<u>The integral from D to E</u>

This integral resembles the previous one, only now we are on the opposite edge of the branch cut where $\theta = 2\pi$ and not 0. Thus the integral becomes (when we set $z = r\, e^{2\pi i}$)

$$\int_R^{\varepsilon} \frac{(r^{-1/2}\, e^{-\pi i})\,(dr\, e^{2\pi i})}{r\, e^{2\pi i} + 1}$$

Since $e^{2\pi i} = 1$, and $e^{-\pi i} = -1$, and since ε approaches 0 and R approaches infinity, this last integral tends to

$$(5) \qquad \int_{E \leftarrow D} = \int_R^{\varepsilon} \frac{-\, r^{-1/2}\, dr}{r + 1} \longrightarrow I$$

because the minus sign and the inverted order of integration cancel.

<u>The integral over the large circle BCD.</u>

We now show why the integral over the large circle tends to zero as R grows . On this circle we have

$$z = R\, e^{i\theta} \quad , \quad dz = i\, R\, e^{i\theta}\, d\theta \text{ (remember } R \text{ is constant)}$$

$$z^{-1/2} = R^{-1/2}\, e^{-i\theta/2}, \text{ and}$$

$$z + 1 \approx z = R\, e^{i\theta} \quad \text{(because } R \text{ is so large)}$$

Substituting these values into the integral we get

$$\int_{C \circlearrowleft^{B}_{D}} \longrightarrow \int_0^{2\pi} \frac{R^{-1/2}\, e^{-i\theta/2}\quad iR\, e^{i\theta}\, d\theta}{R\, e^{i\theta}} = i\, R^{-1/2} \int_0^{2\pi} - \; e^{-i\theta/2}\, d\theta$$

$$= \frac{\text{constant}}{\sqrt{R}}$$

Clearly this last expression tends to zero as R tends to infinity.

The integral over the small circle EFA

We now show that the integral over the small circle tends to zero as the radius ε tends to zero . On this circle we have

$$z = \varepsilon\, e^{i\theta} \quad , \quad dz = i\, \varepsilon\, e^{i\theta}\, d\theta \ ,$$

$$z^{-1/2} = \varepsilon^{-1/2}\; e^{-i\theta/2} \quad \text{and}$$

$$z + 1 \approx 1 \quad \text{because} \quad z \text{ is so very small .}$$

Substituting these values into the integral we have

$$\int_{F \circlearrowleft^{A}_{E}} \rightarrow \int_{2\pi}^{0} (\varepsilon^{-1/2}\; e^{-i\theta/2})\; (\, i\, \varepsilon\, e^{i\theta}\, d\theta\,) = i\, \varepsilon^{1/2} \int_{2\pi}^{0} e^{i\theta/2}\, d\theta$$

$$= (\text{constant}) \sqrt{\varepsilon}$$

Clearly this last expression tends to zero as ε tends to zero.

Now we substitute (3) , (4) and (5) into (2) and get

$$2\pi = I + 0 + I + 0$$

thus we have $I = \pi$.

Remarks

1. In the above example, a critical feature was that the integral over the two sides of the branch cut produced some multiple of the desired integral I . In selecting a branch cut and a contour of integration, we should keep in mind that our real integral I will emerge from the segment or segments of the contour along the branch line.

2. The integral over the large circle tended to zero. Is there a simple way to see this? Yes! The integral is, loosely speaking, the length of the contour times the integrand. Now the length of the contour (circle) is $2\pi R$, and thus the integrand must tend to zero faster than $1/R$ so as to successfully fight of the $2\pi R$ of the contour length. Thus we see that if the integrand, in absolute value, behaves like R^{-2} or $R^{-3/2}$ etc., for large R, then the integral over the large circle will surely tend to zero.

