APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 6

1/ (a) $\int z^5 + 2z + 3 \, dz = \boxed{\frac{z^6}{6} + z^2 + 3z + c}$

(b) $\int z e^{2z} dz = \int z \, d\frac{e^{2z}}{2} =$ INTEGRATING BY PARTS

$$= z\frac{e^{2z}}{2} - \frac{1}{2}\int e^{2z} dz$$

$$= \boxed{\frac{z e^{2z}}{2} - \frac{e^{2z}}{4} + c}$$

(c) $\int \frac{dz}{1+z^2} = \boxed{\arctan z + c,}$

2/ (a) using the result of problem 1(a) we have

$$\left.\frac{z^6}{6} + z^2 + 3z\right|_0^{1+i} = \frac{(\sqrt{2}e^{i\frac{\pi}{4}})^6}{6} + (\sqrt{2}e^{i\frac{\pi}{4}})^2 + 3(1+i)$$

$$= \frac{8e^{i\frac{3\pi}{2}}}{6} + 2e^{i\frac{\pi}{2}} + 3 + 3i$$

$$= \frac{4}{3}(-i) + 2i + 3 + 3i = \boxed{3 + \frac{11}{3}i}$$

(b) using the result of problem 1(b) we have

$$\left.\frac{z e^{2z}}{2} - \frac{e^{2z}}{4}\right|_0^{1+i} = \frac{(1+i)e^{2+2i}}{2} - \frac{e^{2+2i}}{4} + \frac{1}{4}$$

$$= e^2(\cos 2 + i \sin 2)\left(\frac{1}{4} + \frac{i}{2}\right) + \frac{1}{4}$$

$$= \boxed{\frac{1}{4}e^2\cos 2 - \frac{1}{2}e^2\sin 2 + \frac{1}{4} + i\left[\frac{e^2}{2}\cos 2 + \frac{e^2}{4}\sin 2\right]}$$

23/ (continued)

$$I = \frac{\pi i}{2}\left\{\left[i e^{iz}(z+i)^{-2} - 2e^{iz}(z+i)^{-3}\right]_{z=i} - \left[-i e^{-iz}(z-i)^{-2} - 2e^{-iz}(z-i)^{-3}\right]_{z=-i}\right\}$$

$$I = \frac{\pi i}{2} e^{-1}\left\{\left[\frac{i}{-4} - \frac{2}{-8i}\right] - \left[\frac{-i}{-4} - \frac{2}{8i}\right]\right\}$$

$$I = \frac{\pi i}{2e}\left\{\frac{1}{i}\right\} = \boxed{\frac{\pi}{2e}}$$

24/ $I = \int_{-\infty}^{\infty} \frac{\sin^2 x \; dx}{1+x^2}$

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = -\frac{1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right)$$

Thus

$$I = -\frac{1}{4}\int \frac{e^{2iz}\,dz}{1+z^2} + \frac{1}{2}\int \frac{dz}{1+z^2} - \frac{1}{4}\int \frac{e^{-2iz}\,dz}{1+z^2}$$

or

$$I = 2\pi i\left\{\left(-\frac{1}{4}\right)\operatorname{Res}\left(\frac{e^{2iz}}{1+z^2}, i\right) + \frac{1}{2}\operatorname{Res}\left(\frac{1}{1+z^2}, i\right) - \left(-\frac{1}{4}\right)\operatorname{Res}\left(\frac{e^{-2iz}}{1+z^2}, -i\right)\right\}$$

minus because is in the negative sense.

24/ (continued)

$$I = 2\pi i\left\{-\frac{1}{4}\left(\frac{e^{-2}}{2i}\right)+\frac{1}{2}\left(\frac{1}{2i}\right)+\frac{1}{4}\left(\frac{e^{-2}}{-2i}\right)\right\}$$

$$I = \boxed{\frac{\pi}{2}\left(1-e^{-2}\right)}$$

25/ $$I = \int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \oint_{|z|=1}\frac{1}{a+b\,\frac{z-z^{-1}}{2i}}\,\frac{dz}{iz}$$

$$= \frac{2}{b}\oint_{|z|=1}\frac{dz}{z^2+\frac{2ia}{b}z-1}$$

The singularities of the integrand occur at

$z_\pm = \dfrac{-a\pm\sqrt{a^2-b^2}}{b}$, but only z_+ is inside the unit circle.

$$I = \frac{2}{b}\oint\frac{dz}{(z-z_+)(z-z_-)} = (2\pi i)\left(\frac{2}{b}\right)\frac{1}{z_+-z_-}$$

$$= \boxed{\frac{2\pi}{\sqrt{a^2-b^2}}}$$

26/ $$I = \int_0^{2\pi}\frac{d\theta}{a+b\cos\theta+c\sin\theta} = \oint_{|z|=1}\frac{1}{\left[a+b\,\frac{z+z^{-1}}{2}+c\,\frac{z-z^{-1}}{2i}\right]}\frac{dz}{iz}$$

$$I = \frac{-2i}{b-ic}\oint_{|z|=1}\frac{dz}{z^2+\frac{2a}{b-ic}z+\frac{b+ci}{b-ci}}$$

26/ (continued)

The singularities of the integrand occur at

$$z_{\pm} = \frac{-a \pm \sqrt{a^2-b^2-c^2}}{b-ci}, \quad \text{but only } z_+$$

is inside $|z|=1$. Therefore

$$I = \frac{-2i}{b-ic} \oint_{|z|=1} \frac{dz}{(z-z_+)(z-z_-)}$$

$$= \frac{-2i}{b-ic}(2\pi i)\,\mathrm{Res}(z_+) = \frac{-2i}{b-ic}(2\pi i)\frac{1}{z_+-z_-}$$

$$= \boxed{\frac{2\pi}{\sqrt{a^2-b^2-c^2}}}$$

27/ $$I = \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi}(\text{same})$$

$$= \frac{1}{2}\oint_{|z|=1} \frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2}\cdot\frac{dz}{iz}$$

$$= \frac{2}{i}\oint \frac{z\,dz}{(z^2+2az+1)^2}$$

The singularities of the integrand occur at

$$z_{\pm} = -a \pm \sqrt{a^2-1}, \quad \text{and thus}$$

27/ (continued)

$$I = \frac{2}{i}\oint \frac{z\,dz}{(z-z_+)^2(z-z_-)^2}$$

$$= \frac{2}{i}(2\pi i)\left\{\frac{d}{dz}\left[z(z-z_-)^{-2}\right]\right\}\Big|_{z=z_+}$$

$$= 4\pi\left\{(z-z_-)^{-2} - 2(z-z_-)^{-3}z\right\}\Big|_{z=z_+}$$

$$= \frac{-4\pi(z_+ + z_-)}{(z_+ - z_-)^3} = \boxed{\frac{\pi a}{(a^3-1)^{\frac{3}{2}}}}$$

28/ $I = \int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\, d\theta$

$$I = \frac{1}{2}\oint_{|z|=1}\left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz} = \frac{1}{2^{2n+1}(-1)^n i}\oint \frac{(z^2-1)^{2n}dz}{z^{2n+1}}$$

The integrand has a pole of order $2n+1$ at $z=0$. If we expand $(1-z^2)^{2n}$ by means of the binomial theorem, there will appear a term of the form $b\,z^{2n}$, from which we see that b is the desired residue at $z=0$.

$$(1-z^2)^{2n} = 1 + \cdots + \binom{2n}{n}(-z^2)^n + \cdots + (-z^2)^{2n},$$

28/ (continued)

We see now that the residue is $\binom{2n}{n}(-1)^n$.

