

Exercise 7. Suppose that $t_0 = 0$ and that

$$A(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Compute $B(t) = \int_0^t A(s) ds$, and show that A and B commute.

Exercise 8. Suppose A is the coefficient matrix of the companion equation $Y' = AY$ associated with the n th order differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

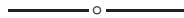
That is,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{pmatrix}.$$

Compute $B(t) = \int_0^t A(s) ds$, and show that A and B commute if and only if all the coefficient functions $p_i(t)$, $i = 1, 2, \dots, n$, are constants.

References

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Extending Theon’s Ladder to Any Square Root

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Introduction. Little is known of the life of Theon of Smyrna (circa 140 AD). At this time in the history of mathematics, there was a tendency to de-emphasize demonstrative and deductive methods in favor of practical mathematics. An excellent example of this is known as Theon’s ladder, which describes a remarkably simple way to calculate rational approximations to $\sqrt{2}$. (See [2], [3], and [5].)

1	1
2	3
5	7
12	17
29	41
⋮	⋮

Each rung of the ladder contains two numbers. Call the left number on the n th rung x_n and the right number y_n . We will use the notation $[x_n, y_n]$ to denote the n th rung of Theon's ladder. We see that $x_n = x_{n-1} + y_{n-1}$ and that $y_n = x_n + x_{n-1}$. So the next rung of the ladder is $[70, 99]$ because $29 + 41 = 70$, and $70 + 29 = 99$. The ratios of the two numbers on each rung give us successively better approximations to $\sqrt{2}$:

1	1	$1/1 = 1.00000 \dots$
2	3	$3/2 = 1.50000 \dots$
5	7	$7/5 = 1.40000 \dots$
12	17	$17/12 = 1.41666 \dots$
29	41	$41/29 = 1.41379 \dots$
70	99	$99/70 = 1.41428 \dots$
169	239	$239/169 = 1.41420 \dots$

Notice that the numbers are alternately above and below $\sqrt{2} = 1.41421 \dots$. The convergence of y_n/x_n to $\sqrt{2}$ is slow. From the above calculations, it appears that we gain an extra decimal digit in $\sqrt{2}$ after calculating another one or two rungs of the ladder.

We will investigate more features of this ladder. We will show how to modify it to calculate the square root of any number, we will look at several recursion relations, and we will show how to increase the speed of the convergence.

First extension: Finding any square root. Suppose we wish to find rational approximations to \sqrt{c} using the basic idea of Theon's ladder. We assume always that $1 < c$. The recursion relations that achieve this end are, for $n > 1$,

$$x_n = x_{n-1} + y_{n-1}, \tag{1}$$

and

$$y_n = x_n + (c - 1)x_{n-1}. \tag{2}$$

(Notice that (1) is the same for any root.) Throughout this paper, we assume that the first rung of the ladder is $[1, 1]$. From the above relations, it follows that $1 \leq x_n$ and $1 \leq y_n$. As an example, to find $\sqrt{3}$ we use $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_n + 2x_{n-1}$ to obtain the ladder

1	1	$1/1 = 1.00000 \dots$
2	4	$4/2 = 2.00000 \dots$
6	10	$10/6 = 1.66666 \dots$
16	28	$28/16 = 1.75000 \dots$
44	76	$76/44 = 1.72727 \dots$
120	208	$208/120 = 1.73333 \dots$
328	568	$568/328 = 1.73170 \dots$

We now show why the recursion relations (1) and (2) always lead to rational approximations of \sqrt{c} . Our examination in this section is simple, but not rigorous, since we are required to assume that $\lim_{n \rightarrow \infty} y_n/x_n$ exists. Later, independent of this section, we will prove that this limit exists.

Dividing (2) by (1) gives us

$$\frac{y_n}{x_n} = \frac{x_n + (c - 1)x_{n-1}}{x_{n-1} + y_{n-1}}.$$

Dividing the numerator and the denominator on the right hand side by x_{n-1} , we get

$$\frac{y_n}{x_n} = \frac{\frac{x_n}{x_{n-1}} + (c - 1)}{1 + \frac{y_{n-1}}{x_{n-1}}}.$$

Replacing x_n on the right hand side by (2), we get

$$\frac{y_n}{x_n} = \frac{1 + \frac{y_{n-1}}{x_{n-1}} + (c - 1)}{1 + \frac{y_{n-1}}{x_{n-1}}} = \frac{c + \frac{y_{n-1}}{x_{n-1}}}{1 + \frac{y_{n-1}}{x_{n-1}}}.$$

Assuming that the limit exists, we let $r = \lim_{n \rightarrow \infty} y_n/x_n$. Then we have

$$r = \frac{c + r}{1 + r},$$

which reduces to $r^2 = c$. Thus we see that the ladder gives rational approximations to \sqrt{c} .

Connection to $(1 + \sqrt{c})^n$ and $(1 - \sqrt{c})^n$. We now show that the rungs of our ladder $[x_n, y_n]$ can be generated by powers of simple binomials

$$y_n + \sqrt{c} x_n = (1 + \sqrt{c})^n \quad \text{and} \quad y_n - \sqrt{c} x_n = (1 - \sqrt{c})^n. \quad (3)$$

We will prove (3) by induction. Notice that (3) is true when $n = 1$. Assume (3) is true for $n = N$. Then

$$\begin{aligned} (1 + \sqrt{c})^{N+1} &= (1 + \sqrt{c})^N (1 + \sqrt{c}) = (y_N + \sqrt{c} x_N) (1 + \sqrt{c}) \\ &= (c x_N + y_N) + \sqrt{c} (x_N + y_N). \end{aligned}$$

But from the recursion relation (1), $x_N + y_N = x_{N+1}$. We now have

$$(1 + \sqrt{c})^{N+1} = (c x_N + y_N) + \sqrt{c} x_{N+1}. \quad (4)$$

Also $(c x_N + y_N) = (c - 1)x_N + x_N + y_N = (c - 1)x_N + x_{N+1} = y_{N+1}$, where we have used (1) and (2). Now (4) becomes

$$(1 + \sqrt{c})^{N+1} = y_{N+1} + \sqrt{c} x_{N+1}, \quad (5)$$

and our induction proof for $(1 + \sqrt{c})^n$ is finished. In the same way we can show that $y_n - \sqrt{c} x_n = (1 - \sqrt{c})^n$.

