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AN UNUSUAL VIEW OF THE ELLIPSE

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1. The astronomer's ellipse

There are two common ways in which students are introduced to the ellipse.

Definition 1: Given two fixed points F_1 and F_2, called the focii of the ellipse. The ellipse is the locus of all points P such the sum of the distances from the focii to the point is a constant , ($F_1P + F_2P = c$).

Definition 2: Given a fixed point F and a fixed line L. (We call F the focus and the line L the directrix.) The ellipse is the locus of all points P such that the ratio FP/PL = e is a constant less than 1.

It is the purpose of this short note to explain a third definition which is familiar to astronomers and workers in celestial mechanics, but which is not usually given in undergraduate text books on mathematics. Look at Figure 1. We see a smaller circle with radius b and a larger circle with radius a. Both circles have the common center O. The radial line segment OBA makes angle E with the positive x axis. This radial line segment intersects the smaller circle at B and the larger circle at A. (We use the notation E rather than θ because this is common in astronomy.) We can now give our new definition.

Definition 3: In Figure 1, construct a horizontal line from B and a vertical line from A. These two lines intersect at point P. The ellipse is the locus of all points P generated in this manner as the angle E varies from 0 to 2π.

It is easy to see that the point P generates an ellipse. Notice that the coordinates of P are $x = a\cos E$ and $y = b\sin E$. From these parametric equations we see at once that $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is the most familar equation for the ellipse.

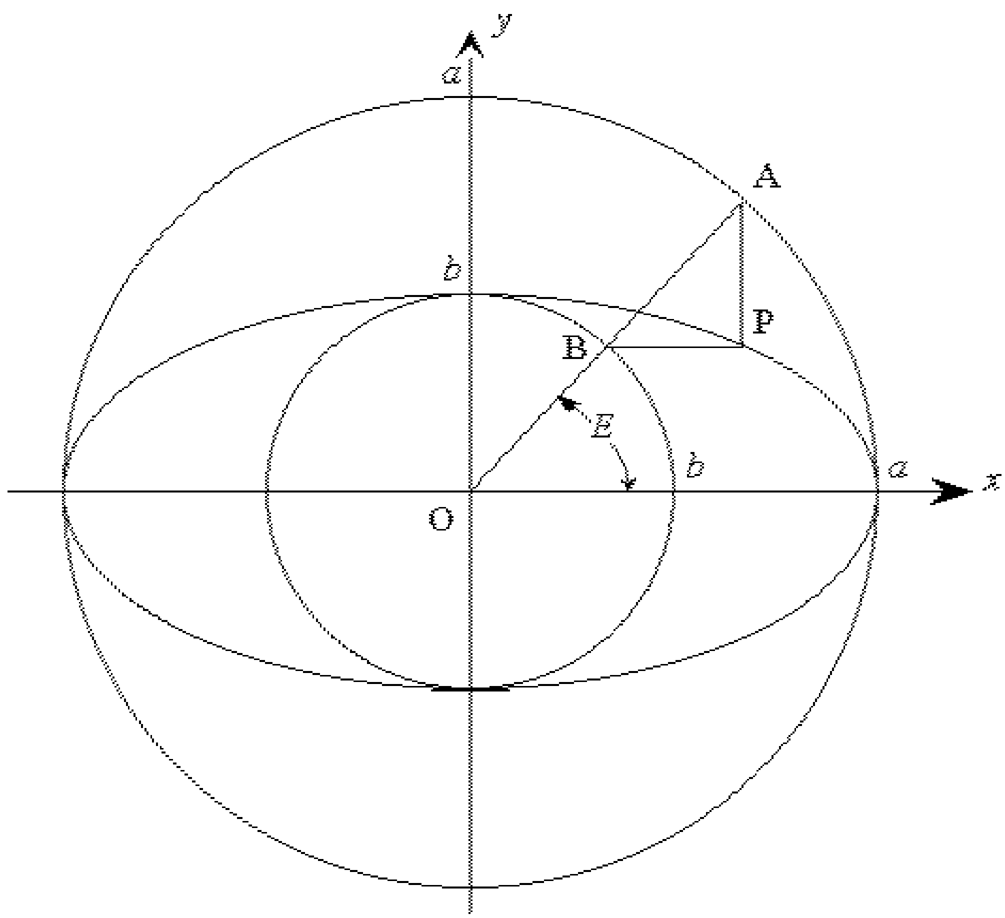

Figure 1

Astronomers call the angle *E* the *eccentric anomaly.* The term *anomaly* usually means *irregularity*. Astronomers have used *anomaly* instead of *angle* for hundreds of years when refering to angles that describe planetary locations. The term originates from the fact that the predicted location of a planet often showed small deviations from the observed data.

2. A little astronomy

Kepler's first law of planetary motion states that planets travel about the sun in elliptical orbits with the sun located at one focus. His second law states that the radius vector from the sun to the planet sweeps out equal areas in equal times.

In the special case that the orbit of the planet is a circle of radius *a,* (with the sun at the origin), the equations describing the motion are

(1) $x = a\cos E,\ \ y = a\sin E$,

(2) $\omega t = E$.

In the case of elliptical orbits we have

(3) $x = a\cos E - ae,\ \ y = b\sin E$

(4) $\omega t = E - e\sin E$.

See [3] for detailed derivations of equations (1) through (4). (The term -*ae* in (3) shifts the center of the ellipse to the left so that the focus is at the origin.) Relation (4) is called *Kepler's equation.* It is derived from Kepler's second law [2]. Here ω is the mean angular velocity of the planet, e is the eccentricity of the elliptical orbit and *E* is the eccentric anomaly describing the location of the planet at time *t.* We call ωt the *mean anomaly.*

Notice that, in the circular case, we can combine (1) and (2) to get simple parametric equations in terms of time

(5) $x = a\cos\omega t,\ \ y = a\sin\omega t$.

In the elliptical case, we cannot analytically combine (3) and (4) as we did in the circular case. This is because Kepler's equation (4) is transcendental, and we cannot solve for E in terms of t using convenient elementary functions.

The eccentric anomaly E is important in astronomy because it helps us find the position of the planet on the elliptical orbit when the time t is specified. Given t, the eccentric anomaly E is calculated using Kepler's equation (4). Then the position of the planet can be found using (3).

For over 300 years, hundreds of papers have been written to give various methods of "solving" Kepler's equation. The book by Colwell [2] traces this remarkable history.

References

[1] Colwell, P. *Solving Kepler's Equation Over Three Centuries*, William-Bell, Inc., Richmond, Virginia, 1993.

[2] Moulton, F. R., *An Introduction to Celestial Mechanics*, 2nd ed., Dover Pub., New York, N.Y., 1970, p. 160.

[3] Osler, T. J., *An unusual approach to Kepler's first law*, American Journal of Physics, (to appear).