Therefore

$$I = \frac{1}{2^{2n+1}(-1)^n i}(2\pi i)\binom{2n}{n}(-1)^n$$

$$= \boxed{\frac{\pi}{2^{2n}} \quad \frac{(2n)!}{n! \cdot n!}}$$

29/ ANSWER IS $\frac{2\pi}{\sqrt{3}}$. THIS INTEGRAL IS A SPECIAL CASE OF PROBLEM 30,

30/ THIS INTEGRAL IS A GENERALIZATION OF EXAMPLE 1. WE WILL USE THE SAME CONTOUR AND BRANCH CUT, ONLY NOW WE HAVE

$$z^{-k} = r^{-k} e^{-ik\theta} \qquad 0 \le \theta < 2\pi,$$

$$(1) \quad \int_{C} \frac{z^{-k}\,dz}{z+1} = \int_{A\to B} + \int_{B \to D \text{ via } C} + \int_{E \leftarrow D} + \int_{E \to A \text{ via } F}$$

THE RESIDUE THEOREM GIVES FOR THE LEFT INTEGRAL IN (1)

$$2\pi i \operatorname{Res}\left(\frac{z^{-k}}{z+1}, -1\right) = 2\pi i\, z^{-k}\Big|_{z=-1=e^{i\pi}} = e^{-ik\pi}\, 2\pi i.$$

30/ (CONTINUED)

ON THE LARGE CIRCLE, $\left|\frac{z^{-k}}{z+1}\right| \approx R^{-k-1}$ AND SINCE $0<k<1$ THIS TENDS TO ZERO FASTER THAN $\frac{1}{R}$,

THUS FROM REMARK 2 THIS INTEGRAL VANISHES AS $R\to\infty$,

ON THE SMALL CIRCLE $\left|\frac{z^{-k}}{z+1}\right| \approx |z^{-k}| = \epsilon^{-k}$

AND BY REMARK 3 THIS INTEGRAL ALSO VANISHES,

$\int_{A\to B} \longrightarrow I$ BECAUSE $\theta=0$ AND $\frac{z^{-k}\,dz}{z+1} = \frac{r^{-k}dr}{r+1}$

ON $E \longleftarrow D$ HOWEVER, $\theta = 2\pi$ AND

$$\int_R^\epsilon \frac{z^{-k}\,dz}{z+1} = \int_R^\epsilon \frac{r^{-k}e^{-i2\pi k}\quad e^{i2\pi}dr}{re^{2\pi i}+1}$$

$$= \int_R^\epsilon \frac{r^{-k}dr}{r+1}\; e^{-i2\pi k}$$

$$\xrightarrow[\substack{\text{as}\\ \epsilon\to 0\\ R\to\infty}]{} -e^{-i2\pi k}\, I$$

THUS (1) BECOMES

$$2\pi i\, e^{-ik\pi} = I + 0 - e^{-i2\pi k} I + 0$$

$$I = \frac{2\pi i\, e^{-ik\pi}}{1-e^{-i2\pi k}} = \frac{2\pi i}{e^{ik\pi}-e^{-ik\pi}} = \boxed{\frac{\pi}{\sin \pi k}}$$

31/ $I = \int_0^\infty \frac{x^{-1/2}\,dx}{1+x^2}$ WE USE THE SAME CONTOUR AND BRANCH CUT AS IN EXAMPLE 1.

$$(1)\quad \oint \frac{z^{-1/2}\,dz}{1+z^2} = \int_{A\to B} + \int_{\substack{\text{LARGE}\\ \text{CIRCLE}}} + \int_{E\leftarrow D} + \int_{\substack{\text{SMALL}\\ \text{CIRCLE}}}$$

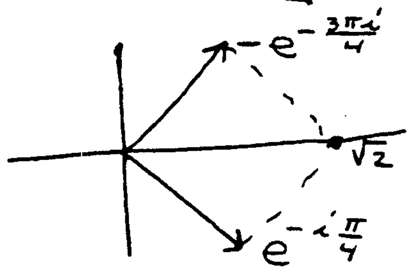
THE INTEGRAL ON THE LEFT OF (1) IS

$$2\pi i\left[\operatorname{Res}\left(\frac{z^{-\frac12}}{1+z^2}, i\right) + \operatorname{Res}\left(\frac{z^{-\frac12}}{1+z^2}, -i\right)\right] =$$

$$2\pi i\left[\left.\frac{z^{-\frac12}}{z+i}\right|_{z=i=e^{i\frac{\pi}{2}}} + \left.\frac{z^{-\frac12}}{z-i}\right|_{z=-i=e^{\frac{3\pi i}{2}}}\right] =$$

$$2\pi i\left[\frac{e^{-\frac{i\pi}{4}}}{2i} + \frac{e^{-\frac{3\pi i}{4}}}{-2i}\right] = \pi\left[e^{-\frac{i\pi}{4}} - e^{-\frac{3\pi i}{4}}\right]$$

$$= \pi\sqrt{2}$$



$\int_{A\to B} \to I$ SINCE $\theta = 0$ HERE,

$$\int_{E\leftarrow D} \to \int_\infty^0 \frac{r^{-\frac12} e^{-\pi i}}{r^2+1}\,dr = I$$

THE INTEGRALS OVER THE TWO CIRCLES VANISH SINCE FOR LARGE R THE INTEGRAND BEHAVES LIKE $\frac{1}{R^{3/2}}$,

31/ (continued)

AND FOR SMALL ϵ IT BEHAVES LIKE $\frac{1}{\sqrt{\epsilon}}$. (See Remarks 2 and 3).

USING (1) WE NOW HAVE

$$\sqrt{2}\,\pi = I + 0 + I + 0 , \quad I = \boxed{\frac{\pi}{\sqrt{2}}} .$$

32/ $I = \int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{(1+x^2)^2}$. WE USE THE SAME CONTOUR AS IN EXAMPLE (1),

$$(1) \quad \int_{C} \frac{z^{1/2}\,dz}{(1+z^2)^2} = \int_{A \to B} + \int_{\substack{\text{LARGE}\\ \text{CIRCLE}}} + \int_{E \leftarrow D} + \int_{\substack{\text{SMALL}\\ \text{CIRCLE}}}$$

$$\int_{C} \frac{z^{1/2}\,dz}{(1+z^2)^2} = 2\pi i \left[\operatorname{Res}(i) + \operatorname{Res}(-i) \right]$$

$$= 2\pi i \left[\frac{d}{dz}\left\{ z^{1/2}(z+i)^{-2} \right\}\Big|_{z=i=e^{i\frac{\pi}{2}}} + \frac{d}{dz}\left\{ z^{1/2}(z-i)^{-2} \right\}\Big|_{z=-i=e^{\frac{3\pi i}{2}}} \right]$$

$$= 2\pi i \left[\left\{ \tfrac{1}{2} z^{-\frac{1}{2}}(z+i)^{-2} - 2 z^{1/2}(z+i)^{-3} \right\}\Big|_{z=i=e^{i\frac{\pi}{2}}} + \left\{ \tfrac{1}{2} z^{-\frac{1}{2}}(z-i)^{-2} - 2 z^{1/2}(z-i)^{-3} \right\}\Big|_{z=-i=e^{i\frac{3\pi}{2}}} \right]$$

32/ (continued)

$$= 2\pi i \left[-\frac{\sqrt{2}\, i}{8} \right] = \frac{\sqrt{2}\,\pi}{4},$$

$$\int_{A \to B} \to I \quad \text{since } \theta = 0 \text{ here,}$$

$$\int_{E \leftarrow D} \longrightarrow \int_{\infty}^{0} \frac{r^{\frac{1}{2}} e^{i\pi}\, dr}{(1+r^2)^2} = I$$

The integrals over the two circles again tend to zero as $R \to \infty$ and $\epsilon \to 0$ since for large R the integrand resembles $R^{-7/2}$ and for small ϵ the integrand behaves like $\sqrt{\epsilon}$.

Combining all these values in (1) we get

$$\frac{\sqrt{2}\,\pi}{4} = I + 0 + I + 0, \quad \text{Thus } I = \boxed{\frac{\pi}{4\sqrt{2}}}.$$

33/ $I = \displaystyle\int_0^{\infty} \frac{\log x \, dx}{(x^2+a^2)^2}$ Here we use the same contour as in Example 2.

$$(1) \quad \int \frac{\log z \, dz}{(a^2+z^2)^2} = \int_{\epsilon}^{R} + \int_{\substack{\text{LARGE}\\\text{ARC}}} + \int_{-R}^{-\epsilon} + \int_{\substack{\text{SMALL}\\\text{ARC.}}}$$

33/ (CONTINUED)

$$\int_{\text{contour}} = 2\pi i \operatorname{Res}\left(\frac{\log z}{(a^2+z^2)^2}, ia\right)$$

$$= 2\pi i \left[\frac{d}{dz}\left\{\frac{\log z}{(ai+z)^2}\right\}\Bigg|_{z=ia} \right]$$

$$= 2\pi i \left[z^{-1}(ai+z)^{-2} - 2\log z \; (ai+z)^{-3} \right]_{z=ia}$$

$$= 2\pi i \left[\frac{-1}{4a^3 i} + \frac{\log a + i\frac{\pi}{2}}{4a^3 i} \right]$$

$$= \frac{\pi}{2a^3}\left[-1 + \log a + i\frac{\pi}{2} \right]$$

Now $\int_\epsilon^R \longrightarrow I$ and

$$\int_{-R}^{-\epsilon} \longrightarrow \int_\infty^0 \frac{\log r + i\pi}{(a^2+r^2)^2}(-dr) = \int_0^\infty \frac{\log r}{(a^2+r^2)^2}\,dr + i\pi\int_0^\infty \frac{dr}{(a^2+r^2)^2}$$

$$= I + i\pi \int_0^\infty \frac{dr}{(a^2+r^2)^2}$$

To evaluate this last integral set $r = au$ and get

$$\int_0^\infty \frac{dr}{(a^2+r^2)^2} = \int_0^\infty \frac{a\,du}{a^4(1+u^2)^2} = \frac{1}{a^3}\int_0^\infty \frac{du}{(1+u^2)^2} = \frac{\pi}{4a^3}$$

where we have used the result of problem 20, Thus

$$\int_{-R}^{-\epsilon} \longrightarrow I + i\frac{\pi^2}{4a^3}$$

33/ (continued)