3. The integral over the small circle also tends to zero. The same method just given also helps to see this simply. The integral is, loosely speaking, the product of the length of the contour times the integrand. The length of the contour is now $2\pi\varepsilon$. Thus so long as the integrand does not tend to infinity at the rate $1/\varepsilon$ or faster, the $2\pi\varepsilon$ of thecontour length will dominate and make the integral tend to zero. Examples are integrands like

$$\frac{1}{\sqrt{z}}, \quad \frac{1 + z^5}{\sqrt{z}}, \quad \frac{\sqrt{z}}{2 + z^2}, \quad \ldots .$$

The first of these functions behaves like $\varepsilon^{-1/2}$, the second does also, and the third behaves like $\varepsilon^{1/2}$. Each of these times $2\pi\varepsilon$ (the circles length) tends to zero as ε tends to zero.

Problems

Evaluate the following integrals:

29. $\displaystyle\int_0^\infty \frac{x^{-1/3}\,dx}{1+x}$ 30. $\displaystyle\int_0^\infty \frac{x^{-k}\,dx}{1+x}$ $0 < k < 1$.

31. $\displaystyle\int_0^\infty \frac{x^{-1/2}\,dx}{1+x^2}$ 32. $\displaystyle\int_0^\infty \frac{x^{1/2}\,dx}{(1+x^2)^2}$

Next we consider an integral featuring a logarithmic branch cut.

Example 2

Evaluate $I = \displaystyle\int_0^\infty \frac{\log x\;dx}{a^2+x^2}$.

Solution

Recall that if we set $z = r\,e^{i\theta}$, then

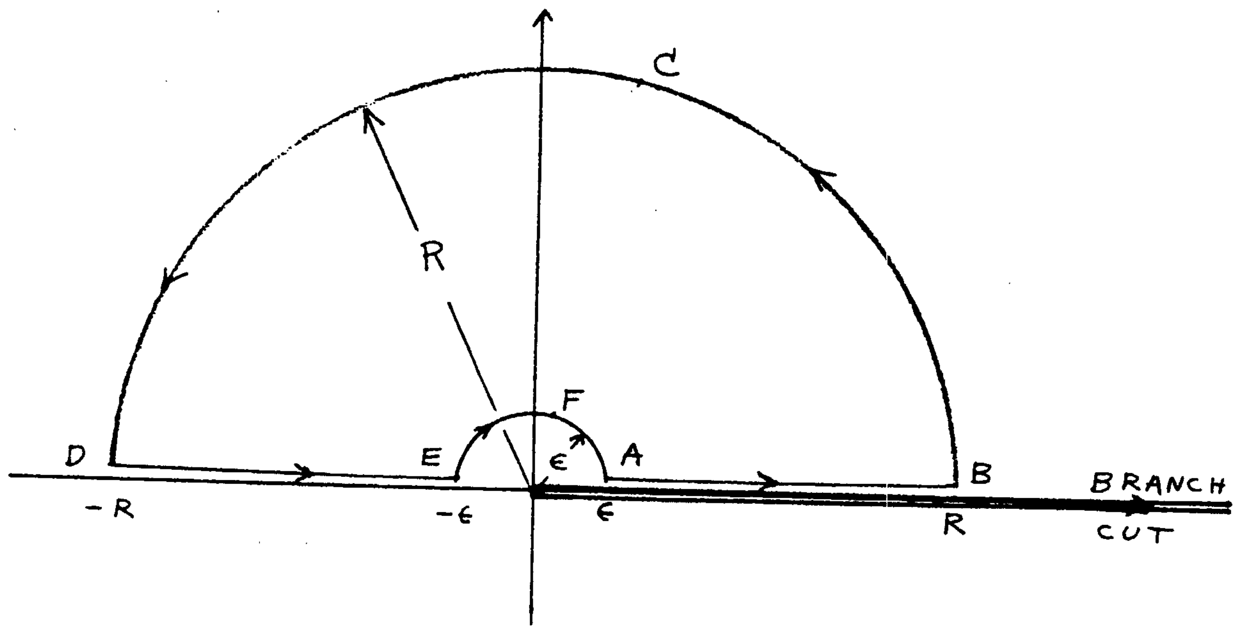
$$\log z = \log r + i\theta \qquad 0 \le \theta < 2\pi \quad .$$

