

**AN INTUITIVE INTRODUCTION TO COMPLEX
ANALYSIS**

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COMPLEX ANALYSIS

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Preface

After having taught the traditional senior level undergraduate complex variables course many times, and after writing some dozen research papers incorporating the elements of this subject, the author became aware of the need for a "down to earth" presentation of the important applicable features. The development here is intuitive and inductive as opposed to the usual rigorous and deductive presentations. Mathematical maturity is not required of the reader, as no use is made of epsilon delta arguments. The inductive exposition offered here requires that the reader first study in detail specific concrete examples, then he is called upon to "conjecture" general truths based on his experience with special cases. In this way the essential facts needed for a good working knowledge of complex analysis are made to stand out clearly, and the intricacies of the subject are mastered from first hand experience. The only background required of the reader is the usual three semester intuitive level calculus course given at most colleges and universities in the freshman and sophomore years. Even then, it is assumed that the reader has only a very vague appreciation for the more subtle aspects of the calculus such as infinite series and improper integrals.

While a rigorous formulation of the subject is absent from these pages, there is no attempt to "water down" the information needed in practical applications. Indeed, use is made of material and intuitive insights which the author has gleaned from his own research in complex variables which is not usually found in text books. As an example, unusual stress is placed upon actually visualizing specific functions through graphical representations. The exact definition of an analytic function is **not** presented until the fourth chapter, even though the concept is used in the second and third chapters. The student is made to see that he can deal with a concept even though it is not precisely formulated, and that definitions often evolve slowly as experience is gained with special cases. Thus the reader gradually develops a "feel" for this subject.

It is hoped that this intuitive presentation will be of value to a wide audience of readers. It can be used as a text book for the usual one semester undergraduate complex variables course given in the junior or senior year. Since this intuitive presentation proceeds at a considerably faster pace than most rigorous texts, advanced topics not usually given in a one semester course can be included. A glance at the Table of Contents shows this. Mathematical maturity is not required of the student, and even advanced sophomores should be able to profit from this course. If the professor also wishes to introduce a rigorous development of complex analysis, this text can serve as a tool for "anchoring the students feet to the ground" so that they will better appreciate the need for a deductive development. Engineers and physicists usually welcome intuitive developments of advanced mathematics, and this presentation might be of value in a one semester course for them. This book is also intended for self-study. There are many example problems, and every

problem posed for the reader is solved in detail in an appendix. In addition, each chapter is followed by review problems which are also solved in full in the appendix. Students who have taken the traditional course in complex analysis might find that reading this book helps to add concreteness to the general theoretical development they have witnessed.

The format usually used in this book is as follows:

- (1) Simple example problems are formulated and solved.
- (2) Problems for the student to solve, similar to the above examples, are posed to reinforce the ideas in the students mind. All problems are fully solved in Appendix I.
- (3) Brief observations are made on these examples and problems.
- (4) The nature of mathematical principles, as yet unknown to the student, can be conjectured from the experience just obtained in problem solving. The student is asked to answer specific conjectures. Answers and remarks concerning the conjectures are provided in Appendix II.

Often intuitive presentations of mathematical subjects are authored by non-mathematicians. The professional mathematician usually objects to these treatments on the ground that they appear to lie to the serious student. Such presentations often present arguments which are called "proofs", when they are really only heuristic reasoning. It is hoped that this intuitive presentation will not be rejected by mathematicians on these grounds. The author is himself a professional mathematician, and he knows well their reaction to false claims of proof. Here the student is constantly reminded of the deficiencies in the arguments presented. When a concept is only partially formulated, the student is so warned, and not made to feel that it is a final definition. When real proofs and definitions are presented, the student is made aware.

CHAPTER 1

COMPLEX NUMBERS AND COMPLEX ARITHMETIC

The reader has certainly encountered the expression $i = \sqrt{-1}$, the "imaginary unit" before. Its occurrence is probably best remembered from the solution of the quadratic equation

$$a x^2 + b x + c = 0$$

given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $4ac$ is larger than b^2 , we can find no real number equal to x .

We have encountered a new type of number which we call "complex".

The number $3 + 4i$ is such a number. We will now present the arithmetic of complex numbers. We will not present anything like an axiomatic development. That is not our purpose. Rather, we will assume that an arithmetic of complex numbers exists in which most of the rules of the real arithmetic already familiar to us carry over. We will proceed often with a "faith" in the symmetry of mathematics, and in the repetition of plausible patterns. After all, it is not unlikely that much that is already familiar to us in the "real domain" will carry over to the "complex domain".

