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**AN EASY INTRODUCTION TO BIPLEX NUMBERS****Revised July 11, 2002**

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**1. INTRODUCTION**

The subject of this paper, the bplex numbers, is an unusual generalization of the familiar complex numbers. This subject is a special case of Clifford Algebras, which is normally only found in advanced textbooks at the graduate level [4, 6, 7, and 9]. We also know of a few research papers [10, 11, and 12] dealing with these numbers. There are applications of bplex numbers in modern physics to special relativity [5]. It is the purpose of this paper to show that the concept of bplex numbers and their arithmetic can be introduced to students at the precalculus level. All that is needed is familiarity with complex numbers and their arithmetic. Good students should profit by discovering for themselves many of the rich properties of these numbers.

In high school algebra courses, the complex numbers are often introduced in a simple natural way. The easy steps are:

- (a) Argue that no number familiar to the students satisfies  $i = \sqrt{-1}$  .

(b) Argue that  $i^2 = -1$ ,  $i^3 = -i$ ,  $i^4 = 1$ , ... .

(c) Introduce the quantity  $x + iy$ , where  $x$  and  $y$  are real as a new kind of number. Call it a complex number.

(d) The arithmetic of these new numbers is invented using a simple principal. Manipulate using all the familiar rules of algebra, treating  $i$  as you would any algebraic variable with the exception that whenever the quantity  $i^2$  occurs, we replace it with  $-1$ .

Using these four easy steps, the arithmetic of the complex number system becomes assessable to precalculus students.

In this paper we will introduce not one, but a family of unusual number systems. Each new number system resembles the complex numbers in that they are written in the form  $z = x + jy$  where  $x$  and  $y$  are real numbers. In these number systems  $j$  is treated as though it were a new kind of number. In place of the relation  $i^2 = -1$  we will use  $j^2 = a + bj$ , (called the *primary relation*), where  $a$  and  $b$  are fixed real constants that never change in a particular bplex number system. For example, when  $a = -1$  and  $b = 0$  we have the familiar complex number system. If we take  $a = 1$  and  $b = 0$  we get the much less familiar spacetime number system [2]. Rather than present the new bplex number systems in a rigorous definition, lemma, theorem approach, we will use the same easy and informal approach listed in the steps (a) to (d) above. In this way we invite the reader to join us on a journey of discovery into this remarkable new world of numbers. Often definitions of new terms are presented in mathematical writing as though they were pulled from a magician's hat. We will allow the mathematical expressions themselves to motivate our new definitions. In this way we will discover a

new arithmetic.

This material can be presented in precalculus courses, just after the arithmetic of complex numbers is discussed. It is also suitable for courses in complex variables as a supplementary topic. In a modern algebra course, it can be used as an example of a number system, which is not a field, but a ring with divisors of zero. Problems for student solution are given throughout the paper with selected answers in the final section.

So let us begin our journey of discovery.

## 2. POWERS OF $j$

We start our discussion by examining the powers of  $j$  in a bplex number system. Naturally we can begin with  $j^0 = 1$  and  $j^1 = j$ . For any given system, with primary relation  $j^2 = a + bj$ , where the real numbers  $a$  and  $b$  are fixed, we then have:

$$j^3 = j \cdot j^2 = j(a + bj) = aj + bj^2 = aj + b(a + bj) = ab + (a + b^2)j$$

Continuing in this manner, always reducing higher powers of  $j$  to lower ones using  $j^2 = a + bj$ , we can construct the following list:

$$j^0 = 1 + 0j$$

$$j^1 = 0 + j$$

$$j^2 = a + bj$$

$$j^3 = ab + (a + b^2)j$$

$$j^4 = a(a + b^2) + (2ab + b^3)j$$

$$j^5 = a(2ab + b^3) + (a^2 + 3ab^2 + b^4)j$$

Similarly by dividing the primary relation  $j^2 = a + bj$  by  $j$  we get  $j = aj^{-1} + b$ .