We cannot use the contour of the previous example, for now when we integrate along the lower side of the branch cut we will have $\theta = 2\pi$ and we will get

$$\int_{E \leftarrow D} \rightarrow \int_\infty^0 \frac{\log r + i2\pi}{a^2+r^2}\,dr = -\int_0^\infty \frac{\log r\,dr}{a^2+r^2} - 2\pi i\int_0^\infty \frac{dr}{a^2+r^2} ,$$

Notice that the minus sign in fromt of the integral which is I in this last expression will cause it to cancel the integral over the segment from A to B. Thus the desired integral disappears from the calculation ! Clearly another contour is needed.

The trouble with the contour of the previous example is that the two integrals yielding I are in reverse directions causing them to cancel. Let us now select a contour in which the two integrals involving I will have the same direction.



As before, we examine the integral as a sum of four segments.

$$(1)\quad \int_{\frown} \frac{\log z \; dz}{a^2 + z^2} = \int_{A \to B} + \int_{\substack{\text{large}\\ \text{arc}}} + \int_{D \to E} + \int_{\substack{\text{small}\\ \text{arc}}}$$

The integral over the entire contour

Here we use the Residue Theorem where we observe that there is but one singularity inside the contour, a simple pole at ia. Thus we have

$$(2)\quad \int \frac{\log z \, dz}{a^2 + z^2} = 2\pi i \, \mathrm{Res}\left(\frac{\log z}{a^2 + z^2}, \ ia \right)$$

$$= 2\pi i \left. \frac{\log z}{z + ia} \right|_{z = ia = ae^{i\pi/2}}$$

$$= 2\pi i \, \frac{\log a + i\pi/2}{2ia}$$

$$= \frac{\pi}{a} \log a + i \frac{\pi^2}{2a}$$

The integral over the line segment from A to B

Here $\log z = \log r + i0 = \log r$. Therefore, as ε approaches zero and R approaches infinity we have

$$(3)\quad \int_{A \to B} \longrightarrow I \, .$$

The integrals over the large and small semicircles

In the integral over the large circle we see that the integrand behaves like $R^{-2} \log R$ for large R. Multiplying this by the contour length of πR gives $\pi R^{-1} \log R$ for the growth of the integral as R grows large. We know from the previous section that R^{-1} dominates $\log R$ and thus the integral tends to zero.

In the integral over the small semicircle we see that the integrand behaves like

$$\frac{\log \varepsilon + i\theta}{a^2}$$

for small ε . Multiplying this by the contour length $\pi\varepsilon$ and letting ε tend to zero we see that the integral approaches zero because ε again dominates $\log \varepsilon$.

<u>The integral over the segment from D to E</u>

Over this segment we have $\log z = \log r + i\pi$, and since $z = r e^{i\pi} = -r$, $dz = -dr$. Thus we get

$$\int_{D \to E} \longrightarrow \int_{\infty}^{0} \frac{\log r + i\pi}{a^2 + r^2}(-dr)$$

$$= \int_0^{\infty} \frac{\log r \, dr}{a^2 + r^2} + i\pi \int_0^{\infty} \frac{dr}{a^2 + r^2}$$

$$= I + i\pi \left. \frac{\tan^{-1} r/a}{a} \right|_{r=0}^{r=\infty}$$

(4) $$= I + i\frac{\pi^2}{2a} .$$

Substituting (2), (3) and (4) into (1) we get

$$\frac{\pi}{a}\log a + i\frac{\pi^2}{2a} = I + 0 + I + i\frac{\pi^2}{2a} + 0$$

which finally yields $I = \frac{\pi}{2a}\log a$.