1.1 Elementary arithmetic

We first consider simple powers of the imaginary unit $i = \sqrt{-1}$.

$$i^0 = 1$$

$$i^1 = i$$

$$i^2 = \sqrt{-1} \sqrt{-1} = -1$$

$$i^3 = i^2 i = -i$$

$$i^4 = i^2 i^2 = (-1)(-1) = 1$$

$$i^5 = i^4 i = i$$

⋮
⋮
⋮

When performing the operations of addition, subtraction, multiplication and division, we treat i as though it was an algebraic constant subject to the interpretation given by the above list of powers of i . For example,

$$(3 + 2i) + (2 - i) = 5 + i,$$

$$(3 + 2i) - (2 - i) = 1 + 3i,$$

$$(3+2i)(2-i) = 6-3i+4i-2i^2 = 6+i+2 = 8 + i .$$

Division requires remembering a simple trick. To compute

$$\frac{3+2i}{2-i}$$

we multiply numerator and denominator by the "complex conjugate" of the denominator. The complex conjugate of the number $a + bi$ is the number $a - bi$, and we denote complex conjugation by $\overline{a+bi} = a-bi$. (Here a and b are real numbers). Therefore

$$\frac{3+2i}{2-i} = \frac{(3+2i)(2+i)}{(2-i)(2+i)} = \frac{6+3i+4i-2}{4+2i-2i+1} = \frac{4+7i}{5} = \frac{4}{5} + \frac{7}{5}i .$$

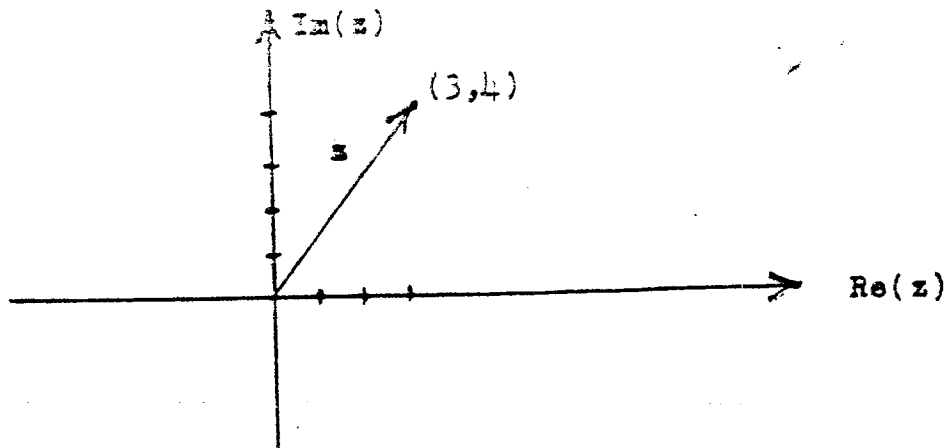
Problem

- Let $z = 3-3i$, and $w = 1 + \sqrt{3}i$. Find (a) $z+w$, (b) $z-w$, (c) zw and (d) z/w .
- Let $z = a + bi$ and $w = c + di$, where a, b, c, d , are real numbers. Show that (a) $\overline{z+w} = \overline{z} + \overline{w}$, (b) $\overline{z-w} = \overline{z} - \overline{w}$, (c) $\overline{zw} = \overline{z} \cdot \overline{w}$, and that (d) $\overline{(z/w)} = \overline{z} / \overline{w}$.

1.2 The Complex Plane

We are familiar with the "real line" as a geometric device for visualizing the real numbers. Since the complex number $z = x + yi$ is a two dimensional item (there is an x and a y), it is not surprising that mathematicians have found it useful to consider

complex numbers as points or vectors on a plane. Thus, the complex number $z = 3+4i$ is represented by the following picture.

