch of our hyperbola (1) in the first quadrant. It is also evident from our construction, that for every point *Q* on the hyperbola in the first quadrant, there is a corresponding point *P* on the asymptote.

We will now demonstrate that the point *Q* is on the hyperbola (1). Let the coordinates of point *P* be (x_P, y_P), and let the coordinates of point Q be (x_Q, y_Q). We now have that

(3) $$y_P = \frac{b}{a}x_P, \text{ and } y_Q = y_P.$$

Note that *Q* is constructed so that $|\overline{OQ}| = |\overline{OP'}|$. But $|\overline{OP'}|^2 = a^2 + |\overline{OP}|^2 = a^2 + x_P^2 + y_P^2$ and $|\overline{OQ}|^2 = x_Q^2 + y_Q^2$. Hence $a^2 + x_P^2 + y_P^2 = x_Q^2 + y_Q^2$. Since $y_Q = y_P$ we have

(4) $$a^2 + x_P^2 = x_Q^2 .$$

Thus using (3) and (4) we have $\frac{x_Q^2}{a^2} - \frac{y_Q^2}{b^2} = \frac{a^2 + x_P^2}{a^2} - \frac{y_P^2}{b^2} = \frac{a^2 + x_P^2}{a^2} - \frac{b^2}{a^2}\frac{x_P^2}{b^2} = 1$. Thus we have shown that the point Q is on the hyperbola (1). It is easy to construct the remaining branches of the hyperbola in the other three quadrants by reflections in the x and y axes.

In our discussion, the x-axis was the axis of the hyperbola and the origin was the center of the hyperbola. Consider now the general case where we are given any two intersecting lines as asymptotes, the axis which is a line that bisects one of the angles of intersection, and the focus, which is a point on the axis. How do we use our method to construct the corresponding hyperbola? Call c the distance from the focus to the center. and call α the angle formed by the axis and an asymptote. Then $a = c\cos\alpha$. We can now construct a line corresponding to $O'A'$ in Figure 1 that is parallel to an asymptote and a distance a from it. Given any point P on the asymptote, the corresponding point Q on the hyperbola will be on a line PH that is parallel to the axis. The reader will have no trouble describing the remaining steps.

Reference

[1] Brannan, D. A., Esplen, M. F., and Gray, J. J., *Geometry*, Cambridge University Press, 1999, p. 5.