Problems

33. Evaluate $\displaystyle\int_0^\infty \frac{\log x \, dx}{(x^2 + a^2)^2}$.

34. Evaluate $\displaystyle\int_0^\infty \frac{\log x \, dx}{x^4 + 1}$

The next example features two multiple valued factors in the integrand. Careful attention must be paid to the branches of each function to be used.

Example 3

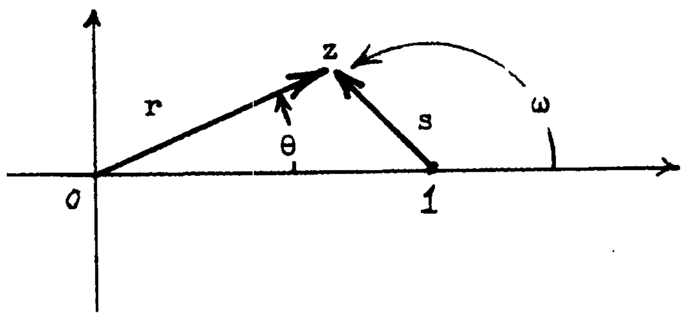
Evaluate $\displaystyle\int_0^1 x^{-1/2}(1 - x)^{-1/2}\, dx$

Solution

The integrand $f(z) = z^{-1/2}$ ~~$(1 - z)$~~ $(z - 1)^{-1/2}$ consists of two factors, and we begin by defining separate single valued branches for each one.

Set $z = r e^{i\theta}$, with $0 \le \theta < 2\pi$ and then

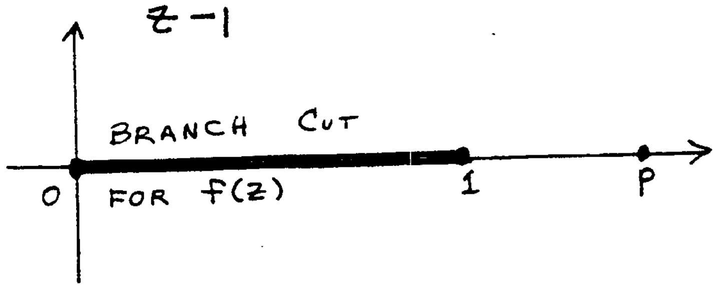
(1) $z^{-1/2} = r^{-1/2} e^{-i\theta/2}$.



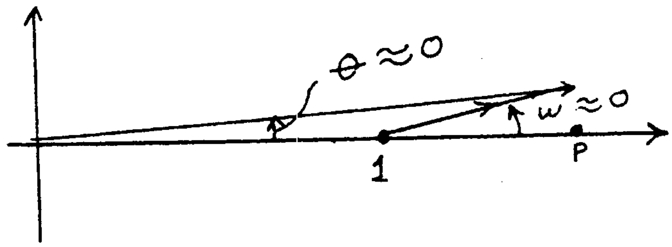
Set $1 - z = s e^{i\omega}$ where $0 \le \omega < 2\pi$. Then

(2) ~~$(1 - z)$~~ $(z - 1)^{-1/2} = s^{-1/2} e^{-i\omega/2}$.

With this selection of branches for the factors of the integrand, the integrand itself, $f(z) = z^{-1/2}(z-1)^{-1/2}$ has a branch cut extending along the real axis from $z = 0$ to $z = 1$. We are not surprised to find the cut along the segment from 0 to 1 because the angle θ is discontinuous here. However, we might also expect the function $f(z)$ to be discontinuous across the real axis from $z = 1$ to $z = +\infty$ because both the angles θ and ω are discontinuous across this segment. To see why $f(z)$ does not change value when we cross this segment of the real axis we look at values of $f(z)$ just above the point P in the figure and values just below this point.