Solving for  $j^{-1}$  we get  $j^{-1} = -a^{-1}b + a^{-1}j$ . Dividing again by  $j$  we get

$$j^{-2} = -a^{-1}bj^{-1} + a^{-1} = -a^{-1}b(-a^{-1}b + a^{-1}j) + a^{-1} = a^{-2}(a + b^2) - a^{-2}bj.$$

Continuing in this way we generate the following list of negative powers of  $j$ .

$$j^{-1} = -a^{-1}b + a^{-1}j$$

$$j^{-2} = a^{-2}(a + b^2) - a^{-2}bj$$

$$j^{-3} = -a^{-3}(2ab + b^2) + a^{-3}(a + b^2)j$$

$$j^{-4} = a^{-4}(a^2 + 3ab^2 + b^3) - a^{-4}(2ab + b^2)j$$

It is convenient to define  $F_n$  and  $G_n$  by

$$(2.1) \quad j^n = F_n + G_n j$$

for all integer  $n$ , where  $F_n$  and  $G_n$  are the real and bplex part of  $j^n$  respectively. If we examine the powers of  $j$  carefully in the above lists, we can see two recursive relations emerging.

$$(2.2) \quad F_n = bF_{n-1} + aF_{n-2}$$

$$(2.3) \quad G_n = bG_{n-1} + aG_{n-2}.$$

Starting with the initial conditions:  $F_0 = G_1 = 1$  and  $F_1 = G_0 = 0$ , these relations can be used to find the powers of  $j$  very easily. We will prove (2.2) and (2.3) in the following problems.

As with complex numbers, we say that  $x + yj = X + Yj$  if and only if  $x = X$  and  $y = Y$ .

### Problems:

2.1 Let  $j^2 = 1 + 0j$ . Show that  $j^{2k} = 1$  and  $j^{2k+1} = j$  for all integer  $k$ .

2.2 Let  $j^2 = 1 + j$ . Find  $j^n$  for  $n = 0, 1, 2, 3, 4, 5$ .

2.3 Let  $j^2 = 1 + j$ . Find  $j^{-n}$  for  $n = 1, 2, 3, 4, 5$ .

2.4 Assume  $F_n$  and  $G_n$  are defined by (2.1). Prove that

$$F_n = aG_{n-1}, \text{ and } G_n = F_{n-1} + bG_{n-1}.$$

(Hint: Start with  $j^{n-1} = F_{n-1} + G_{n-1}j$ , then multiply by  $j$ .)

2.5 Prove that  $G_n = bG_{n-1} + aG_{n-2}$  is true for all  $n$ .

(Hint: Use the results of the previous problem.)

2.6 Prove that  $F_n = bF_{n-1} + aF_{n-2}$  is true for all  $n$ .

(Hint: Start with  $G_n = bG_{n-1} + aG_{n-2}$ , then use  $F_n = aG_{n-1}$ .)

2.7 Let  $\alpha$  and  $\beta$  represent the roots of the equation  $j^2 = a + bj$ , and assume that they are

distinct. Show that  $\frac{\alpha^n - \beta^n}{\alpha - \beta} = G_n$ .

Hint: First show that  $a = -\alpha\beta$  and  $b = \alpha + \beta$ . Next call  $g(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$  and show that

$g(0) = 0$  and  $g(1) = 1$ . Finally show that the recursion relation

$g(n) = bg(n-1) + ag(n-2)$  is satisfied.

2.8 Assume that  $\lim_{n \rightarrow \infty} \frac{G_{n+1}}{G_n} = \gamma$  exists. Prove that  $\gamma$  is a root of the primary relation. In

other words, prove that  $\gamma^2 - b\gamma - a = 0$ .

Hint: Start with  $G_{n+2} = bG_{n+1} + aG_n$  and divide by  $G_{n+1}$ .