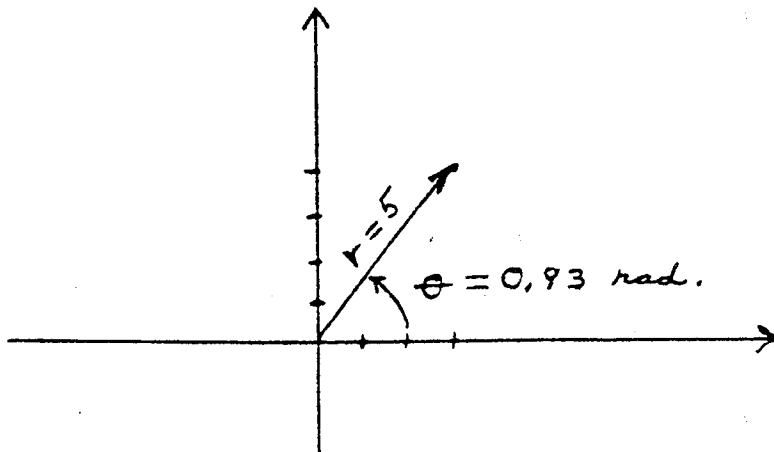


$z = 3+4i$ can be thought of as the point having Cartesian coordinates $(3,4)$, or as the vector drawn from the origin to $(3,4)$. We call the horizontal axis the "real axis" and the vertical axis the "imaginary axis". The entire plane is known as the "complex plane" or the "Argand diagram". In the case of the number $z = 3 + 4i$ we also write:

"real part of z " = $\text{Re}(z) = 3$, and

"imaginary part of z " = $\text{Im}(z) = 4$.

The complex number $z = 3+4i$ can also be defined by its polar coordinates r and θ .



Here $r = \sqrt{3^2 + 4^2} = 5$ and $\theta = \arctan(4/3) = 0.93$ radians.

As in analytic geometry, the angle θ is not unique. We can add to θ any multiple of 2π radians. Thus

$$\theta = 0.93 + 2n\pi, \text{ where } n = 0, \pm 1, \pm 2, \dots$$

We often call r the "modulus of z ", and denote it by use of absolute value symbol $|z|$. Thus $|3+4i| = 5$. The angle θ is often called the "argument of z ", and is denoted by $\arg(z) = \theta$. Thus $\arg(3+4i) = 0.93 + 2n\pi$, where $n = 0, \pm 1, \pm 2, \dots$.

Problem

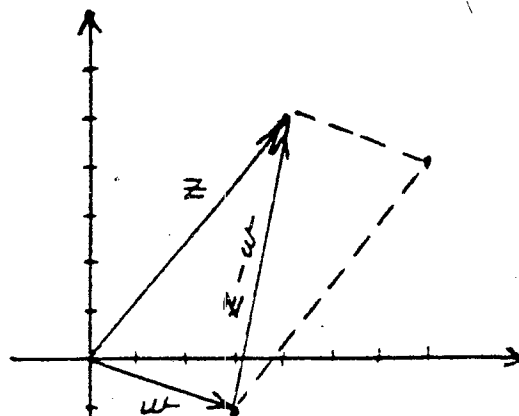
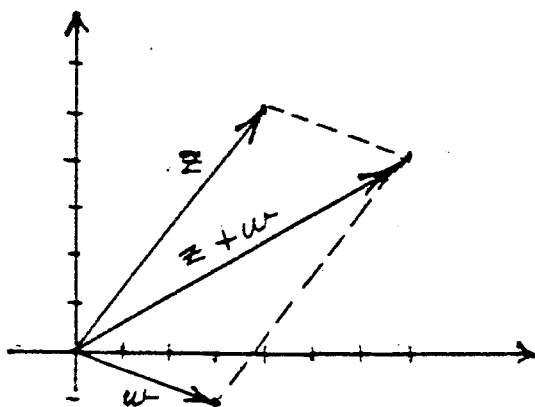
3. Let $z = -4 - 4\sqrt{3}i$. Find (a) $\operatorname{Re}(z)$, (b) $\operatorname{Im}(z)$, (c) $|z|$ and (d) $\arg(z)$.

1.3 Geometric aspects of addition and subtraction

The reader is familiar with the addition and subtraction of vectors from earlier courses in physics and mathematics. Since complex numbers are represented by vectors on the Argand diagram, it comes as no surprise that the geometric description of the addition and subtraction of complex numbers is the same as that previously examined for ordinary vectors in other courses. We merely review the process by means of an example.

Example

Let $z = 4+5i$ and $w = 3-i$. Form $z+w$ and $z-w$.



From the above two diagrams we recall that the vectors $z+w$ and $z-w$ form the two diagonals of the parallelogram having as adjacent sides z and w . Note that the vector $z-w$ is drawn from the head of w to the head of z .

