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A PROOF OF THE CONTINUED FRACTION EXPANSION OF $e^{1/M}$

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1. INTRODUCTION. This paper gives another proof for the remarkable simple continued fraction

$$e^{1/M}=1+\cfrac{1}{M-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3M-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5M-1+\cfrac{1}{1+\cdots}}}}}}}}.$$

Here M is any positive number. We use the notation $x=[a_0;a_1,a_2,a_3,\cdots]$ for the continued fraction

$$x=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}.$$

The i th convergent of $x=[a_0;a_1,a_2,a_3,\cdots]$ is the finite continued fraction $p_i/q_i=\left[a_0;a_1,\cdots,a_i\right]$. (If the denominator of a convergent is zero, we can simply skip it in the sequence of convergents.) It is easy to see that $p_0/q_0=a_0/1$, and $p_1/q_1=\left(a_0a_1+1\right)/a_1$. It is then natural to define $p_0=a_0$, $q_0=1$, $p_1=a_0a_1=1$ and $q_1=a_1$. To find the remaining convergents we use the following theorem (see Hardy and Wright [4, Theorem. 149, page 130]):

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Theorem 1: *For* $n = 2, 3, 4, \ldots$ *it is true that* $p_n = a_n p_{n-1} + p_{n-2}$ *and*

$q_n = a_n q_{n-1} + q_{n-2}$.

Olds [6] gave an expository proof of the continued fraction

$$\frac{e^{2/k}+1}{e^{2/k}-1} = [k; 3k, 5k, 7k, \ldots].$$

He then showed that for $k = 2$, it could be algebraically manipulated into $e = [2; \overline{1, 2n, 1}]_{n=1}^{\infty}$. (Here the bar notation means $e = [2;\ 1,2,1,\ 1,4,1,\ 1,6,1,\cdots]$.) He also indicated that his result could be converted to a simple continued fraction for $e^{2/k}$. The work of Olds is based on the earlier work of Hermite [5]. Cohn [1] streamlined Olds's proof into a short presentation. It is the purpose of this paper to extend Cohn's argument to the more general case of $e^{1/M}$. We note that Euler [2] , [3] found $2/(e-1) = [1; \overline{4n+2}]_{n=1}^{\infty}$. Our proof differs from other proofs we have seen in that we obtain the result directly, without having to convert some other continued fraction. Perhaps the most remarkable feature of the proof is that we are able to express the error ε_k, which is given by $\varepsilon_k = p_k / q_k - e^{1/M}$, exactly in terms of integrals. (Other proofs listed in the references also share this feature.)

2. THE CONTINUED FRACTION FOR $e^{1/M}$. We now prove that

$$e^{1/M} = [1;\ M-1,\ 1,1,3M-1,1,1,5M-1,1,1,7M-1,1,1,9M-1,\ldots].$$

For the remainder of this paper p_n and q_n refer to the continued fraction for $e^{1/M}$. From Theorem 1, we obtain

$$p_{3n} = p_{3n-1} + p_{3n-2}, \qquad q_{3n} = q_{3n-1} + q_{3n-2}; \tag{1}$$

$$p_{3n+1} = \big((2n+1)M-1\big) p_{3n} + p_{3n-1}, \qquad q_{3n+1} = \big((2n+1)M-1\big) q_{3n} + q_{3n-1}; \tag{2}$$

$$p_{3n+2} = p_{3n+1} + p_{3n}, \qquad q_{3n+2} = q_{3n+1} + q_{3n} \tag{3}$$

for $n = 0, 1, 2, 3, \ldots$. Let

$$A_n = \int_0^1 \frac{x^n (x-1)^n}{n! M^{n+1}} e^{x/M} dx,$$

$$B_n = \int_0^1 \frac{x^{n+1} (x-1)^n}{n! M^{n+1}} e^{x/M} dx,$$

and

$$C_n = \int_0^1 \frac{x^n (x-1)^{n+1}}{n! M^{n+1}} e^{x/M} dx.$$

We will show that for $n = 0, 1, 2, 3, \ldots$,

$$\frac{p_{3n}}{q_{3n}} - e^{1/M} = -\frac{A_n}{q_{3n}}, \tag{4}$$

$$\frac{p_{3n+1}}{q_{3n+1}} - e^{1/M} = \frac{B_n}{q_{3n+1}}, \tag{5}$$

and

$$\frac{p_{3n+2}}{q_{3n+2}} - e^{1/M} = \frac{C_n}{q_{3n+2}}. \tag{6}$$

It is not hard to see that as $n \to \infty$ the right-hand sides of (4), (5), and (6) all approach zero. Thus once we prove (4) - (6) (Theorem 3), we will have shown that the continued fraction expansion converges to the number $e^{1/M}$.