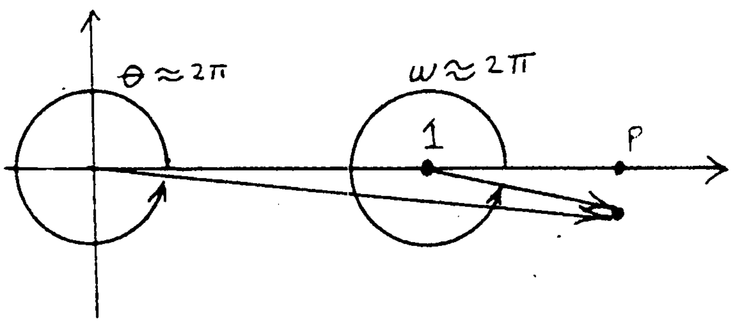


Just above P, both θ and ω are zero and thus $f(P) = r^{-1/2}\, s^{-1/2}$, a real value.



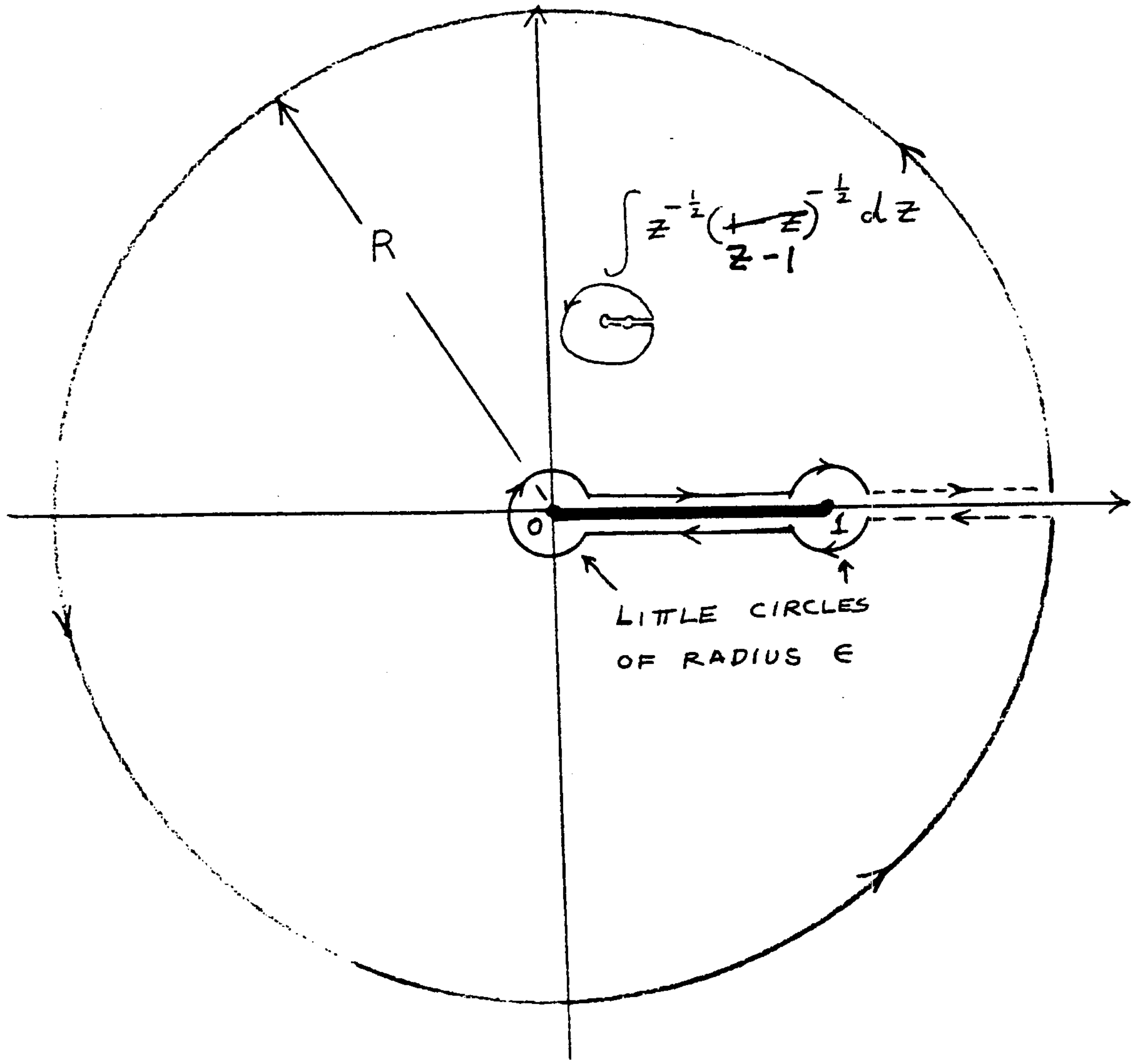
Now, for the value of $f(P)$ just below P we have both θ and ω equal to 2π. We get using (1) and (2)

