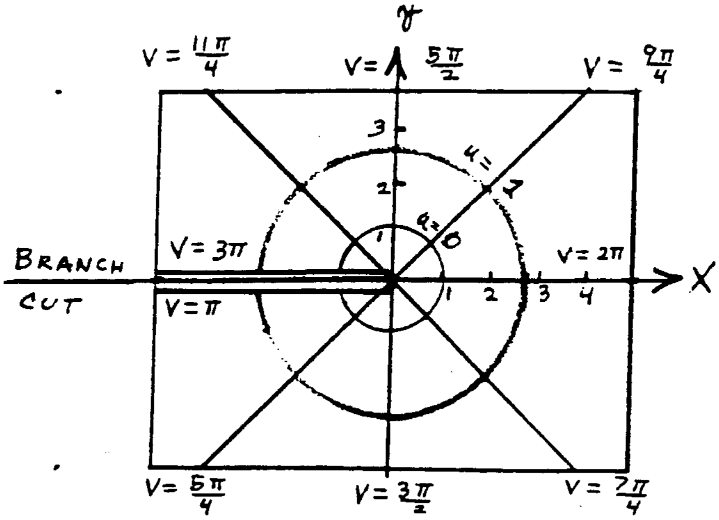
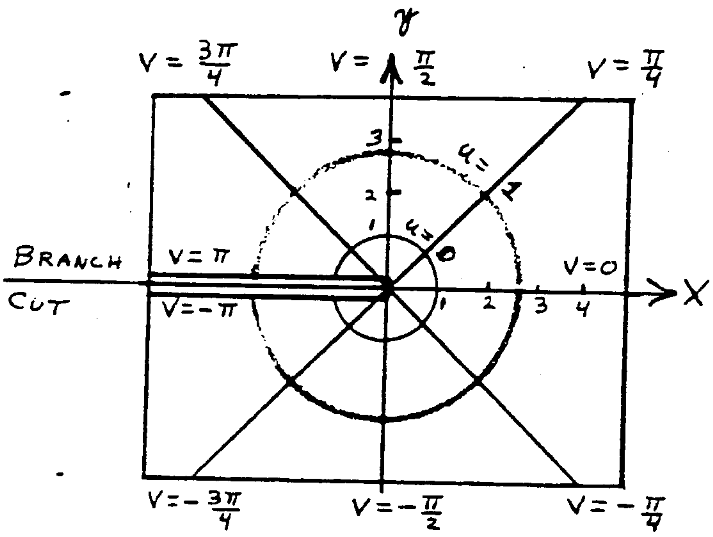
Figure 2.11 Branches for the function $w = \log z$. Only three of the infinitely many branches are shown.

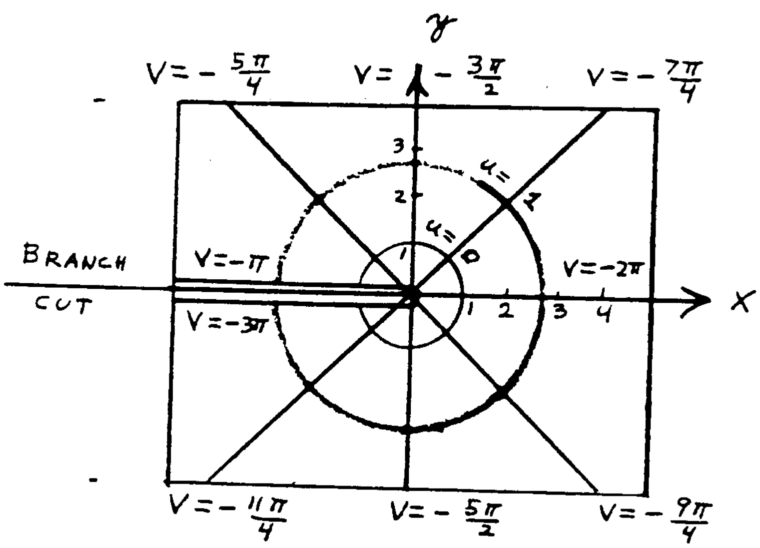


$\pi < v \leq 3\pi$



$-\pi < v \leq \pi$

(Principal Branch, $w = \text{Log } z$)



$-3\pi < v \leq -\pi$

Figure 2.11 shows one method of introducing branches for $w = \log z$. The negative real axis has been selected as the branch cut. Notice that lines of constant u and v have been plotted over each z - plane (in contrast to lines of constant ρ and ϕ used previously).

