

### Binomials to Binomials

The College Mathematics Journal, Vol. 31, No. 3, (2000), pp. 211-212.

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The familiar binomial theorem expands  $(a + b)^n$  into a series involving  $n + 1$  terms.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k .$$

The result may then be reduced to the form of a binomial once again, in examples such as

$$\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^9 = 38 + 17\sqrt{5} \quad \text{and} \quad (2 + i)^6 = -117 + 44i .$$

When the binomial theorem is used to evaluate this last example, first seven terms are calculated, then the four real numbers are added to get -117 and finally the three

imaginary numbers are added to get  $44i$ . In general we write  $(a + b)^n = a_n + b_n$ , where

$$a_n = \sum_{k \text{ even}} \binom{n}{k} a^{n-k} b^k \quad \text{and} \quad b_n = \sum_{k \text{ odd}} \binom{n}{k} a^{n-k} b^k .$$

We are looking for a way to recursively

generate the terms of these “ $a$  and  $b$  sequences”. Note that  $(a + b)^0 = a_0 + b_0 = 1 + 0$

and  $(a + b)^1 = a_1 + b_1 = a + b$  so  $\{a_n\} = \{1, a, a_2, a_3, \dots\}$  and  $\{b_n\} = \{0, b, b_2, b_3, \dots\}$ .

We will describe our method for generating successive terms of the  $a$  and  $b$  sequences, and then show why it works. Suppose you are given the numbers  $a$  and  $b$ . Calculate the values  $C = 2a$ , and  $D = b^2 - a^2$ . Now successive values of the  $a$  and  $b$  sequences can be calculated from the recursion relations

$$(1) \quad a_n = C a_{n-1} + D a_{n-2} \quad \text{and} \quad b_n = C b_{n-1} + D b_{n-2} .$$

To justify (1) we first observe that

$$a_n = \frac{1}{2}((a+b)^n + (a-b)^n) \text{ and } b_n = \frac{1}{2}((a+b)^n - (a-b)^n).$$

This is so because  $(-b)^{\text{even}} = b^{\text{even}}$ , while  $(-b)^{\text{odd}} = -b^{\text{odd}}$ . Now

$$\begin{aligned} (a+b)^n &= (a+b)^2(a+b)^{n-2} = (a^2 + 2ab + b^2)(a+b)^{n-2} = \\ (2a^2 + 2ab + b^2 - a^2)(a+b)^{n-2} &= (2a(a+b) + (b^2 - a^2))(a+b)^{n-2} = \\ 2a(a+b)^{n-1} + (b^2 - a^2)(a+b)^{n-2}. \end{aligned}$$

In the notation of the  $a$  and  $b$  sequences, it follows that

$$\begin{aligned} a_n + b_n &= 2a(a_{n-1} + b_{n-1}) + (b^2 - a^2)(a_{n-2} + b_{n-2}) = \\ C(a_{n-1} + b_{n-1}) + D(a_{n-2} + b_{n-2}) &= (Ca_{n-1} + Da_{n-2}) + (Cb_{n-1} + Db_{n-2}). \end{aligned}$$

In exactly the same way,

$$a_n - b_n = (Ca_{n-1} + Da_{n-2}) - (Cb_{n-1} + Db_{n-2}).$$

Solving for  $a_n$  and  $b_n$  gives (1).

For example, let us evaluate  $(2+i)^6$ . Here  $a=2$  and  $b=i$ . So  $C=4$  and  $D=-5$ . Now the recursion relations (1) are

$$-5a_{n-2} + 4a_{n-1} = a_n \text{ and } -5b_{n-2} + 4b_{n-1} = b_n.$$

Since the first two terms of the  $a$  sequence are 1 and 2, the next is found to be 3 from the recursion relation. The first two terms of the  $b$  sequence are 0 and  $i$ , so the next term is  $4i$ . Continuing in this way we get further terms of the sequences:

$$\begin{array}{rcccccc} n = & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ a_n = & 1 & 2 & 3 & 2 & -7 & -38 & -117 \\ b_n = & 0 & i & 4i & 11i & 24i & 41i & 44i \end{array}$$

Thus we have all the powers of  $2+i$  up to the sixth calculated in succession and

$$(2+i)^6 = -117+44i.$$

Notice that the real and imaginary parts are calculated independently. If we only need the real part of  $(2+i)^6$  we can ignore the calculation of the  $b$  sequence completely.

For another example we show that  $\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^9 = 38 + 17\sqrt{5}$ . Here  $a = 1/2$  and

$b = \sqrt{5}/2$ . ( $a+b$  is the golden section.) So  $2a = C = 1$  and  $b^2 - a^2 = D = 1$ . The

recursion relations are particularly nice, being the same as for the Fibonacci sequence:

$$a_{n-2} + a_{n-1} = a_n \quad \text{and} \quad b_{n-2} + b_{n-1} = b_n.$$

The first two terms of the  $a$  sequence are 1 and  $1/2$ , so the next is  $3/2$ . The first two

terms of the  $b$  sequence are 0 and  $\sqrt{5}/2$  so the next is  $\sqrt{5}/2$ . The following terms

of the sequences are

$n =$	0	1	2	3	4	5	6	7	8	9
$a_n =$	1	$1/2$	$3/2$	2	$7/2$	$11/2$	9	$29/2$	$47/2$	38
$b_n =$	0	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{5}}{2}$	$\sqrt{5}$	$\frac{3\sqrt{5}}{2}$	$\frac{5\sqrt{5}}{2}$	$4\sqrt{5}$	$\frac{13\sqrt{5}}{2}$	$\frac{21\sqrt{5}}{2}$	$17\sqrt{5}$

We have easily calculated all the powers of the golden section up to the ninth.

The reader may have observed that both of our examples are of the following type: Let  $F$  be a field and let  $d \in F$  be chosen so that  $\sqrt{d} \notin F$ . If  $\alpha$  and  $\beta$  are elements of  $F$  and  $n$  is a positive integer, we want to express  $(\alpha + \beta\sqrt{d})^n$  in the form  $\alpha_n + \beta_n\sqrt{d}$ , where  $\alpha_n$  and  $\beta_n$  are in  $F$ . Our method applies in all such cases, producing the sequences  $a_n = \alpha_n$  and  $b_n = \beta_n\sqrt{d}$ .