$$f(P) = r^{-1/2}\, e^{-i\pi}\, s^{-1/2}\, e^{-i\pi}$$
$$= r^{-1/2}\, s^{-1/2}$$



which is the same value that we had just above P. Thus there is no discontinuity across the x-axis beyound the point x=1.

Now examine the contour shown.



We notice at once that the two integrals along the dotted portion of the contour cancel each other since they are in opposite directions and there is no branch cut separating them. Thus they can be ignored. Thus we have

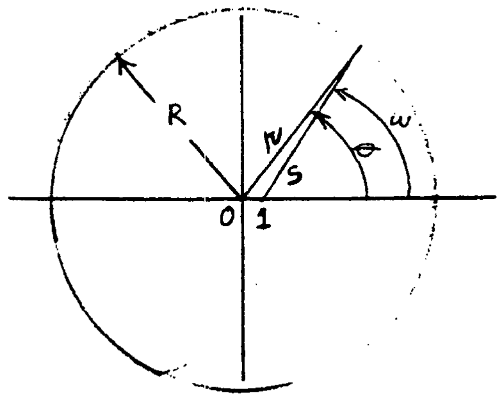
$$(3) \quad \oint z^{-1/2}(z-1)^{-1/2}dz = \underset{\text{large circle}}{\int} + \underset{\text{above branch}}{\int_{\epsilon}^{1-\epsilon}} + \underset{\text{below branch}}{\int_{1-\epsilon}^{\epsilon}} + \underset{\text{two small circles}}{\int + \int}$$

The integral over the entire contour

The integral on the left side of (3) is zero by Cauchy's integral theorem since the integrand has no singularities inside the contour.

The integral over the large circle

On the large circle, where we assume the radius R is quite large, we have $z = R\, e^{i\theta}$, and thus $dz = i\, R\, e^{i\theta}\, d\theta$. Since R is very large, $r \approx R$ and $s \approx R$. Also $\omega \approx \theta$. therefore we have



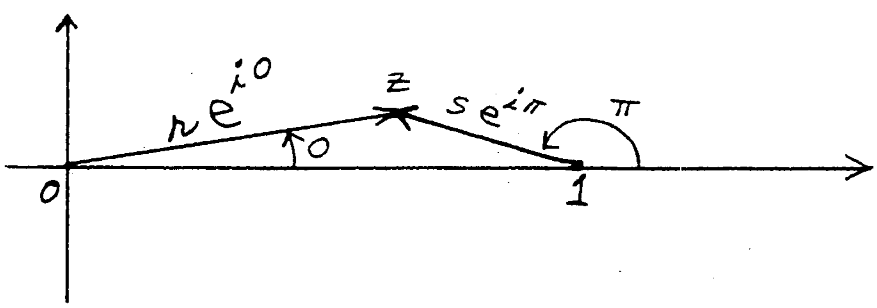
$$(4) \qquad \int_{\substack{\text{large}\\ \text{circle}}} \longrightarrow \int_0^{2\pi} (R^{-1/2}\, e^{-i\theta/2})(R^{-1/2}\, e^{-i\theta/2})\, i\, R\, e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} d\theta = 2\pi i \quad .$$

Thus as R approaches infinity, the integral over the large circle approaches the value $2\pi i$.