### 3. THE ARITHMETIC OF BIPLEX NUMBERS

We now investigate the arithmetic of bplex numbers. We already know how to add, subtract, multiply and divide complex numbers, so we will use this experience to guide us to the same operations with bplex numbers. The following table summarizes

the results. Notice that addition and subtraction are exactly the same whether we use complex or bplex numbers. Multiplication is a bit different, and results from the fact that  $i^2 = -1$ , while  $j^2 = a + bj$ . The reader will have no difficulty verifying our result for multiplication.

<b>Complex vs. Bplex Arithmetic</b>		
Notation	<b>Complex numbers</b> $z = x + yi, Z = X + Yi$ $i^2 = -1$	<b>Bplex numbers</b> $z = x + yj, Z = X + Yj$ $j^2 = a + bj$ (primary relation)
Addition	$z + Z = (x + X) + (y + Y)i$	$z + Z = (x + X) + (y + Y)j$
Subtraction	$z - Z = (x - X) + (y - Y)i$	$z - Z = (x - X) + (y - Y)j$
Multiplication	$zZ = (x + yi)(X + Yi)$ $= (xX - yY) + (xY + yX)i$	$zZ = (x + yj)(X + Yj)$ $= (xX + ayY) + (xY + yX + byY)j$
Conjugate	$\bar{z} = \overline{x + yi} = x - yi$ with $z\bar{z} = x^2 + y^2$	$\bar{z} = \overline{x + yj} = (x + by) - yj$ , with $z\bar{z} = x^2 + bxy - ay^2$
Division	$\frac{Z}{z} = \frac{Z\bar{z}}{z\bar{z}} = \frac{X + Yi}{x + yi} \cdot \frac{x - yi}{x - yi}$ $= \frac{xX + yY}{x^2 + y^2} + \frac{xY - yX}{x^2 + y^2}i$	$\frac{Z}{z} = \frac{Z\bar{z}}{z\bar{z}} = \frac{X + Yj}{x + yj} \cdot \frac{(x + by) - yj}{(x + by) - yj}$ $= \frac{xX - ayY + byX}{x^2 + bxy - ay^2} + \frac{xY - yX}{x^2 + bxy - ay^2}j$

In complex arithmetic we introduce the complex conjugate of the number

$z = x + yi$  and call it  $\bar{z} = x - yi$ . The product  $z\bar{z} = x^2 + y^2$  is a real number. What is the

*biplex conjugate* of the number  $z = x + yj$ ? Suppose we write  $\bar{z} = X + Yj$ , then we would expect

$$z\bar{z} = (x + yj)(X + Yj) = xX + ayY + (xY + yX + byY)j$$

to be a real number. This means that the  $j$  part must be zero.

$$xY + yX + byY = yX + (x + by)Y = 0.$$

We see at once that

$$\frac{Y}{X} = \frac{-y}{x + by}.$$

The simplest solution is  $X = x + by$  and  $Y = -y$ . So our definition of the *biplex conjugate* of  $z = x + jy$  is

$$\bar{z} = X + Yj = (x + by) - yj.$$

This gives us  $z\bar{z} = x^2 + bxy - ay^2$ . (Notice that when  $a = -1$  and  $b = 0$ , then  $j = i$  and these reduce to the familiar expressions for complex numbers.)

We can now calculate division of two biplex numbers in the way familiar from complex arithmetic.

$$(3.1) \quad \frac{Z}{z} = \frac{Z\bar{z}}{z\bar{z}} = \frac{X + Yj}{x + yj} \cdot \frac{(x + by) - yj}{(x + by) - yj} = \frac{xX - ayY + byX}{x^2 + bxy - ay^2} + \frac{xY - yX}{x^2 + bxy - ay^2} j.$$

The division of two biplex numbers is *not* always possible. The division fails if  $x^2 + bxy - ay^2 = 0$ . We will examine this problem with division in the next section.