We notice in particular that :

1. There are infinitely many distinct branches of $w = \log z$, since there are infinitely many values of $\log z$ for each z.
2. The branch points are $z = 0$ and $z = \infty$.
3. The branch line is, to a certain degree, arbitrary. We could have selected the negative y-axis as the branch line, or any other line from $z = 0$ to $z = \infty$. (Review problem 27)
3. The function $w = \log z$ is discontinuous at the branch line, because the values of v jump by 2π as we cross the negative x-axis.

We shall arbitrarily call the branch for which $-\pi < v \leq \pi$ the "principal branch" of logarithum of z, and denote it by $\text{Log } z$.

Example

Find (a) $\text{Log } i$ and (b) $\text{Log}(-e)$.

Solution

(a) Since we want the "principal value" of the logarithum, we take $i = e^{i\pi/2}$. The value of θ was selected as $\pi/2$ since this value is in the correct range for the principal branch which is $(-\pi, \pi]$. From (3) we have $\text{Log } i = i\pi/2$.

(b) We select $-e = e\, e^{i\pi}$, and get from (3) that $\text{Log}(-e) = \log e + i\pi = 1 + i\pi$.

We recall the following rules for logarithums from study in elementary courses:

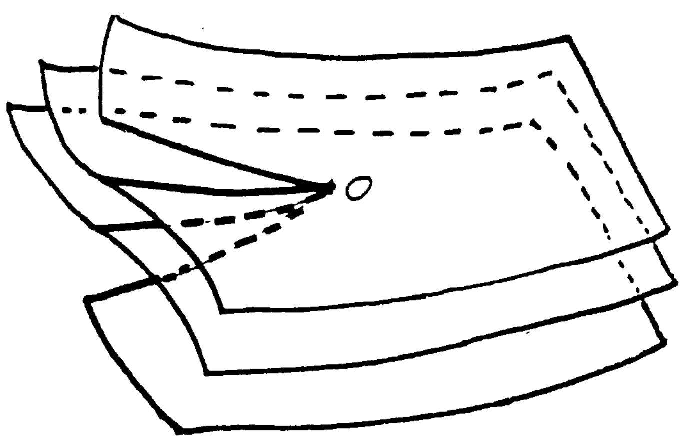
(4) $\log zw = \log z + \log w$

(5) $\log z/w = \log z - \log w$

(6) $\log z^p = p \log z$.

Are these three relations true for complex values of the variables ? Since log z for complex z was defined in a natural way, we at once anticipate that these relations should be valid. However, when we reflect upon the fact that each of the logarithums in these relations can assume infinitely many values for each fixed z and w , we do indeed anticipate some difficulty. What is true is the following : " For appropriately selected branches of each of the logarithums occuring, the relations (4), (5) and (6) are valid".

We can construct a Riemann surface for $w = \log z$ from the various branches shown in Figure 2.11. The method is the same as that used in the previous section for $w = \sqrt{z}$, only now we have infinitely many sheets for the surface,rather that two. Also, we never join the <u>first</u> and <u>last</u> sheets together (since there is no first or last sheet) !



We will now define z^p, where both z and p are complex numbers. Since by definition

$$c = e^{\log c},$$

we replace c by z^p and get

$$z^p = e^{\log z^p}.$$

Using (6) this last relation becomes

(7) $$z^p = e^{p \log z}$$

which we will use as our defining relation for z^p.

Example

Find all values of i^i.

Solution

From (7) we get $i^i = e^{i \log i} = e^{i(\pi/2 + 2\pi n)i} = e^{-\pi/2 - 2\pi n}$, where $n = 0, \pm 1, \pm 2, \ldots$.

We see that z^p has infinitely many values for each z and p. We define the "principal value" of z^p to be

(8) $$z^p = e^{p \operatorname{Log} z} \quad (\text{principal value}).$$

Problems:

34. Find (a) Log 1, (b) Log(-1), (c) Log(-ei), (d) Log(1+i).

35. Find all values of (a) 1^2, (b) $(-1)^2$, (c) $(-ei)^2$, (d) $(1+i)^{1+i}$

36. Find the "principal value" of (a) 1^2, (b) $(-1)^2$, (c) $(-ei)^2$, (d) $(1+i)^{1+i}$.

Review problems for Chapter 2

1. Find the values of each of the following expressions using only the appropriate analytic definitions. Express the results in Cartesian form. (a) $e^{1 + i\pi/4}$, (b) $\sin(\pi/3 + i)$, (c) $\cosh(i\pi/3)$, (d) $\mathrm{Log}(-2 + 2\sqrt{3}\, i)$, (e) $(-2+2\sqrt{3}\, i)^{i}$.