More recursion relations. In addition to the recursion relations (1) and (2), we can easily derive others. For example, eliminating x_n from (2) using (1), we get

$$y_n = c x_{n-1} + y_{n-1}. \quad (6)$$

Relations (1) and (6) allow us to calculate the next rung using only the two numbers on the present rung.

If we reindex (2) to read $y_{n-1} = x_{n-1} + (c - 1)x_{n-2}$ and use it to eliminate y_{n-1} from (1), we get

$$x_n = 2 x_{n-1} + (c - 1)x_{n-2}. \quad (7)$$

From (6), we have

$$cx_n = y_{n+1} - y_n. \quad (8)$$

Multiply (1) by c and get $cx_n = cx_{n-1} + cy_{n-1}$. Now use (8) to remove cx_n and cx_{n-1} to get

$$y_n = 2y_{n-1} + (c-1)y_{n-2}. \quad (9)$$

Relations (7) and (9) form an interesting pair. They allow us to calculate the x 's without reference to the y 's and vice versa.

A rigorous proof that the ladder converges to \sqrt{c} . Earlier, in section 2, we gave a simple non-rigorous demonstration that our extended ladder converges to \sqrt{c} . Now we are able to give a rigorous demonstration. Dividing (3) by x_n and recalling that $1 \leq c$, we have

$$\left| \frac{y_n}{x_n} - \sqrt{c} \right| = \frac{(\sqrt{c}-1)^n}{x_n}.$$

From (7) we know that $x_n > (c-1)x_{n-2}$. So it follows that $x_n > (c-1)^2x_{n-4}$, and $x_n > (c-1)^3x_{n-6}$. Continuing in this way, we see that $x_{2n} > (c-1)^n$ and $x_{2n+1} > (c-1)^n$. It now follows that for even rungs

$$\left| \frac{y_{2n}}{x_{2n}} - \sqrt{c} \right| = \frac{(\sqrt{c}-1)^{2n}}{x_{2n}} < \frac{(\sqrt{c}-1)^{2n}}{(c-1)^n} = \left(\frac{\sqrt{c}-1}{\sqrt{c}+1} \right)^n,$$

and for odd rungs

$$\left| \frac{y_{2n+1}}{x_{2n+1}} - \sqrt{c} \right| = \frac{(\sqrt{c}-1)^{2n+1}}{x_{2n+1}} < \frac{(\sqrt{c}-1)^{2n+1}}{(c-1)^n} = (\sqrt{c}-1) \left(\frac{\sqrt{c}-1}{\sqrt{c}+1} \right)^n.$$

Thus it follows that $\lim \frac{y_n}{x_n} = \sqrt{c}$.

Second extension: Leaping over rungs. We saw earlier that the convergence of y_n/x_n to \sqrt{c} was slow. We now show how to radically accelerate this speed by jumping over many rungs of the ladder. Using (3), we have

$$\begin{aligned} y_{2n} + \sqrt{c}x_{2n} &= (1 + \sqrt{c})^{2n} = \left((1 + \sqrt{c})^n \right)^2 = (y_n + \sqrt{c}x_n)^2 \\ &= (cx_n^2 + y_n^2) + \sqrt{c}(2x_ny_n). \end{aligned}$$

Now we see that

$$x_{2n} = 2x_ny_n \quad \text{and} \quad y_{2n} = cx_n^2 + y_n^2. \quad (10)$$

Using (10), we could start on the 10th rung $[x_{10}, y_{10}]$ and jump immediately to the 20th rung $[x_{20}, y_{20}]$ then to $[x_{40}, y_{40}]$, then $[x_{80}, y_{80}]$, \dots . In only 4 steps we jump from rung 10 to rung 80!

As a numerical example, consider the ladder given in section 1 (with $c = 2$), where our last entry was $x_7 = 169$, $y_7 = 239$, and $y_7/x_7 = 1.414201183$. Using (10),

we calculate $x_{14} = 80782$ and $y_{14} = 114243$. Now our calculator gives $y_{14}/x_{14} = 1.414213562$ and $\sqrt{2} = 1.414213562$. We have jumped from 4 accurate decimal places to 9 in one step.

From (10), we get

$$\frac{y_{2n}}{x_{2n}} = \frac{y_n^2 + cx_n^2}{2x_n y_n} = \frac{1}{2} \left(\frac{y_n}{x_n} + \frac{cx_n}{y_n} \right).$$

Let

$$z_k = \frac{y_{2^k n}}{x_{2^k n}},$$

and obtain

$$z_{k+1} = \frac{1}{2} \left(z_k + \frac{c}{z_k} \right).$$

This is the iteration obtained when using Newton's method [1, p. 68] for finding the roots of $z^2 - c = 0$. It is well known that this iteration converges "quadratically." Roughly speaking, we double the number of accurate decimal digits of \sqrt{c} with each iteration.

References

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To call in the statistician after the experiment is done may be no more than asking him to perform a postmortem examination: he may be able to say what the experiment died of.

—R. A. Fisher

(Indian Statistical Congress, Sankhya, ca. 1938)