Problem

4. Let $a = 1+i$, $b = -1 + \sqrt{3}i$ and $c = -\sqrt{3} - i$. Find $a+b-c$ geometrically.

1.4 Important regions in the complex plane

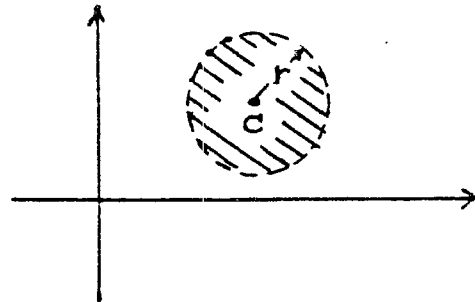
In the study of the calculus, we became familiar with certain important subsets of the real line:

open intervals such as $x-a < b$, $x < a$, and $x > b$;

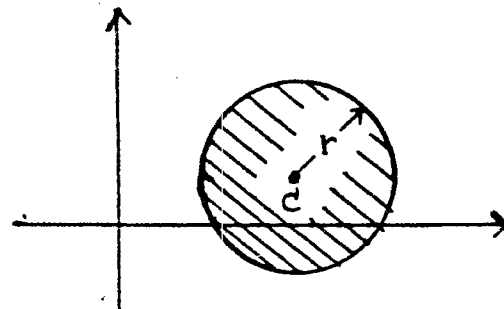
closed intervals such as $x-a \leq b$, $x \leq a$, and $x \geq b$.

In the complex plane we usually denote $z = x + iy$ as the "complex variable", where both x and y are real variables. The most important subsets of the complex plane are shown below.

