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Fun with 0.999...

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Thomas J. Osler
Mathematics Department
Rowan University
Glassboro NJ 08028
osler@rowan.edu

1. Introduction

For many years I have used the infinite decimal $0.999\dots$ as an entertaining and informative way to begin the discussion of such difficult concepts as infinite series, sequences, and limits. The number $0.999\dots$ looks rather innocent to most students, and the fact that the number is exactly one is very non intuitive. It is the purpose of this short note to show how this number can be used as an introduction to the difficult infinite concepts of mathematical analysis. I especially enjoy using it on the first day of our undergraduate Real Analysis course, but I use it in Calculus II also when we discuss infinite series.

2. The quiz

I begin the discussion by announcing a surprise quiz. I tell them not to worry as it involves nothing more than elementary school decimal numbers and fractions. They take out a piece of paper and answer the following questions which I write on the blackboard:

$$x = 0.999\dots$$

1. What is the meaning of the three dots?

2. Which of the following are true?

(a) $x = 1$

(b) $x < 1$

(c) x is the largest number less than one.

I then collect the papers and ask for an answer to question 1. Everyone gets this correct, and it insures that the students are aware that the nines continue without end. I then ask how many gave (a) as an answer to question 2. In a class of thirty mathematics majors at any level I usually get one, two or three hands raised. Almost all the class agrees with choices (b) and (c).

Now the fun begins! I tell them that mathematics is not a democracy and the correct answer is $x = 1$. I am sure at this time many think I am joking, as I find it difficult to restrain a broad smile. I then proceed to give them not one, but the following five short arguments that prove my claim!

3. The arguments

At this time my students are eager to see what I am up to, and I begin with my simplest, and perhaps best argument.

Argument 1: Division by three.

Start with the defining expression $x = 0.999\dots$, and divide both sides by three.

We get

$$\begin{aligned} \frac{x}{3} &= \frac{0.999\dots}{3} \\ &= 0.333\dots \end{aligned}$$

Since everyone knows that

$$0.333\dots = \frac{1}{3},$$

we have

$$\frac{x}{3} = \frac{1}{3}.$$

Multiply both sides by 3 to get $x = 1$.

At this point I see many amazed faces. No doubt some are wondering what trick, or possibly incorrect step was performed.

Argument 2: Averaging

If $x < 1$ then the average of x and 1 should be between x and 1. But the average is

$$\begin{aligned} \frac{1+x}{2} &= \frac{1.999\dots}{2} \\ &= 0.999\dots \end{aligned}$$

Thus we see that $(1+x)/2 = x$ from which we conclude that $x = 1$.

Argument 3: Find the fraction

Since $x = 0.999\dots$ is a repeating decimal, there is a common fraction which it equals. This is a good opportunity to review how such fractions are found. We first multiply x by 10 and then subtract x .

$$\begin{aligned}
 10x &= 9.999\dots \\
 -x &= -0.999\dots \\
 9x &= 9 \\
 x &= 1
 \end{aligned}$$

Argument 4: The geometric series

This argument provides an excuse to review the most important of all series, the geometric series. We begin by writing

$$\begin{aligned}
 (1) \quad 0.999\dots &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots \\
 &= 9\left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots\right)
 \end{aligned}$$

The sum of the geometric series is (for $-1 < y < 1$)

$$\frac{y}{1-y} = y + y^2 + y^3 + \dots.$$

We can replace y by $1/10$ to get

$$\frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots.$$

Substituting the geometric series in (1) by the fraction $1/9$ as calculated above we get

$$0.999\dots = 9(1/9) = 1.$$

Argument 5: Another average

We conclude with another averaging argument. If $x < 1$ then it must be the largest number less than one since there is no way to write a bigger number as a decimal.

Now consider the average of x and 1. Since the average is between x and 1 it must be bigger than x . This contradicts the argument that x is the largest number less than 1.

At times a discussion like the following emerges:

Professor: How can you write the decimal expansion of a number greater than 0.999... but less than one?

Student: Put another nine after the *last nine* in 0.999... .

Professor: Where is this last nine? Is in the hundredth position, or in the hundred billionth position, or where?

Student: In the last position!

Professor: In everyday calculation when you write a decimal number, there will be a last digit that you write. But in mathematics we can imagine nines without end. We are now in the world of the infinite which is different from our finite experience. To refer to the *last nine* in this context is to make a meaningless statement. In this course you will learn to recognize correct statements about the infinite.

This completes the arguments.

4. Two ways to write terminating decimals

Before ending the hour, I believe it is important to show the class that certain numbers have two different decimal expansion. This is easy to do after the above arguments that the number one can be written as 1.0 and as 0.999... . First I ask the class, “What other numbers could be written in two ways as decimals?” A discussion usually emerges that every integer, except zero, has this property. For example

$$2 = 1.999\dots, 125 = 124.999\dots, -4 = -3.999\dots$$

Finally we observe that every terminating decimal has this property like

$$0.5 = 0.4999\dots, 13.25 = 13.24999\dots, -123.456 = -123.455999\dots$$

This ability to write certain rational numbers in two ways as decimal expansions seems to be at the heart of students reluctance to accept the above arguments. Finally I explain that the real number system is a fundamental tool. There is no easier way to visualize it than through the decimal representation. If the Pythagoreans had known of decimal numbers, the irrationality of $\sqrt{2}$ would not have disturbed them, and the history of mathematics would have taken a different road.