The integral over the large semicircle behaves like

$$\frac{\log R}{R^4}\,\pi R = \frac{\log R}{R^3} \longrightarrow 0 \quad \text{as} \quad R \to \infty,$$

The integral over the small semicircle behaves like

$$\left(\frac{\log \epsilon + i\theta}{a^4}\right)\pi\epsilon \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0,$$

Combining these values in (1) we get

$$-\frac{\pi}{2a^3} + \frac{\pi \log a}{2a^3} + i\,\frac{\pi^2}{4a^3} = I + 0 + I + i\,\frac{\pi^2}{4a^3} + 0$$

which yields

$$I = \boxed{\frac{\pi}{4a^3}\left[-1 + \log a\right]}$$

34/ $I = \int_0^\infty \frac{\log x \, dx}{x^4+1}$, We use the same contour as in Example 2,

$$(1) \quad \int_{\text{contour}} \frac{\log z \, dz}{z^4+1} = \int_\epsilon^R + \int_{\substack{\text{large}\\\text{semicircle}}} + \int_{-R}^{-\epsilon} + \int_{\substack{\text{small}\\\text{arc}}}$$

Now

$$(2) \quad \int_{\text{contour}} = 2\pi i\left[\operatorname{Res}\left(\frac{\log z}{z^4+1}, e^{i\pi/4}\right) + \operatorname{Res}\left(e^{i\frac{3\pi}{4}}\right)\right]$$

34/ (continued)

$$\text{Res}\left(e^{i\pi/4}\right) = \lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})\log z}{z^4+1} = \left(\begin{matrix}\text{using}\\ \text{L'Hospital's}\\ \text{Rule}\end{matrix}\right)$$

$$= \left.\frac{\log z}{4z^3}\right|_{z=e^{i\pi/4}} = \frac{i\,\pi/4}{4e^{\frac{3\pi i}{4}}}$$

$$\text{Res}\left(e^{\frac{i3\pi}{4}}\right) = \left.\frac{\log z}{4z^3}\right|_{z=e^{\frac{3i\pi}{4}}} = \frac{i\,\frac{3\pi}{4}}{4e^{i\frac{9\pi}{4}}}$$

Thus from (2) we have

$$(3)\quad \int = 2\pi i\left[\frac{i\pi}{16}e^{-\frac{3\pi}{4}i} + \frac{3\pi i}{16}e^{-\frac{i\pi}{4}}\right]$$

$$= -\frac{\pi^2}{8}\left[e^{-\frac{3\pi}{4}i} + 3e^{-i\frac{\pi}{4}}\right]$$

Now

$$\int_\epsilon^R \longrightarrow I \quad , \text{ and}$$

$$\int_{-R}^{-\epsilon} \longrightarrow \int_\infty^0 \frac{\log r + i\pi}{r^4+1}(-dr) = \int_0^\infty \frac{\log r\,dr}{r^4+1} + i\pi\int_0^\infty \frac{dr}{1+r^4}$$

$$(4)\quad \int_{-R}^{-\epsilon} \longrightarrow I + \frac{i\pi^2}{2\sqrt2}$$ (where we use the solution to problem 19)

As before, the integrals over the circular arcs tend to zero. Substituting (3) and (4) into (1) we get

$$-\frac{\pi^2}{8}\left[-\frac{1}{\sqrt2} - \frac{i}{\sqrt2} + \frac{3}{\sqrt2} - \frac{3i}{\sqrt2}\right] = I + 0 + I + \frac{i\pi^2}{2\sqrt2}$$

$$-\frac{\pi^2\sqrt2}{8} = 2I\ , \quad \text{Thus}\quad I = \boxed{-\frac{\pi^2\sqrt2}{16}}$$

35/ $I = \int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}\,dx$

As in Example 3, we define

$$z^{-\frac{2}{3}} = r^{-\frac{2}{3}} e^{-i\frac{2\theta}{3}} \quad \text{where} \quad 0 \le \theta < 2\pi$$

and

$$(1-z)^{-\frac{1}{3}} = s^{-\frac{1}{3}} e^{-i\frac{\omega}{3}} \quad \text{where} \quad 0 \le \omega < 2\pi$$

so that

(1) $$z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} = r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{-i\left(\frac{2}{3}\theta + \frac{1}{3}\omega\right)}$$

We must first locate the branch line for the function (1). Since θ is discontinuous on the positive real axis, it comes as no surprise that (1) is discontinuous on the line segment from $z=0$ to $z=1$. However, it is not discontinuous on the real axis beyond the point $z=1$. To see this, note that as we approach this segment from above, both θ and ω approach zero and thus (1) is a real number. As we approach from below, both θ and ω are 2π and the exponential term in (1) becomes

$$e^{-i\left(\frac{2}{3}(2\pi) + \frac{1}{3}(2\pi)\right)} = e^{-i2\pi} = 1,$$

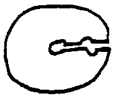
which means that (1) is again real.

Thus (1) has the very same branch cut as was found in Example 3.

Using the same contour as in Example 3 we have

35/ (continued)

(2) $$\oint z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}}\,dz = \underset{\substack{\text{large}\\\text{circle}}}{\int} + \underset{\substack{\text{above}\\\text{cut}}}{\int_{\epsilon}^{1-\epsilon}} + \underset{\substack{\text{below}\\\text{cut}}}{\int_{1-\epsilon}^{\epsilon}} + \underset{\substack{\text{Two small}\\\text{circles}}}{\int + \int}$$



The left side of (2) is zero by Cauchy's integral theorem since there are no singularities inside the contour.

On the large circle, $\theta \simeq w$ and $r = R$ and $s \simeq R$.

Thus (1) becomes

$$(z)^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} \simeq R^{-1}e^{-i\theta}, \quad \text{Since } dz = iRe^{i\theta}d\theta$$

we see that

(3) $$\underset{\substack{\text{large}\\\text{circle}}}{\int} \longrightarrow \int_0^{2\pi}\left(R^{-1}e^{-i\theta}\right)\left(iRe^{i\theta}d\theta\right) = i\int_0^{2\pi}d\theta = 2\pi i.$$

On the line segment above the cut, $\theta = 0$ and $w = \pi$,

Thus (1) becomes

$$z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} = r^{-\frac{2}{3}}s^{-\frac{1}{3}}\,e^{-i\frac{\pi}{3}} = x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}e^{-i\frac{\pi}{3}},$$

Since $dz = dx$ we have as $\epsilon \to 0$

(4) $$\underset{\substack{\text{above}\\\text{cut}}}{\int_{\epsilon}^{1-\epsilon}} \xrightarrow[\epsilon\to 0]{} e^{-i\frac{\pi}{3}}\int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}dx = e^{-i\frac{\pi}{3}}I$$

On the line segment below the cut,

$\theta = 2\pi$, but $w = \pi$ and thus (1) becomes,

$$z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} = r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{-i\left(\frac{2}{3}(2\pi) + \frac{1}{3}(\pi)\right)}$$
$$= r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{-\frac{5\pi}{3}i}$$
$$= r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{\frac{\pi}{3}i}$$
$$= x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}} e^{\frac{\pi}{3}i}$$

Since $dz = dx$ we have as $\epsilon \to 0$

$$(5) \quad \int_{\substack{1-\epsilon \\ \text{below} \\ \text{cut}}}^{\epsilon} \xrightarrow[\epsilon \to 0]{} \int_1^0 e^{\frac{\pi}{3}i} x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}} dx$$
$$= -e^{\frac{\pi}{3}i} I$$

The integrals over the two small circles tend to zero as $\epsilon \to 0$ as in Example 3,