The integral from 0 to 1 above the branch cut

Since $\theta = 0$ and $\omega = \pi$ on this segment we have from (1) and (2)



$$z^{-1/2}(z-1)^{-1/2} = r^{-1/2} \quad s^{-1/2} \, e^{-i\pi/2}$$

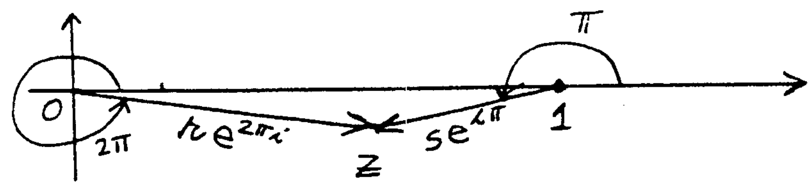
$$= x^{-1/2} \quad (1 - x)^{-1/2} \, (-i)$$

Since $dz = dx$ we see that as ε tends to zero, the integral on the straight line segment abpve the branch cut approaches

(5) $$-i \int_0^1 x^{-1/2}(1 - x)^{-1/2} \, dx \quad = \quad - i \, I \, .$$

<u>The integral from 1 to 0 below the branch cut</u>

On this segment of the contour $\theta = 2\pi$ and $\omega = \pi$. Thus we have



$$z^{-1/2} \, (z-1)^{-1/2} = r^{-1/2} \quad e^{-i\pi} \quad s^{-1/2} \, e^{-i\pi/2}$$

$$= r^{-1/2} \quad s^{-1/2} \quad i$$

$$= i \quad x^{-1/2} \quad (1-x)^{-1/2} \, .$$

Now again $dz = dx$ and as ε approaches zero the integral below the branch cut approaches

(6) $$i \int_1^0 x^{-1/2}(1-x)^{-1/2} \, dx \quad = \quad - i \, I \, .$$

The integrals over the two small circles

Consider the small circle centered at $z = 0$. A glance at the integrand shows that it behaves like $\varepsilon^{-1/2}$ when z is on this circle. Since the product of $\varepsilon^{-1/2}$ and the length of the contour $2\pi\varepsilon$ gives $2\pi\varepsilon^{1/2}$, we see that the integral approaches zero as ε approaches zero.

The integral centered at $z = 1$ vanishes for the same reason.

Substituting (4), (5) and (6) into (3) we get

$$0 = 2\pi i - i\,I - i\,I + 0 + 0$$

Therefore $I = \pi$.

Problem

35. Evaluate $\int_0^1 x^{-2/3}(1 - x)^{-1/3}\,dx$.

Review Problems for Chapter 6

1. Use Cauchy's integral formula to find $\oint_{|z|=2\pi} \frac{\sin z \; dz}{(z-\pi)^5}$

2. Find the residue of each of the following functions at the indicated points.

(a) $(a^2 + z^2)^{-4}$ at ia

(b) $\frac{z}{z^4 + 6z^2 + 1}$ at the two points $\pm\sqrt{3 \mp 2\sqrt{2}}\, i$

(c) $\frac{z^{-k} \log z}{1 + z}$ at -1, where $z^{-k} = r^{-k} e^{-i\theta k}$ and $\log z = \log r + i\theta$ with $0 \le \theta < 2\pi$.

3. Evaluate $\int_0^\infty \frac{dx}{(a^2 + x^2)^4}$

4. Evaluate $\int_0^\infty \frac{x^{-k} \ln x \; dx}{1 + x}$, $0 < k < 1$.

5. Evaluate $\int_0^{2\pi} \frac{d\theta}{1 + \cos^2\theta}$