We begin with a theorem relating the three integrals.

Theorem 2. *For $n = 0, 1, 2, 3, \ldots$, the following are true:*

(a) $A_{n+1} = -B_n - C_n$,

(b) $B_{n+1} = -\left((2n+1)M - 1\right) A_{n+1} + C_n$, *and*

(c) $C_n = B_n - A_n$.

Proof. It is easy to see that (c) is true by expanding $(x-1)^{n+1} = x(x-1)^n - (x-1)^n$ in the integral defining C_n. Next notice that

$$\frac{d}{dx}\left(\frac{x^n(x-1)^n}{n!M^{n+1}}\,e^{x/M}\right) = \frac{x^{n-1}(x-1)^n}{M(n-1)!M^n}\,e^{x/M} + \frac{x^n(x-1)^{n-1}}{M(n-1)!M^n}\,e^{x/M} + \frac{x^n(x-1)^n}{Mn!M^{n+1}}\,e^{x/M}.$$

Integrating from 0 to 1 we get $0 = C_{n-1}/M + B_{n-1}/M + A_n/M$, and thus (a) is proved.

Finally, we start with

$$\frac{d}{dx}\left(\frac{x^n(x-1)^{n+1}}{n!M^{n+1}}\,e^{x/M}\right) = \frac{x^{n-1}(x-1)^{n+1}}{(n-1)!M^{n+1}}\,e^{x/M} + \frac{(n+1)x^n(x-1)^n}{n!M^{n+1}}\,e^{x/M} + \frac{x^n(x-1)^{n+1}}{Mn!M^{n+1}}\,e^{x/M},$$

which after some manipulation reduces to

$$\frac{d}{dx}\left(\frac{x^n(x-1)^{n+1}}{n!M^{n+1}}\,e^{x/M}\right) =$$

$$\frac{\big((2n+1)M-1\big)x^n(x-1)^n}{M(n!)M^{n+1}}\,e^{x/M} - \frac{x^{n-1}(x-1)^n}{M(n-1)!M^n}\,e^{x/M} + \frac{x^{n+1}(x-1)^n}{Mn!M^{n+1}}\,e^{x/M}.$$

Integrating from 0 to 1 we get

$$0 = \left(\frac{(2n+1)M-1}{M}\right)A_n - \frac{C_{n-1}}{M} + \frac{B_n}{M},$$

which proves (c).

Theorem 3. *Equations* (4), (5), and (6) *are true for* $n = 0, 1, 2, 3, \ldots$.

Proof. Our proof proceeds by induction on *n*. We leave it to the reader to show that (4), (5), and (6) are true when $n = 0$. We assume that (4) - (6) are satisfied for $n = N$.

We begin by proving (4) for $n = N+1$. Using (1) we see that

$$p_{3(N+1)} - q_{3(N+1)}e^{1/M} = \left(p_{3N+2} - q_{3N+2}e^{1/M}\right) + \left(p_{3N+1} - q_{3N+1}e^{1/M}\right).$$

By the induction hypothesis in combination with (5) and (6), we get

$p_{3(N+1)} - q_{3(N+1)} e^{1/M} = C_N + B_N$. Invoking Theorem 2(a) we arrive at (4).

Next we establish (5) for $n = N+1$. From (2) we infer that

$$p_{3(N+1)+1} - q_{3(N+1)+1} e^{1/M} = \left((2N+3)M - 1\right)\left(p_{3(N+1)} - q_{3(N+1)} e^{1/M}\right) + \left(p_{3N+2} - q_{3N+2} e^{1/M}\right).$$

Appealing to the identity $p_{3(N+1)} - q_{3(N+1)} e^{1/M} = -A_{N+1}$, (6), Theorem (2b), and the induction hypothesis, we see that (5) is true.

Finally, we prove (6) for $n = N+1$. It follows from (3) that

$$p_{3(N+1)+2} - q_{3(N+1)+2} e^{1/M} = \left(p_{(3N+1)+1} - q_{(3N+1)+1} e^{1/M}\right) + \left(p_{(3N+1)} - q_{(3N+1)} e^{1/M}\right).$$

Using $p_{(3N+1)+1} - q_{(3N+1)+1} e^{1/M} = B_{N+1}$ and Theorem (2c), we see at once that (6) holds. This completes the proof of the Theorem 3. □

In [1] Cohn suggests the motivation that may have led Hermite to examine integrals of the form $\int_0^1 x^n (x-1)^m e^x \, dx$ when he examined the continued fraction for *e*.

References

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