Combining (3), (4) and (5) in (2) we get

$$0 = 2\pi i + e^{-i\frac{\pi}{3}} I - e^{\frac{\pi}{3}i} I + 0 + 0$$

$$I = \frac{2\pi i}{e^{\frac{\pi}{3}i} - e^{-\frac{\pi}{3}i}} = \frac{\pi}{\sin\frac{\pi}{3}} = \boxed{\frac{2\pi}{\sqrt{3}}}.$$

Solutions to Review Problems from Chapter 6

1/ $$\oint_{|z|=2\pi} \frac{\sin z\, dz}{(z-\pi)^5} = \frac{2\pi i}{4!} \left.\frac{d^4 \sin z}{dz^4}\right|_{z=\pi}$$

$$= \frac{2\pi i}{24} \sin z \Big|_{z=\pi} = \boxed{0}$$

2/ (a) $$\frac{1}{(a^2+z^2)^{+4}} = \frac{1}{(z-ia)^4 (z+ia)^4}$$

$$\text{Res}\,(ia) = \frac{1}{3!} \left.\frac{d^3 (z+ia)^{-4}}{dz^3}\right|_{z=ia}$$

$$= \frac{1}{3!} (-4)(-5)(-6)(z+ia)^{-7}\Big|_{z=ia}$$

$$= -\frac{20}{(2ia)^7} = \frac{-20}{-2^7 a^7 i} = \boxed{\frac{-5i}{32a^7}}$$

(b) z^4+6z^2+1 has roots at $z^2 = -3 \pm \sqrt{9-1}$
$= -3 \pm 2\sqrt{2}$

$$z^4+6z^2+1 = (z^2+3+2\sqrt{2})(z-\sqrt{3-2\sqrt{2}}\,i)(z+\sqrt{3-2\sqrt{2}}\,i)$$

WE SEE THAT THE ROOTS ARE SIMPLE,

$$\text{Res}\,(\sqrt{3-2\sqrt{2}}\,i) = \left.\frac{z}{(z^2+3+2\sqrt{2})(z+\sqrt{3-2\sqrt{2}}\,i)}\right|_{z=\sqrt{3-2\sqrt{2}}\,i}$$

$$= \frac{\sqrt{3-2\sqrt{2}}\,i}{(4\sqrt{2})\,2\sqrt{3-2\sqrt{2}}\,i} = \boxed{\frac{1}{8\sqrt{2}}}$$

$$\text{Res}\,(-\sqrt{3-2\sqrt{2}}\,i) = \left.\frac{z}{(z^2+3+2\sqrt{2})(z-\sqrt{3-2\sqrt{2}}\,i)}\right|_{z=-\sqrt{3-2\sqrt{2}}\,i}$$

$$= \frac{-\sqrt{3-2\sqrt{2}}\,i}{(4\sqrt{2})(-2\sqrt{3-2\sqrt{2}}\,i)} = \boxed{\frac{1}{8\sqrt{2}}}$$

2/ (continued)

(c) The pole is simple, therefore

$$\operatorname{Res}(-1) = z^{-k} \log z \Big|_{z=-1=e^{i\pi}}$$

$$= e^{-ik\pi}\left[\log 1 + i\pi\right]$$

$$= \boxed{i\pi e^{-ik\pi}}$$

3/ $I = \int_0^{\infty} \frac{dx}{(a^2+x^4)^4} = \frac{1}{2}\int_{-\infty}^{\infty}$

Using the contour of Section 6.7, Example 1 we have

$$\int \frac{dz}{(a^2+z^2)^4} = \int_{-\infty}^{\infty} + \int$$

$$2\pi i \operatorname{Res}\left(\frac{1}{(a^2+z^2)^4}, ia\right) = 2I + 0$$

Use review problem 2 (a)

$$2\pi i \left(\frac{-5i}{32a^7}\right) = 2I$$

$$I = \boxed{\frac{5\pi}{32a^7}}$$

4/ $I = \int_0^{\infty} \frac{x^{-k} \log x \, dx}{1+x}$

Use the contour of Example 1, Section 6.8.

(1) $$\int \frac{z^{-k} \log z \, dz}{1+z} = \underset{\text{above cut}}{\int_0^{\infty}} + \underset{\text{below cut}}{\int_{\infty}^{0}} + \underset{\text{small circle}}{\int} + \underset{\text{big circle}}{\int}$$

4/ (continued)

The left side of (1) is evaluated with the help of the Residue Theorem and problem 2(c) above

$$\int_{G} = 2\pi i \left[i\pi e^{-ik\pi} \right] = -2\pi^2 e^{-ik\pi}$$

Since $\theta = 0$ above the cut, $\displaystyle\int_{0\ \text{above cut}}^{\infty} = I$

Below the cut, $\theta = 2\pi$ and

$$\frac{z^{-k}\log z}{1+z}\,dz = \frac{r^{-k} e^{-i2\pi k}\left[\log r + 2\pi i\right]}{1+r} (+dr).$$

Thus

$$\int_{\infty\ \text{below cut}}^{0} = e^{-i2\pi k}\int_{\infty}^{0} \frac{r^{-k}\log r}{r+1}\,dr + 2\pi i\, e^{-i2\pi k}\int_{\infty}^{0}\frac{r^{-k}\,dr}{1+r}$$

$$= -e^{-i2\pi k} I - 2\pi i\, e^{-i2\pi k}\underbrace{\int_0^\infty \frac{r^{-k}\,dr}{1+r}}_{\frac{\pi}{\sin \pi k}\ \text{problem 30 Chapter 6}}$$

$$= -e^{-i2\pi k} I - \frac{2\pi^2 i\, e^{-i2\pi k}}{\sin \pi k}$$

Now (1) becomes

$$-2\pi^2 e^{-ik\pi} = I - e^{-i2\pi k} I - \frac{2\pi^2 i\, e^{-i2\pi k}}{\sin \pi k} + 0 + 0$$

$$I = \left[\frac{2\pi^2 i\, e^{-i2\pi k}}{\sin \pi k} - 2\pi^2 e^{-ik\pi}\right]\frac{1}{1-e^{-i2\pi k}}$$

$$= 2\pi^2\left[\frac{2e^{-i2\pi k}}{e^{i\pi k}-e^{-i\pi k}} - e^{-ik\pi}\right]\frac{1}{1-e^{-i2\pi k}}$$

$$= 2\pi^2\left[\frac{1+e^{-i2\pi k}}{e^{i\pi k}-e^{-i\pi k}}\right]\frac{1}{1-e^{-i2\pi k}}$$

$$= \frac{2\pi^2\left(e^{i\pi k}+e^{-i\pi k}\right)}{\left(e^{i\pi k}-e^{-i\pi k}\right)^2} = \boxed{\frac{-\pi^2\cos \pi k}{\sin^2 \pi k}}$$

5/ $$I = \int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta} = \oint_{|z|=1} \frac{1}{1+\left(\frac{z+z^{-1}}{2}\right)^2} \frac{dz}{iz}$$

$$= \frac{1}{i} \oint_{|z|=1} \frac{4z\,dz}{4z^2 + z^4 + 2z^2 + 1}$$

$$= \frac{4}{i} \oint_{|z|=1} \frac{z\,dz}{z^4 + 6z^2 + 1}$$

$$= \frac{4}{i}\, 2\pi i \left[\text{sum of the residues of } \frac{z}{z^4+6z^2+1} \text{ inside } |z|=1 \right]$$

$$= 8\pi \left[\text{sum of the two residues computed above in problem } 2(b) \right]$$

$$= 8\pi \left[\frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}} \right] = \frac{2\pi}{\sqrt{2}} = \boxed{\sqrt{2}\,\pi}$$

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 6

6.1 The indefinite integral

Let $f(z)$ be an analytic function defined on the open set R. Then there exists an analytic function $F(z)$ such that for each z on R, $F'(z) = f(z)$. We call $F(z)$ an indefinite integral of $f(z)$ and denote the family of all functions which when differentiated yield $f(z)$ by $\int f(z)\,dz$. It can be shown that this family is

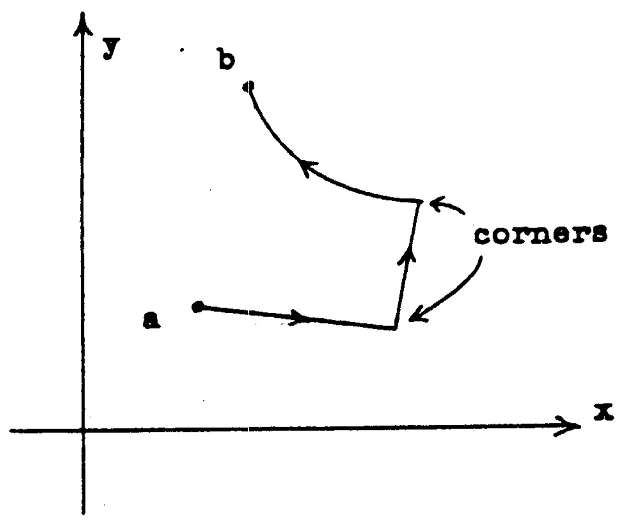
$$\int f(z)\,dz = F(z) + c,$$

where c is an arbitrary (complex) constant.