(i) open disk $|z - c| < r$

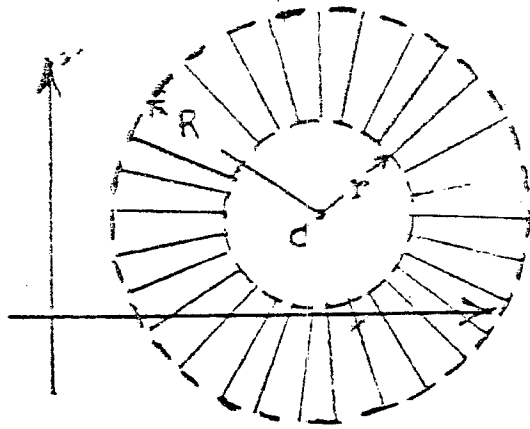


(ii) closed disk $|z - c| \leq r$



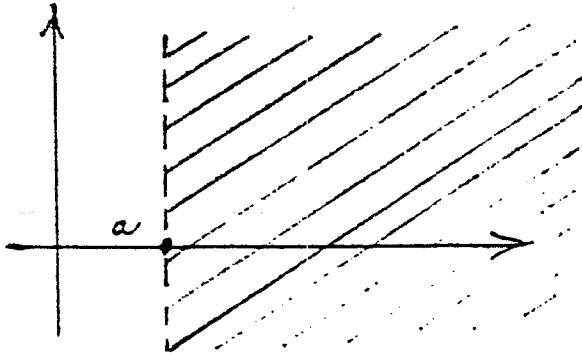
(iii) open annulus

$$r < |z - c| < R$$



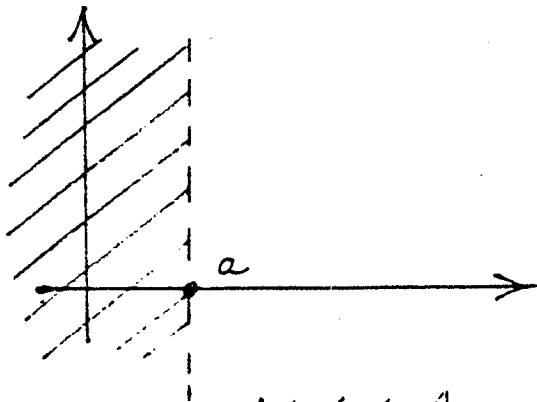
(iv) right half plane

$$a < \operatorname{Re}(z)$$



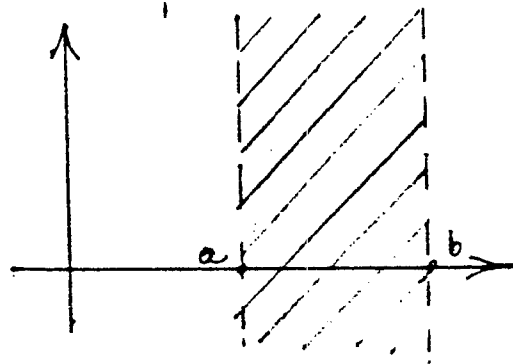
(v) left half plane

$$\operatorname{Re}(z) < a$$



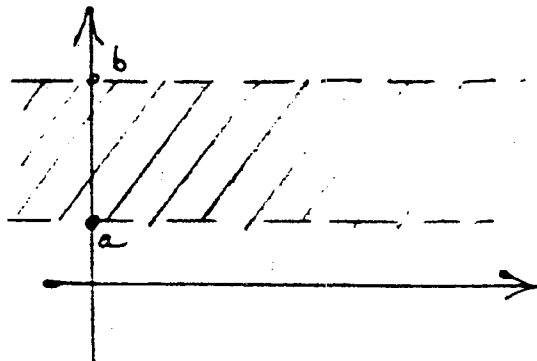
(vi) vertical strip

$$a < \operatorname{Re}(z) < b$$



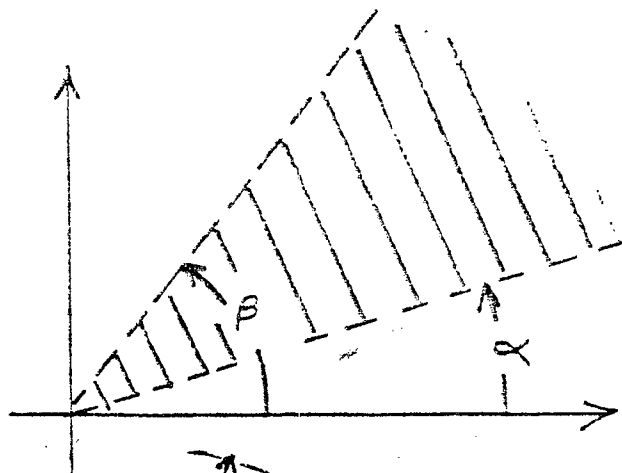
(vii) horizontal strip

$$a < \operatorname{Im}(z) < b$$



(viii) Infinite open sector

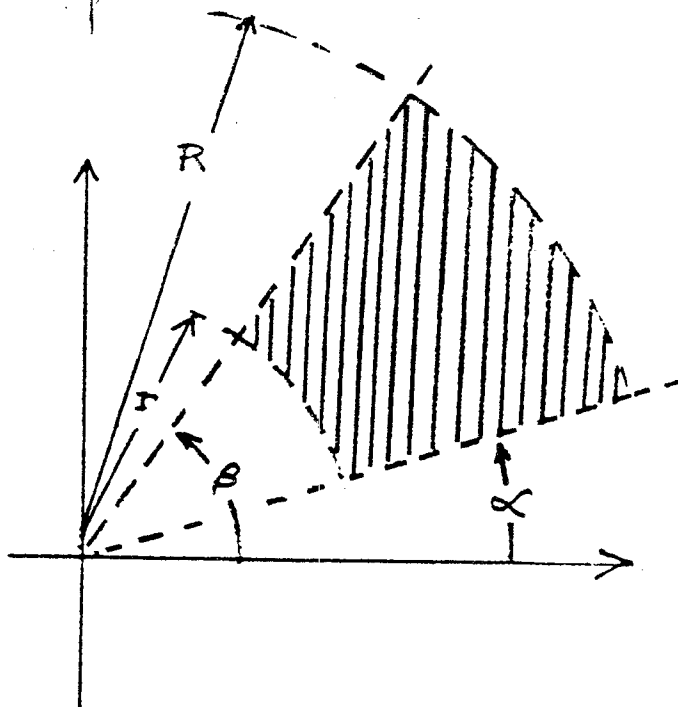
$$\alpha < \arg(z) < \beta$$



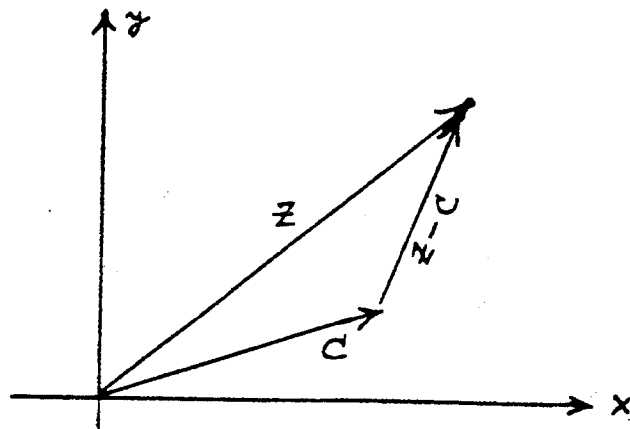
(ix) Finite open sector

$$\alpha < \arg(z) < \beta$$

$$r < |z| < R$$



It is easy to see how the analytical inequalities given in (iv) through (viii) above yield the corresponding geometric regions. The remaining regions are easily understood when we examine the geometric meaning of $|z-c|$. The diagram below shows the three vectors c , z , and $z-c$. The expression $|z-c|$ is the length of



the vector $z-c$. Now let us consider the inequality described in (i). If c is held fixed, and the vector z is allowed to move under the restriction that the length of $z-c$ must stay less than some fixed given number r ($|z-c| < r$), we see at once that z may assume any location inside the circle centered at c of radius r as shown in (i).