2. Find u and v for $w = (z-1)^3$, where $w = u+iv$.

3. Map the region $\{ z \mid 0 < \arg(z) < \pi/4 , \text{ and } |z| < 2 \}$ onto the w-plane by (a) $w = z^2$, (b) $w = 1/z$, (c) $w = \mathrm{Log}\ z$.

4. Map the region $\{z \mid 1 < \mathrm{Re}(z) < 2, \text{ and } \pi/2 < \mathrm{Im}(z) < \pi\}$ onto the w plane by $w = e^z$.

5. (a) Describe the Riemann surface for $w = \sqrt{z-1}$. Where are the branch points ? (b) How many sheets are needed to form the Riemann surface for $\sqrt[6]{z}$?

6. Find all values of z such that $e^z = 1 - i$.

SUPPLEMENTARY PROBLEMS

2.1.1 Let $z = x + i y$ and $w = u + i v$. Find u and v as functions of x and y.

(a) $w = 2 z^2 - 3$; (b) $w = (z - 1)(z + 3)$;

(c) $w = z^3 - 3 z^2 + 3 z - 1$, (d) $w = \frac{1}{(z - 1)^3}$.

2.1.2 Let $z = r e^{i\theta}$ and $w = \rho e^{i\phi}$. Find ρ and ϕ as functions of r and θ . (a) $w = \frac{1}{z^2}$; (b) $w = z^5$; (c) $w = 2 z^2 - 3$

(d) $w = \frac{1}{(z - 1)^3}$.

2.1.3 Use Figure 2.1 to estimate the value of z^2 at the following points: (a) $z = 2 + 2 i$, (b) $z = - 1 - 2 i$, (c) $z = 2 i$, (d) $z = - 2 + i$.

2.1.4 Use Figure 2.1 to estimate the values of z associated with the following values of w given by the equation $w = z^2$: (a) $w = - 4$, (b) $w = 2 + 4 i$, (c) $w = - 4 + 4 i$, (d) $w = z + 5 i$.

2.1.5 From Figure 2.2 , estimate the values of the function $w = z^2$ at:

(a) $z = 2i$, (b) $z = 1.75 - i$, (c) $z = 0.7 - 1.2i$.

2.1.6 From Figure 2.2 , determine the values of z associated with each of the following values of w governed by the equation $w = z^2$. (For each value of w , there are two values of ϕ , one in the range $-2\pi < \phi \leq 0$, and the other in $0 < \phi \leq 2\pi$.)

(a) $w = 1 - \sqrt{3}\,i$, (b) $w = 4\,e^{-i\,\pi/3}$, (c) $w = 2$.

2.1.7 Map the following regions onto the w - plane under the mapping $w = z^2$.

(a) $x^2 + y^2 < 4$, $0 < \arg(z) < \frac{\pi}{4}$

(b) $0 < x < y$ in the first quadrant

(c) $\text{Im}(z) < 0$

(d) $\frac{\pi}{4} < \arg(z) < \frac{\pi}{2}$

(e) $0 < xy < 1$ in the third quadrant

2.1.8 Determine the regions in the z - plane which corresponds to the following regions in the w - plane under the mapping $w = z^2$.

(a) $0 < u < 1$, $0 < v < 1$

(b) $0 < \arg(w) < \pi$ (c) $-1 < u < 1$, $4 < v < 6$

2.1.9 Let $w = u + i v = \dfrac{z - 1}{z + 1}$.

Show that $u = \dfrac{x^2 + y^2 - 1}{(x + 1)^2 + y^2}$ and $v = \dfrac{2y}{(x + 1)^2 + y^2}$.

Also show that the level lines u = constant are circles with center on the real axis passing through $(-1, 0)$, and that the lines v = constant are circles with center on $x = -1$ and passing through $(-1, 0)$. Construct a contour map similar to Figure 2.1 illustrating these level lines over the complex z - plane.