6.2 The Riemann integral

In real analysis we integrated over a segment of the x-axis from $x=a$ (the starting point) to $x=b$, (the final point) to form the definite integral $\int_a^b f(x)dx$. In the extension to the complex plane, this directed segment of the x-axis is replaced by a directed curve starting at $z=a$ and ending at $z=b$.

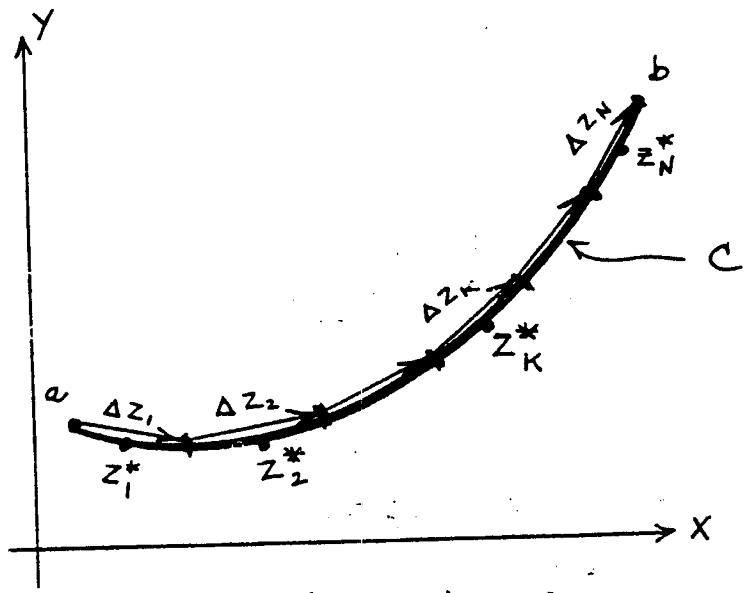
A directed curve (like the one in the figure) is called "nice" if it is smooth at most points. We do allow a few corners as shown in the figure where the curve fails to be smooth. (An exact definition of a nice curve would be one that is described by



parametric equations $x = x(t)$ and $y = y(t)$, $t_0 \le t \le t_1$, where $x(t)$ and $y(t)$ have piecewise continuous derivatives.) We will not, however, have occasion to worry about the precise nature of a "nice" curve in this book, as the curves we usually encounter are composed of straight line segments and circular arcs.

Now let $f(z)$ be a complex valued function defined for all z on the nice directed curve C which starts at $z=a$ and ends at $z=b$.

(i) Subdivide the curve C into N small consecutive arcs as shown in the figure.



(ii) On each small arc select a point z_k^* and form $f(z_k^*)\Delta z_k$.

(iii) Form $\sum_{k=1}^{N} f(z_k^*)\Delta z_k$.

(iv) Continue to sub-divide the curve C into more and more arcs $(N \to \infty)$ and simultaneously let the length of each small arc approach zero $(\Delta z_k \to 0)$. If the limit

$$\lim_{\substack{\Delta z_k \to 0 \\ N \to \infty}} \sum_{k=1}^{N} f(z_k^*)\Delta z_k$$

exists and gives but one value regardless of the manner in which the subdivision of the curve C is made, then we call this limit the Riemann integral of $f(z)$ over the directed curve (or contour) from $z=a$ to $z=b$ and denote it by

$$\int_C f(z)\,dz = \int_{a \atop C}^{b} f(z)\,dz = \int_a^b f(z)dz \quad .$$

Remarks

1. Each of the three expressions above for the definite integral over the curve C from a to b require that some description be given of the contour C which joins a to b . Sometimes other notations are also used which are often self-explanatory. For example, the notation

$$\oint_{|z|=1} f(z)\,dz$$

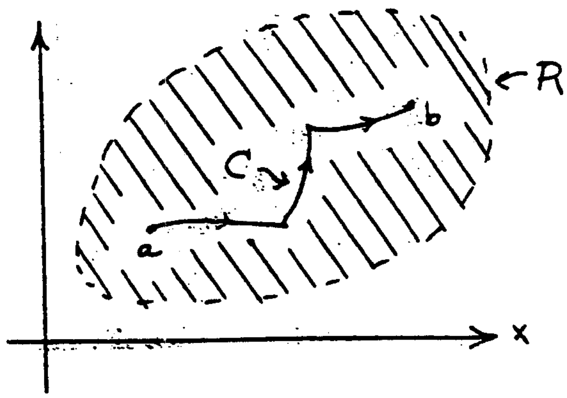
means that $f(z)$ is integrated over the closed curve which is the circle of radius one centered at the origin in the counter-clockwise sense.

2. Note that there is no need to require that $f(z)$ be analytic on the contour of integration C in the above definition of the Riemann integral. If $f(z)$ is simply continuous on C, then the Riemann integral always exists provided the length of C is finite.

6.3 The Fundamental Theorem of the Integral Calculus

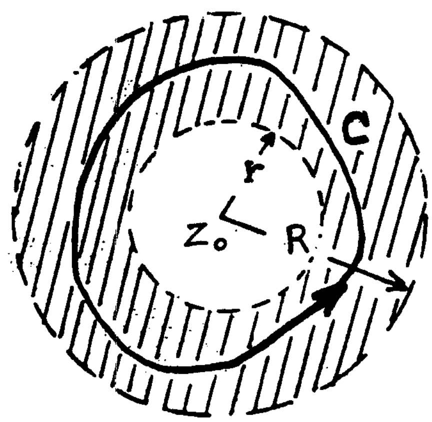
Let $f(z)$ be an analytic function for each z on an open set R and let C denote a directed contour from $z=a$ to $z=b$ in the set R. Then there exists an analytic function $F(z)$ on R such that $F'(z) = f(z)$ and

$$\int_{C\,a}^{b} f(z)\,dz = F(b) - F(a) .$$



6.4 Laurent's Theorem

Let $f(z)$ be analytic for all z in the annulus $r < |z-z_0| < R$. Then $f(z)$ can be expanded in a convergent series at each point of this annulus given by



$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{(t-z)^{n+1}} \,(z-z_0)^n ,$$

where the simple closed curve C circles the annulus as shown and does not extend outside the annulus.

3/ (a) $\displaystyle\int_{C_2} 2f(z) - 3g(z)\,dz = 2\int_{C_2} f(z) - 3\int_{C_2} g(z)\,dz$

$= 2(-i) - 3(2) = \boxed{-6-2i}$

(b) $\displaystyle\int_{1+i}^{1} 2f(z) - 3g(z)\,dz = -\int_{C_2} 2f(z) - 3g(z)\,dz$

$= -(-6-2i) = \boxed{6+2i}$

(c) $\displaystyle\int_{C} 2f(z) - 3g(z)\,dz = \int_{C_1} 2f(z) - 3g(z)\,dz + \int_{C_2} 2f(z) - 3g(z)\,dz$

$\displaystyle= 2\int_{C_1} f(z)\,dz - 3\int_{C_1} g(z)\,dz \quad -6-2i$

$= 2(3) - 3(2i) - 6 - 2i = \boxed{-8i}$

4/(a) Since $\cos z$ is analytic for all z, this integral is zero by Cauchy's Integral Theorem.

(b) The singularity is outside the contour, and thus the integral vanishes.

(c) The singularity is inside the contour and thus

$2\pi i \cos z \big|_{z=0} = 2\pi i$ is the value of the integral.