**Problems:**

3.1 Evaluate  $(5 + 3j) + (6 - 5j)$ , where the primary relation is  $j^2 = 5 - 6j$ .

3.2 Evaluate  $(7 - 12j) - (15 + 7j)$ , where the primary relation is  $j^2 = 4 - 5j$ .

3.3 Evaluate  $(5 + 3j)(9 - 12j)$  where  $j^2 = 3 + 7j$ .

3.4 Evaluate  $\frac{6-7j}{2+3j}$  where  $j^2 = 1 + 2j$ .

3.5 Evaluate  $\frac{3-5j}{4+j}$  where  $j^2 = 4 - 3j$ .

3.6 Verify the multiplication of two bplex numbers from the table.

3.7 Let  $z = x + jy$  be a bplex number defined within a system with primary relation  $j^2 = a + bj$ . Let the multiplicative inverse of  $z$  be  $z^{-1} = A + Bj$ . Find  $A$  and  $B$  in terms of  $x, y, a$ , and  $b$  without using the bplex conjugate. Start with  $1 = zz^{-1} =$

$(x + jy)(A + jB)$ , and show that  $A = \frac{x + by}{x^2 + bxy - ay^2}$ , and  $B = \frac{-y}{x^2 + bxy - ay^2}$ , provided  $(x^2 + bxy - ay^2) \neq 0$ .

3.8 Verify the division of two bplex numbers using the result of the previous problem.

(Hint:  $Z/z = Zz^{-1}$ .)

#### 4. THE THREE TYPES OF BIPLEX NUMBERS

In the previous section we saw that the inverse of the bplex number  $z = x + jy$  in a system in which the primary relation is  $j^2 = a + bj$  will not exist if the expression appearing in the denominator of (3.1) is zero. That expression is

$$(4.1) \quad x^2 + bxy - ay^2 = 0.$$

Let  $\alpha$  and  $\beta$  be the roots of our primary system relation  $j^2 = a + bj$ . This means that

$$j^2 - bj - a = (j - \alpha)(j - \beta) = 0.$$

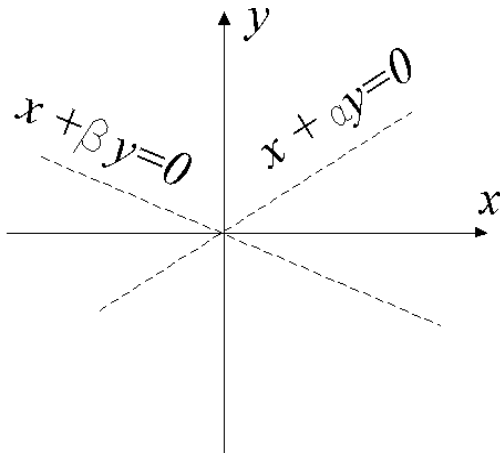
It is clear that  $\alpha + \beta = b$  and  $a = -\alpha\beta$ . We can now rewrite (4.1) as

$$(4.2) \quad x^2 + (\alpha + \beta)xy + \alpha\beta y^2 = (x + \alpha y)(x + \beta y) = 0.$$

It follows that the inverse,  $z^{-1}$ , does not exist whenever  $\alpha$  and  $\beta$  are real and the following equations are satisfied:

$$(4.3) \quad y = -\frac{1}{\alpha}x \text{ or } y = -\frac{1}{\beta}x.$$

We can represent our bplex numbers  $z = x + yj$  geometrically as points in an  $x, y$  plane just as we do complex numbers. In Figure 1 we see the two lines (4.3) graphed in this bplex number plain. (Here we assume that  $\alpha \neq \beta$  and that both are real numbers.) If our bplex number  $z$  falls on one of these lines, its inverse does not exist.