Often the notions of open sets and closed sets of points in a plane are introduced in courses in Real Analysis, and in Topology. We will have need of this terminology for sets of points in the complex plane, but there is no need for us to give difficult general definitions. For our purposes an open set is a two dimensional region with the boundary points removed. The open set is made closed by adding to it the points on its boundary. As an example, the figure for region (i) is open because the circumference of the circle is excluded, while the figure for region (ii) is closed because the circumference is included.

Problem

5. Show the following regions in the complex plane and determine if the regions are (i) open, (ii) closed, or (iii) neither open nor closed. (a) $2 < |z|$, (b) $2 \leq \operatorname{Re}(z)$, (c) $0 < \arg(z) \leq \pi/4$, (d) $0 < |z| < 1$, (e) $1 \leq |z-2| \leq 2$, (f) $\{z \mid 0 < \arg(z) < \pi/4, |z| > 1\}$.

We will sometimes have need to simplify an inequality before we attempt to visualize the corresponding region in the complex plane.

Example

Describe the region defined by $|3z - 6| \leq 9$.

Solution

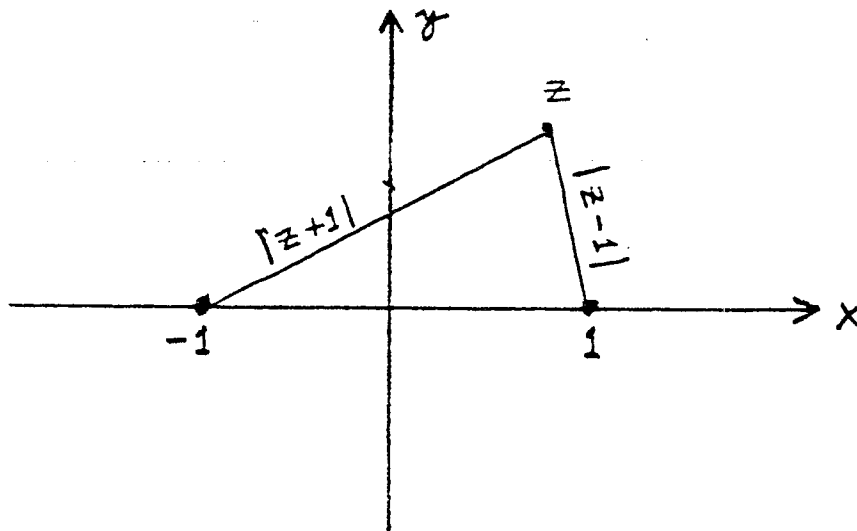
We can factor 3 from inside the absolute value sign to get $3|z-2| \leq 9$, or $|z-2| \leq 3$. This is the closed circle with center at $z = 2$ having radius 3.

Example

Describe the region in the z -plane defined by $\left| \frac{z-1}{1+z} \right| < 1$.

Solution

The inequality can be written as $|z-1| < |z+1|$. This inequality says "the distance of the point z from the point 1 is less than the distance of the point z from the point -1 ". We see



from the figure that this is true for points z in the right half plane described by $\operatorname{Re}(z) > 0$. This is an open set.