(d) $\displaystyle 2\pi i \left.\frac{D^2 \cos z}{2!}\right|_{z=0} = \left.\frac{2\pi i(-\cos z)}{2}\right|_{z=0} = -\pi i$

4/(e) $$\oint_{|z-1|=1} \frac{(z+1)^{-1}\,dz}{z-1} = 2\pi i\,(z+1)^{-1}\Big|_{z=1} = \cancel{4}\pi i.$$

(f) $$2\pi i\,\frac{D^N e^z}{N!}\Big|_{z=0} = 2\pi i\,\frac{e^z}{N!}\Big|_{z=0} = \frac{2\pi i}{N!}$$

(g) $$2\pi i\,\frac{D^N e^z}{N!}\Big|_{z=2} = 2\pi i\,\frac{e^z}{N!}\Big|_{z=2} = \frac{2\pi i\,e^2}{N!}$$

(h) This integral has two simple poles inside the contour at $z=i$ and $z=-i$. The methods of this section do not apply to this problem. In the next section we will learn that

$$I = \oint_{|z|=2} \frac{z\,dz}{z^2+1} = \oint_{C_1} \frac{z(z+i)^{-1}}{z-i}\,dz + \oint_{C_2} \frac{z(z-i)^{-1}\,dz}{z+i}$$

where C_1 contains only the singularity at i,
and C_2 " " " " $-i$,

$$I = 2\pi i\,z(z+i)^{-1}\Big|_{z=i} + 2\pi i\,z(z-i)^{-1}\Big|_{z=-i}$$

$$= 2\pi i\left(\tfrac{1}{2}\right) + 2\pi i\left(\tfrac{1}{2}\right) = 2\pi i.$$

5/ $$\oint_{|z|=4} = \oint_{|z|=\frac{1}{2}} + \oint_{|z-1|=\frac{1}{2}}$$

5/ (continued)

$$= 2\pi i \left.\frac{\cos z}{z-1}\right|_{z=0} + 2\pi i \left.\frac{\cos z}{z}\right|_{z=1}$$

$$= -2\pi i + 2\pi i \cos 1 = 2\pi i\,(\cos(1) - 1),$$

6/ $$\oint_{|z|=2} = \oint_{|z|=\frac{1}{2}} + \oint_{|z-1|=\frac{1}{2}}$$

$$= 2\pi i \left.\frac{\frac{d}{dz}\left(\frac{1}{z-1}\right)}{1!}\right|_{z=0} + 2\pi i \left.\frac{1}{z^2}\right|_{z=1}$$

$$= 2\pi i \left.\left(\frac{-1}{(z-1)^2}\right)\right|_{z=0} + 2\pi i = 0$$

7/ $$\oint_{|z|=2} \frac{dz}{z(z+3)(z-1)} = \oint_{|z|=\frac{1}{2}} + \oint_{|z+3|=\frac{1}{2}} + \oint_{|z-1|=\frac{1}{2}}$$

$$= 2\pi i \left.\left(\frac{1}{(z+3)(z-1)}\right)\right|_{z=0} + 2\pi i \left.\frac{1}{z(z-1)}\right|_{z=-3} + 2\pi i \left.\frac{1}{z(z+3)}\right|_{z=1}$$

$$= 2\pi i\left(-\frac{1}{3}\right) + 2\pi i\,\frac{1}{12} + 2\pi i\,\frac{1}{4}$$

$$= 2\pi i\,\frac{0}{12} = 0$$

8/ $\dfrac{z}{z^2+1} = \dfrac{z}{(z+i)(z-i)}$

$$\operatorname{Res}\left(\frac{z}{z^2+1}, -i\right) = \left.\frac{z}{z-i}\right|_{z=-i} = \frac{1}{2}$$

$$\operatorname{Res}\left(\frac{z}{z^2+1}, i\right) = \left.\frac{z}{z+i}\right|_{z=i} = \frac{1}{2}$$

$$\oint_{|z|=2} \frac{z}{z^2+1}\,dz = 2\pi i\,(\text{sum of residues})$$

$$= 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i$$

9/ $\operatorname{Res}(\csc z, \pi) = -1$ from Example 2,

$$\operatorname{Res}(\csc z, 0) = \lim_{z\to 0} \frac{z}{\sin z}$$

$$= \lim_{z\to 0} \frac{\frac{d}{dz}(z)}{\frac{d}{dz}\sin z} = \left.\frac{1}{\cos z}\right|_{z=0} = 1$$

$$\operatorname{Res}(\csc z, -\pi) = \lim_{z\to -\pi} \frac{z+\pi}{\sin z}$$

$$= \lim_{z\to -\pi} \frac{1}{\cos z} = \frac{1}{-1} = -1,$$

$$\oint_{|z|=4} \csc z\,dz = 2\pi i\left[\operatorname{Res}(-\pi) + \operatorname{Res}(0) + \operatorname{Res}(\pi)\right]$$

$$= 2\pi i\left[-1 + 1 - 1\right] = -2\pi i$$

10/ $$\frac{1}{z^3-1} = \frac{1}{(z-1)(z-e^{2\pi i/3})(z-e^{-2\pi i/3})}$$

$$\text{Res}\left(\frac{1}{z^3-1}, 1\right) = \lim_{z\to 1} \frac{z-1}{z^3-1}$$

$$= \lim_{z\to 1} \frac{1}{3z^2} = \boxed{\frac{1}{3}}$$

$$\text{Res}\left(\frac{1}{z^3-1}, e^{\frac{2\pi i}{3}}\right) = \lim_{z\to e^{\frac{2\pi i}{3}}} \frac{z-e^{\frac{2\pi i}{3}}}{z^3-1}$$

$$= \lim_{z\to e^{\frac{2\pi i}{3}}} \frac{1}{3z^2}$$

$$= \frac{1}{3e^{4\pi i/3}} = \boxed{\frac{e^{2\pi i/3}}{3}}$$

$$\text{Res}\left(\frac{1}{z^3-1}, e^{-\frac{2\pi i}{3}}\right) = \lim_{z\to e^{-\frac{2\pi i}{3}}} \frac{z-e^{-\frac{2\pi i}{3}}}{z^3-1}$$

$$= \lim_{z\to e^{-\frac{2\pi i}{3}}} \frac{1}{3z^2}$$

$$= \lim_{z\to e^{-\frac{2\pi i}{3}}} \frac{1}{3e^{-\frac{4\pi i}{3}}} = \boxed{\frac{e^{-\frac{2\pi i}{3}}}{3}}$$

$$\oint_{|z|=2} \frac{dz}{z^3-1} = 2\pi i\left[\frac{1}{3} + \frac{1}{3}e^{2\pi i/3} + \frac{1}{3}e^{-\frac{2\pi i}{3}}\right]$$

$$= 0$$

11/ $$\operatorname{Res}\left(\frac{e^z}{z^2(z-2)^3}, 0\right) = \left.\frac{\frac{d}{dz}\left(\frac{e^z}{(z-2)^3}\right)}{1!}\right|_{z=0}$$

$$= \left.\frac{(z-2)^3 e^z - e^z 3(z-2)^2}{(z-2)^6}\right|_{z=0} = \frac{e^0(-8-12)}{2^6}$$

$$\cancel{= -\frac{1}{2^5} = -\frac{1}{32}} = -\frac{5}{16}$$

Since only the singularity at $z=0$ is inside $|z|=1$ we get

$$\oint_{|z|=1} \frac{e^z}{z^2(z-2)^3}\,dz = 2\pi i \operatorname{Res}(0) = 2\pi i\left(\cancel{-\frac{1}{32}} -\frac{5}{16}\right)$$

$$= \cancel{-\frac{\pi i}{16}} - \frac{5\pi i}{8}$$

12/ $$\operatorname{Res}\left(\frac{e^z}{z^2(z-2)^3}, 2\right) = \left.\frac{\frac{d^2}{dz^2}\left(\frac{e^z}{z^2}\right)}{2!}\right|_{z=2} =$$

$$= \frac{1}{2}\frac{d}{dz}\left.\left(\frac{e^z(z-2)}{z^3}\right)\right|_{z=2} = \frac{1}{2}\left.\left[\frac{e^z(z^2-4z+6)}{z^4}\right]\right|_{z=2}$$

$$= \frac{1}{2}\left[\frac{e^2(2)}{16}\right] = \frac{e^2}{16}$$

$$\oint \frac{e^z}{z^2(z-2)^3}\,dz = 2\pi i\left[\operatorname{Res}(0) + \operatorname{Res}(2)\right]$$

$$= 2\pi i\left[\cancel{-\frac{1}{32}} -\frac{5}{16} + \frac{e^2}{16}\right]$$

$$= \cancel{\frac{\pi i}{16}\left[2e^2 - 1\right]} = \frac{\pi i}{8}\left[e^2 - 5\right]$$

13/ $\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{6z^3} + \cdots$

Thus $\text{Res}\left(\sin \frac{1}{z}, 0\right) = 1$,

$$\oint_{|z|=4} \sin \frac{1}{z} \, dz = 2\pi i \, \text{Res}(0) = 2\pi i$$

14/ Multiply the series for $\sin \frac{1}{z}$ by itself and get

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{6z^3} + \cdots$$

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{6z^3} + \cdots$$

$$\frac{1}{z^2} - \frac{1}{6z^4} + \cdots$$

$$- \frac{1}{6z^4} + \cdots$$

$$\cdots$$

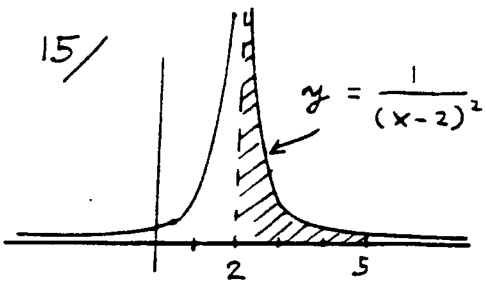
$$\sin^2 \frac{1}{z} = \frac{1}{z^2} - \frac{1}{3z^4} + \cdots$$

Since there is no term involving $\frac{1}{z}$ in this last series, $\text{Res}\left(\sin^2 \frac{1}{z}, 0\right) = 0$.