**Figure 1: The two lines on which inverses of  $z$  fail to exist.**

This discussion leads us to distinguish three types of bplex numbers:

Type I: *The roots  $\alpha$  and  $\beta$  of the primary relation  $j^2 = a + bj$  are complex numbers.*

In this case, inverses always exist and the bplex number system is a field.

Type II: *The roots  $\alpha$  and  $\beta$  of the primary relation  $j^2 = a + bj$  are real and distinct.*

In this case there are two lines (4.3) in the bplex plane on which inverses fail to exist. In courses in modern algebra, such numbers are called *divisors of zero*. This bplex number system is a ring, but not a field.

Type III: *The roots  $\alpha$  and  $\beta$  of the primary relation  $j^2 = a + bj$  are real and equal.*

In this case there is only one line (4.3) in the bplex plane on which inverses fail to exist. Again, such numbers are called *divisors of zero*, and this bplex number system is a ring and not a field.

In elementary algebra we learn that the relation  $\alpha x = \alpha y$ , ( $\alpha \neq 0$ ), implies that  $x = y$ . This is no longer true with bplex numbers of type II and III if  $\alpha$  is a divisor of zero.

### Problems

- 4.1 Suppose the primary relation is  $j^2 = 1$ . Show that the lines  $y = \pm x$  define the numbers  $z = x + yj$  in the bplex plane where inverses fail to exist. This is an important system of numbers called *spacetime numbers* or *hyperbolic numbers*.
- 4.2 In each of the following a primary relation is given. Decide if the associated bplex number system is of type I, II or III. (a)  $j^2 = -1$ , (b)  $j^2 = 1$ , (c)  $j^2 = 1 + j$ , (d)  $j^2 = -1 + j$ , (e)  $j^2 = -1 - 2j$ .
- 4.3 Show that a bplex number system with primary relation  $j^2 = a + bj$  is (1) of type I if  $4a + b^2 < 0$ , (2) of type II if  $4a + b^2 > 0$ , and (3) of type III if  $4a + b^2 = 0$ .

## 5. FINAL COMMENTS

The reader who wishes to explore this subject further could begin with the expository papers [2] and [17]. These are concerned with details of the *spacetime* or *hyperbolic numbers* in which the primary relation is  $j^2 = 1$ . The papers and letters in the American Journal of Physics [1, 5, 13, 14, 15, and 16] could be explored next. The papers by Majernik [10, 11, 12, 13] are more advanced, but deal closely with our subject.

The subject of bplex numbers is a special case of *Clifford algebras*. The pioneering paper by Clifford, in 1878, seems to be [3] and is conveniently available from JSTOR on the web. Graduate level textbooks on Clifford algebras are [4, 6, 7, and 9]. An amusing commentary on this general enterprise is found in [8]. Our bplex numbers are Clifford algebras of dimension one. Those of higher dimension are important in physics.

Finally we mention that there appears to be no generally accepted terminology associated with our subject. The terms *bplex numbers* and *primary relation* used in this paper were invented by the authors. All other terms we found used in the literature seemed to us to be inadequate. For example in [13] we find our bplex numbers called *binary numbers*, and in [10] we find them called *multicomponent numbers*.

## 6. ANSWERS TO SELECTED PROBLEMS

$$\mathbf{2.2} \quad j^0 = 1, j^1 = j, j^2 = 1 + j, j^3 = 1 + 2j, j^4 = 2 + 3j, j^5 = 3 + 5j.$$

$$\mathbf{2.3} \quad j^{-1} = -1 + j, j^{-2} = 2 - j, j^{-3} = -3 + 2j, j^{-4} = 5 - 3j, j^{-5} = -8 + 5j.$$

3.1  $11 - 2j$ . 3.2  $-8 - 19j$ . 3.3  $-63 - 285j$ . 3.4  $\frac{69}{7} - \frac{32}{7}j$ . 3.5 Does not exist.

4.2 (a) Type I, (b) Type II, (c) Type II, (d) Type I, and (e) Type III.

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