Problem

6. Describe the region in the complex plane expressed by the

inequality. (a) $|4 - 4z| > 8$, (b) $\left| \frac{z - z_0}{1 - z_0} \right| < 1$,

(c) $|z| \leq |1 - z|$.

1.5 The complex exponential

In a previous course we learned that the functions e^x , $\sin x$ and $\cos x$ could be expanded in power series as

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots, \text{ and}$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots.$$

If we now set $x = i\theta$ into the above power series for e^x we get

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots$$

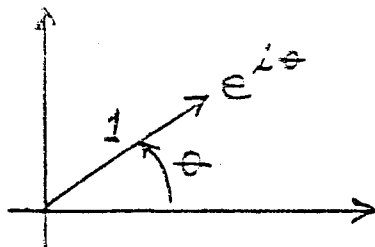
$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \dots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right).$$

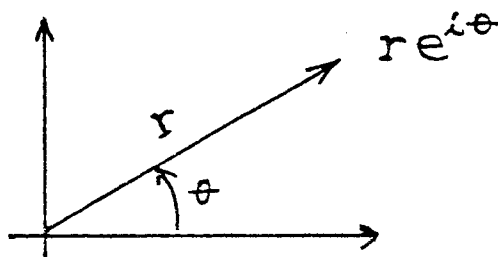
Comparing this last expression with the above power series for $\sin \theta$ and $\cos \theta$ we see that

$$\boxed{e^{i\theta} = \cos \theta + i \sin \theta} \quad (\text{Euler's formula}).$$

Euler's formula defines $e^{i\theta}$ for real θ as the vector of unit length making angle θ with the real axis.



Note that multiplying $e^{i\theta}$ by the real number r gives $r e^{i\theta} = r \cos \theta + i r \sin \theta$, and since $|r e^{i\theta}| = r$, we see that $r e^{i\theta}$ is a complex number having modulus r and argument θ .



Thus the expression $z = r e^{i\theta}$ is convenient when we wish to display the "polar coordinates" of z , and the notation $z = x+iy$ is convenient when we wish to display the "Cartesian components" of z . For example,

$$1+i = \sqrt{2} e^{i\pi/4} = -\sqrt{2} e^{i5\pi/4} = -\sqrt{2} e^{-i3\pi/4}$$

are all equivalent ways of representing the same complex number.

(Recall that the $x+iy$ form of the complex number is unique; while the $r e^{i\theta}$ form is not unique.)

Problems

7. Express the following numbers in the form $r e^{i\theta}$, with $r > 0$, and $0 \leq \theta < 2\pi$. (a) $5i$, (b) $-7+7i$, (c) $-4-4\sqrt{3}i$, (d) -1 .

8. Express the following numbers in the form $x+iy$.

(a) $2e^{3\pi i}$, (b) $e^{13\pi i/4}$, (c) $7e^{-\pi i/6}$, (d) $e^{-\pi i/2}$.

1.6 Geometric aspects of multiplication and division.

Conjecture 1

Based on rules familiar from the study of exponents involving real numbers, conjecture the results of the following operations involving imaginary exponents:

$e^{ai} e^{bi}$, e^{ai} / e^{bi} , and $(e^{ai})^b$, where a and b are real numbers. (See Appendix II for the answers to Conjectures.)

Let us now examine multiplication. Let $z = r e^{i\theta}$ and $w = s e^{i\omega}$. Then

$$\begin{aligned} zw &= r e^{i\theta} s e^{i\omega} \\ &= rs e^{i\theta} e^{i\omega} \\ &= rs e^{i(\theta+\omega)}. \end{aligned}$$

This last expression tells us that $|zw| = |z| \cdot |w|$ and that $\arg(zw) = \arg(z) + \arg(w)$. In other words, when we multiply the vectors z and w , we form a new vector whose length is the product of the lengths of the original two vectors, and whose angle is the sum of the angles of the original two vectors.

Division is similar.

$$\frac{z}{w} = \frac{r e^{i\theta}}{s e^{i\omega}} = \frac{r}{s} e^{i(\theta-\omega)}$$

Thus $|z/w| = |z| / |w|$, and $\arg(z/w) = \arg(z) - \arg(w)$.

Example

Let $z = 6 e^{i\pi/3}$ and $w = 3 e^{i\pi/6}$. Describe zw and z/w . zw has modulus 18 and argument $\pi/2$. Thus $zw = 18i$. z/w has

modulus 2 and argument $\pi/6$, which means that $z/w = \sqrt{3} + i$.