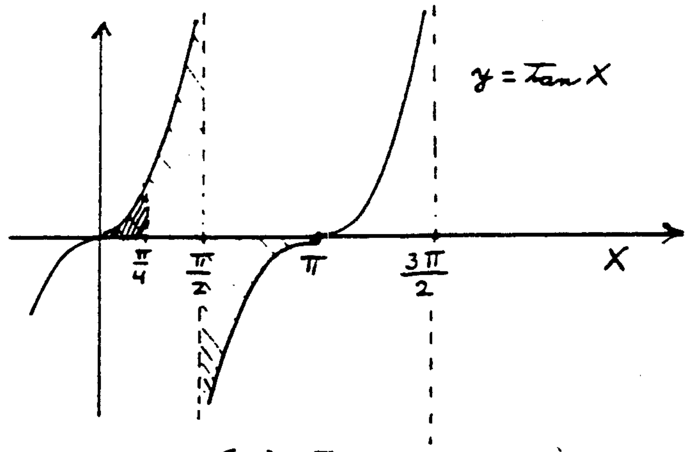
Thus

$$\oint_{|z|=4} \sin^2\left(\frac{1}{z}\right) dz = 0.$$

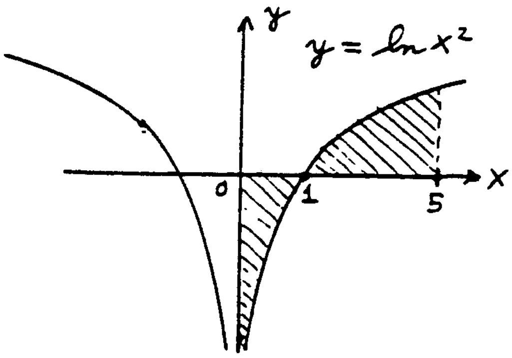
15/



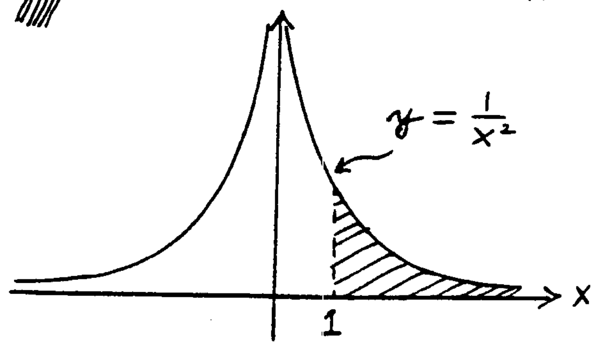
(a) IMPROPER



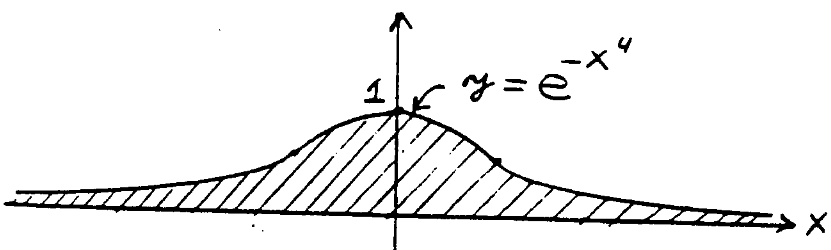
(b) PROPER, (c) IMPROPER



(d) IMPROPER



(e) IMPROPER



(f) IMPROPER

16/ (a) CONVERGES BECAUSE $p = -5 < -1$.

(b) DIVERGES BECAUSE $\frac{x}{5+x^2} \approx \frac{1}{x}$ FOR LARGE x.

(c) LET $u = x-2$, THEN WE GET $\int_0^3 u^{-\frac{1}{2}}\,du$ WHICH CONVERGES BECAUSE $p = -\frac{1}{2} > -1$.

(d) CONVERGES BECAUSE $\left|\frac{\sin x}{3+x^3}\right| < \frac{1}{x^3}$.

16/ (continued)

(e) LET $u = x-1$ AND GET $\int_{-1}^{0} \frac{du}{u}$ WHICH DIVERGES,

(f) NEAR $x=0$, $\sin x \approx x$. THUS $\frac{1}{\sin x} \approx \frac{1}{x}$ NEAR $x=0$,
THEREFORE THE INTEGRAL DIVERGES.

17/ (a) SINCE $\cos x$ VARIES BETWEEN -1 and $+1$, $e^{\cos x}$ VARIES BETWEEN e^{-1} AND e^{1}. THUS

$$\frac{e^{\cos x}}{x} \geq \frac{e^{-1}}{x}$$ AND SINCE $\int_2^{\infty} \frac{e^{-1}}{x}\,dx$ DIVERGES,

OUR INTEGRAL ALSO DIVERGES.

(b) e^{-2t} DOMINATES, OUR INTEGRAL CONVERGES SINCE $\int_0^{\infty} e^{-2t}\,dt$ CONVERGES,

(c) e^{-x} DOMINATES AND THE INTEGRAL CONVERGES.

(d) FOR SMALL x, $$\frac{1}{\sqrt{x+x^3}} = \frac{1}{\sqrt{x}\sqrt{1+x^2}} \approx \frac{1}{\sqrt{x}}$$

THUS NEAR $x=0$, OUR INTEGRAND BEHAVES LIKE $x^{-\frac{1}{2}}$
AND THE AREA IS FINITE HERE,

FOR LARGE x, $$\frac{1}{\sqrt{x+x^3}} \approx \frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$$

THUS THE AREA IS FINITE FOR LARGE x ALSO,
THUS THE INTEGRAL CONVERGES.

(e) SET $4-x=u$ AND GET $\int_0^4 u^{-\frac{1}{2}}\,du$ WHICH CONVERGES.

(f) $\frac{1}{e^x+6} \approx e^{-x}$ FOR LARGE x. THUS IT CONVERGES.

17/ (continued)

(g) x^x DOMINATES AND THUS $\dfrac{e^{2x}}{x^x+8} \approx x^{-x}$ FOR LARGE x. THE INTEGRAL <u>CONVERGES</u>.

(h) FOR LARGE z, $\dfrac{1}{(x^3+z^3)^{5/3}} \approx z^{-5}$

THUS IT <u>CONVERGES</u>.

(i) FOR LARGE x, $\ln x > e$, THUS $(\ln x)^x > e^x$.

THEREFORE $\dfrac{x^4}{(\ln x)^x} < \dfrac{x^4}{e^x}$. THE INTEGRAL <u>CONVERGES</u>.

(j) $\tan x = \dfrac{\sin x}{\cos x}$. NEAR $x = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$ AND $\cos x \approx \frac{\pi}{2} - x$. THUS NEAR $x = \frac{\pi}{2}$, $\tan x \approx \dfrac{1}{\frac{\pi}{2}-x}$.

THE INTEGRAL THEREFORE <u>DIVERGES</u>.