2.1.10 Construct a contour map for the function $w = \dfrac{z - 1}{z + 1}$ similar to Figure 2.2 in which lines of constant ρ and constant ϕ are sketched over the complex z - plane. (The lines of constant ρ are circles with centers on the real axis. The lines of constant ϕ are circular arcs with centers on the imaginary axis, and which begin and end at the two points $(-1, 0)$ and $(1, 0)$.)

2.1.11 Use the contour map constructed in problem 2.1.9 to demonstrate that any circle in the z - plane maps onto a circle in the w-plane under the mapping $w = \frac{z-1}{z+1}$. (In fact, the general bilinear function $w = \frac{az+b}{cz+d}$ is circular.)

2.2.1 Using the function $w = e^z$ and Figure 2.5 , map the following regions onto the complex w - plane :

(a) $-\pi < y < 0$; (b) $x < 0$, $0 < y < \frac{\pi}{2}$;

(c) $0 < x < 2$, $\pi < y < 2\pi$.

2.2.2 Find the points on the z - plane which map onto the third quadrant of the w - plane under the mapping $w = e^z$.

2.2.3 Prove that $|e^z| = e^x$ and $|e^{iz}| = e^{-y}$.

2.2.4 Prove that there is no value of z such that $e^z = 0$.

2.2.5 Find all values of z such that (a) $e^z = 1$, (b) $e^{2z} = i$,

answers (a) $z = 2\pi n i$, (b) $z = \frac{\pi}{4} + \pi n$, $n = 0, \pm 1, \pm 2, \ldots$.

2.3.1 Compute sin i and check the result with Figure 2.7 .

2.3.2 Prove that (a) $\sin(-z) = -\sin z$, (b) $\cos(-z) = \cos z$.

2.3.3 Prove that $1 + \tan^2 z = \sec^2 z$

2.3.4 Find $u(x, y)$ and $v(x, y)$ such that $w = u + iv = \sin 2z$

2.3.5 Prove that $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

2.4.1 Find (a) $\sinh(-\pi i)$, (b) $\coth(-\pi i)$, (c) $\cosh(-\pi i)$,
(d) $\cosh(1 + i)$

2.4.2 Prove that (a) $\sinh(-z) = -\sinh z$, (b) $\cosh(-z) = \cosh z$,
(c) $\tanh(-z) = = \tanh z$.

2.4.3 Find all values of z such that $\sinh z = o$.

2.4.4 Find all values of z such that $\tanh z = o$.

2.5.1 Find all values of $\sqrt{4\sqrt{3} - 4i}$.

2.5.2 Construct a Riemann surface for the function $\sqrt{z + 3}$.

2.5.3 Construct a Riemann surface for the function $\sqrt[3]{z + 3}$.

2.5.4 Construct a Riemann surface for the function $\sqrt{z^2 - 1}$.

2.6.1 Find (a) Log e , (b) Log e i , (c) Log ($\sqrt{3}$ + i),

(d) Log (-2 π i) .

answers (a) 1 , (b) $1 + \frac{\pi i}{2}$, (c) Log 2 + $\frac{\pi i}{6}$,

(d) Log $2\pi - \frac{\pi i}{2}$.

2.6.2 Find all values of (a) 2^2 , (b) $(-2)^2$, (c) $(ei)^2$.

2.6.3 Find the principal value of (a) 2^2 , (b) $(-2)^2$, (c) $(ei)^2$.

2.6.4 Let $w = \sin^{-1} z$. (a) Show that $e^{2iw} - 2i\, e^{iw} - 1 = 0$.

(b) Show that $\sin^{-1} z = -i \log (iz + \sqrt{1 - z^2})$.

2.6.5 Show that $\cos^{-1} z = -i \log (z + \sqrt{z^2 - 1})$.

2.6.6 Show that $\tan^{-1} z = \frac{1}{2i} \log \left[\frac{1 + iz}{1 - iz}\right]$.

2.6.7 Show that $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$.

2.6.8 There are several places in the following manipulations where the steps are questionable. Find them.

$$
\begin{aligned}
1 &= \sqrt{1} \\
&= \sqrt{(-1)(-1)} \\
&= \sqrt{-1}\,\sqrt{-1} \\
&= i \cdot i \\
&= -1
\end{aligned}
$$

Thus we have $1 = -1$?