Problems

9. Returning to problems 1(c) and 1(d), form the required product and quotient using the $r e^{i\theta}$ form of the numbers, and compare the results with the computation performed previously using the $x+iy$ form of the numbers.

10. By squaring $e^{i\theta} = \cos \theta + i \sin \theta$, find formulas for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\sin \theta$ and $\cos \theta$.

11. Find formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\sin \theta$ and $\cos \theta$.

1.7 Roots of complex numbers

Suppose we were asked for the cube root of $8i$. We would seek z such that $z^3 = 8i$. Writing $8i = 8 e^{i(\frac{\pi}{2} + 2\pi n)}$, $n = 0, \pm 1, \pm 2, \dots$, we are led quite naturally to write

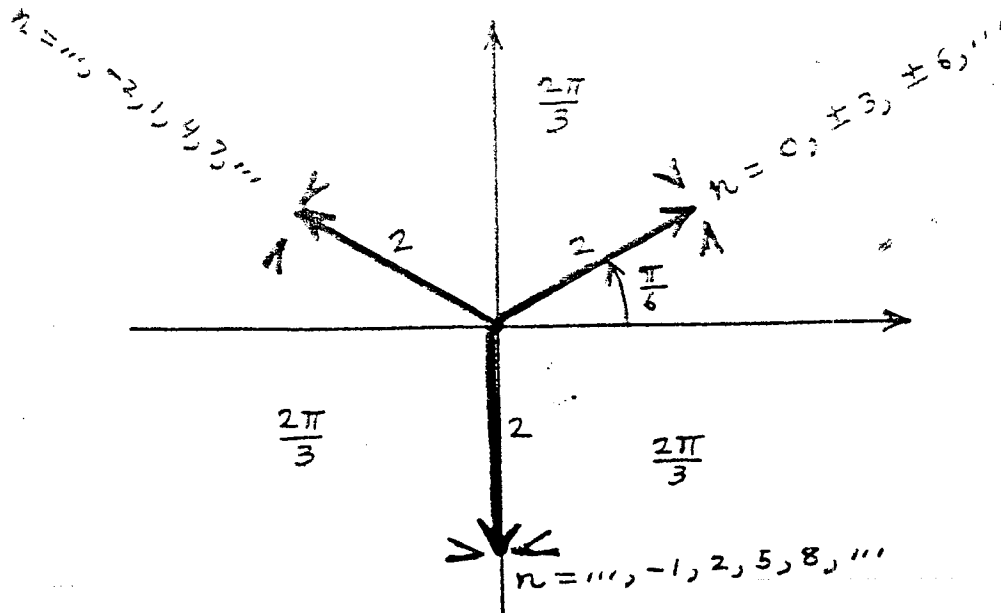
$$z^3 = 8 e^{i(\frac{\pi}{2} + 2\pi n)}$$

$$z = [8 e^{i(\frac{\pi}{2} + 2\pi n)}]^{\frac{1}{3}}$$

$$z = 8^{\frac{1}{3}} e^{i(\frac{\pi}{2} + 2\pi n) \frac{1}{3}}$$

$$z = 2 e^{i(\frac{\pi}{6} + \frac{2\pi n}{3})}, \quad n = 0, \pm 1, \pm 2, \dots$$

At first glance, it looks as though there are infinitely many possible answers for z since n can be any integer. However, if we examine the answers on the Argand diagram we see that we have



These three values for the cube root of $8i$ can also be written as $\sqrt{3} + i$, $-\sqrt{3} + i$ and $-2i$. Thus there are three distinct answers. In a way, this is not surprising, since $z^3 = 8i$ is a cubic equation and we therefore expect three answers.

Problems

Solve the following equations and show the answers on the Argand diagram.

12. $z^2 + 4 = 0$, 13. $z^3 - 8 = 0$, 14. $z^4 = 81(1+i)/\sqrt{2}$,

15. $z^3 + i = 0$, 16. $z^6 = -64$.

Conjecture 2

You wish to find all z such that $z^N = a e^{ib}$, where N is a positive integer, a and b are real numbers, and $0 \leq b < 2\pi$.

(i) What is the number of distinct answers? N

(ii) What is the modulus of each answer? a

(iii) When the answers are displayed on the Argand diagram, what is the smallest positive angle, and what angle separates each of the vectors.