18/ The solution here is similar to the solution of Example 1, only now the integrand

$$\frac{1}{a^2+z^2} = \frac{1}{(z+ia)(z-ia)}$$

has a singularity at $z = ia$ inside the contour. Thus

$$\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \oint \frac{dz}{a^2+z^2} = 2\pi i \operatorname{Res}\left(\frac{1}{a^2+z^2},\ ia\right)$$

$$= 2\pi i \frac{1}{ia+ia} = \frac{2\pi i}{2ia} = \boxed{\frac{\pi}{a}}.$$

19/ The solution is similar to the solution of Example 2, only now the integrand

$$\frac{1}{a^4+z^4}$$ has singularities at

$z = a e^{i\frac{\pi}{4}}$ and $z = a e^{\frac{3\pi i}{4}}$. Thus

$$\int_{-\infty}^{\infty} \frac{dx}{a^4+x^4} = 2\pi i\left[\operatorname{Res}\left(\frac{1}{a^4+z^4},\ a e^{i\frac{\pi}{4}}\right) + \operatorname{Res}\left(a e^{\frac{i3\pi}{4}}\right)\right].$$

$$\operatorname{Res}\left(a e^{i\frac{\pi}{4}}\right) = \lim_{z\to a e^{i\frac{\pi}{4}}} \frac{z - a e^{i\frac{\pi}{4}}}{a^4+z^4}$$

$$= \left.\frac{1}{4z^3}\right|_{z=a e^{i\frac{\pi}{4}}} = \frac{e^{-i\frac{3\pi}{4}}}{4a^3}$$

$$\operatorname{Res}\left(a e^{\frac{3\pi i}{4}}\right) = \left.\frac{1}{4z^3}\right|_{z=a e^{\frac{3\pi i}{4}}} = \frac{e^{-\frac{i\pi}{4}}}{4a^3}$$

$$\int_{-\infty}^{\infty} \frac{dx}{a^4+x^4} = 2\pi i\left[\frac{e^{-i\frac{3\pi}{4}} + e^{-\frac{i\pi}{4}}}{4a^3}\right]$$

$$= \frac{2\pi i}{4a^3}\left[-\sqrt{2}\, i\right] = \boxed{\frac{\pi}{\sqrt{2}\, a^3}}.$$

20/ We use the same contour as before. The integrand $\frac{1}{(1+z^2)^2} = \frac{1}{(z-i)^2(z+i)^2}$ has a pole of order two at $z=i$. Thus

20/ (continued)

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \oint \frac{dz}{(1+z^2)^2} = 2\pi i \operatorname{Res}\left(\frac{1}{(1+z^2)^2}, i\right)$$

$$= 2\pi i \left\{ \frac{d}{dz} (z+i)^{-2} \right\}\Bigg|_{z=i} = 2\pi i \left\{ -2 (z+i)^{-3} \right\}\Bigg|_{z=i}$$

$$= \frac{2\pi i(-2)}{(2i)^3} = \frac{-4\pi i}{-8i} = \frac{\pi}{2},$$

Since $\int_0^{\infty} = \frac{1}{2}\int_{-\infty}^{\infty} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$.

21/ Use the contour as before. The integrand

$\dfrac{z^2}{(1+z^2)^2} = \dfrac{z^2}{(z-i)^2(z+i)^2}$ has a singularity of order two at $z = i$ inside. Thus

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)^2} = 2\pi i \operatorname{Res}\left(\frac{z^2}{(1+z^2)^2}, i\right)$$

$$= 2\pi i \left\{ \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right] \right\}\Bigg|_{z=i}$$

$$= 2\pi i \left\{ -2(z+i)^{-3} z^2 + 2z(z+i)^{-2} \right\}\Bigg|_{z=i}$$

$$= 2\pi i \left\{ \frac{2}{-8i} + \frac{2i}{-4} \right\} = 2\pi i \left\{ \frac{i}{4} - \frac{i}{2} \right\}$$

$= \frac{\pi}{2}$. Thus $\int_0^{\infty} = \frac{1}{2}\int_{-\infty}^{\infty} = \boxed{\frac{\pi}{4}}$.

22/ As in Example 3 we have

$$I = \int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^4} = \lim_{R\to\infty}\left\{\frac{1}{2}\int \frac{e^{iz}\,dz}{1+z^4} + \frac{1}{2}\int \frac{e^{-iz}\,dz}{1+z^4}\right\}$$

But $z^4+1 = \left(z-e^{i\frac{\pi}{4}}\right)\left(z-e^{i\frac{3\pi}{4}}\right)\left(z-e^{-i\frac{\pi}{4}}\right)\left(z-e^{-\frac{3\pi i}{4}}\right)$.

Thus

$$I = 2\pi i\left\{\frac{1}{2}\,\mathrm{Res}\left(\frac{e^{iz}}{1+z^4},\ e^{i\frac{\pi}{4}}\right) + \frac{1}{2}\,\mathrm{Res}\left(\frac{e^{iz}}{1+z^4},\ e^{i\frac{3\pi}{4}}\right)\right\}$$

$$-2\pi i\left\{\frac{1}{2}\,\mathrm{Res}\left(\frac{e^{-iz}}{1+z^4},\ e^{-i\frac{\pi}{4}}\right) + \frac{1}{2}\,\mathrm{Res}\left(\frac{e^{-iz}}{1+z^4},\ e^{-i\frac{3\pi}{4}}\right)\right\}$$

Note that the minus sign is used because is in the negative sense.

To find the residue of $\dfrac{e^{\pm iz}}{1+z^4}$ at z_0 we use

$$\lim_{z\to z_0}\frac{(z-z_0)e^{\pm iz}}{1+z^4} = \left.\frac{e^{\pm iz} \pm i(z-z_0)e^{\pm iz}}{4z^3}\right|_{z=z_0}$$

$$= \frac{e^{\pm i z_0}}{4z_0^3}$$

Thus we have

$$I = \frac{\pi i}{4}\left\{e^{-\frac{3\pi i}{4}}\,e^{i e^{i\frac{\pi}{4}}} + e^{-\frac{\pi i}{4}}\,e^{i e^{i\frac{3\pi}{4}}} - e^{\frac{3\pi i}{4}}\,e^{-i e^{-i\frac{\pi}{4}}} - e^{\frac{\pi i}{4}}\,e^{-i e^{-i\frac{3\pi}{4}}}\right\}$$

22/ (continued)

This last expression is of the form

$$I = \frac{\pi i}{4}\left\{ a + b - \bar{a} - \bar{b} \right\}$$

where the bar denotes "complex conjugate".

Recall that $a - \bar{a} = 2i\,\mathrm{Im}(a)$. Thus

$$I = \frac{\pi i}{4}\left\{ 2i\,\mathrm{Im}\left(e^{-\frac{3\pi i}{4}} e^{ie^{i\frac{\pi}{4}}}\right) + 2i\,\mathrm{Im}\left(e^{-\frac{\pi i}{4}} e^{ie^{i\frac{3\pi}{4}}}\right)\right\}$$

$$I = \frac{\pi i}{4}\left\{ 2i\,\mathrm{Im}\left(\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}\right) + 2i\,\mathrm{Im}\left(\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{i\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}\right)\right\}$$

$$I = \frac{-\pi}{2}\left\{ \mathrm{Im}\left(\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)\right) + \mathrm{Im}\left(\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)\right)\right.$$

$$I = -\frac{\pi}{2}\,\mathrm{Im}\left(-\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)\right) e^{-\frac{1}{\sqrt{2}}}$$

This last expression is of the form

$$I = -\frac{\pi}{2}\,\mathrm{Im}\left(-b + \bar{b}\right) e^{-\frac{1}{\sqrt{2}}}$$ which simplifies to

22/ (continued)

$$I = +\frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}} \operatorname{Im}\left(b - \bar{b}\right) = \frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}} \operatorname{Im}\left(2 i \operatorname{Im}(b)\right)$$

$$= \frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}}\, 2 \operatorname{Im}(b)$$

$$= \pi\, e^{-\frac{1}{\sqrt{2}}} \operatorname{Im}\left(\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\cos\frac{1}{\sqrt{2}} + i \sin\frac{1}{\sqrt{2}}\right)\right)$$

$$= \pi\, e^{-\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2}} \left(\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right) = \boxed{\pi e^{-\frac{1}{\sqrt{2}}} \sin\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\right)}$$

23/ $I = \int_0^\infty \frac{\cos x}{(1+x^2)^2}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\, dx$

This problem is similar to Example 3, only now the poles at $z = \pm i$ are of order two.

$$I = \frac{1}{2} \left\{ \frac{1}{2} \int_{\circlearrowleft} \frac{e^{iz}\, dz}{(1+z^2)^2} + \frac{1}{2} \int_{\circlearrowright} \frac{e^{-iz}\, dz}{(1+z^2)^2} \right\}$$

Since $(1+z^2)^2 = (z-i)^2 (z+i)^2$ we have

$$I = \frac{1}{4}\, 2\pi i \left\{ \operatorname{Res}\left(\frac{e^{iz}}{(1+z^2)^2}, i\right) - \operatorname{Res}\left(\frac{e^{-iz}}{(1+z^2)^2}, -i\right) \right\}$$

$$I = \frac{\pi i}{2} \left\{ \frac{d}{dz}\left[\frac{e^{iz}}{(z+i)^2}\right]\Bigg|_{z=i} - \frac{d}{dz}\left[\frac{e^{-iz}}{(z-i)^2}\right]\Bigg|_{z=-i} \right\}$$