1.8 The Riemann sphere of numbers

We have seen that the complex plane is a convenient device for visualizing the operations of addition, subtraction, multiplication and division. For some purposes, it is convenient to use points on a sphere rather than points on the plane to visualize operations involving complex numbers. To see how this is achieved, look at Figure 1. Here a sphere having unit diameter is placed directly over the origin of coordinates of the complex z-plane. Each point P on the complex plane is made to correspond to a point P′ on the surface of the sphere in the following way:

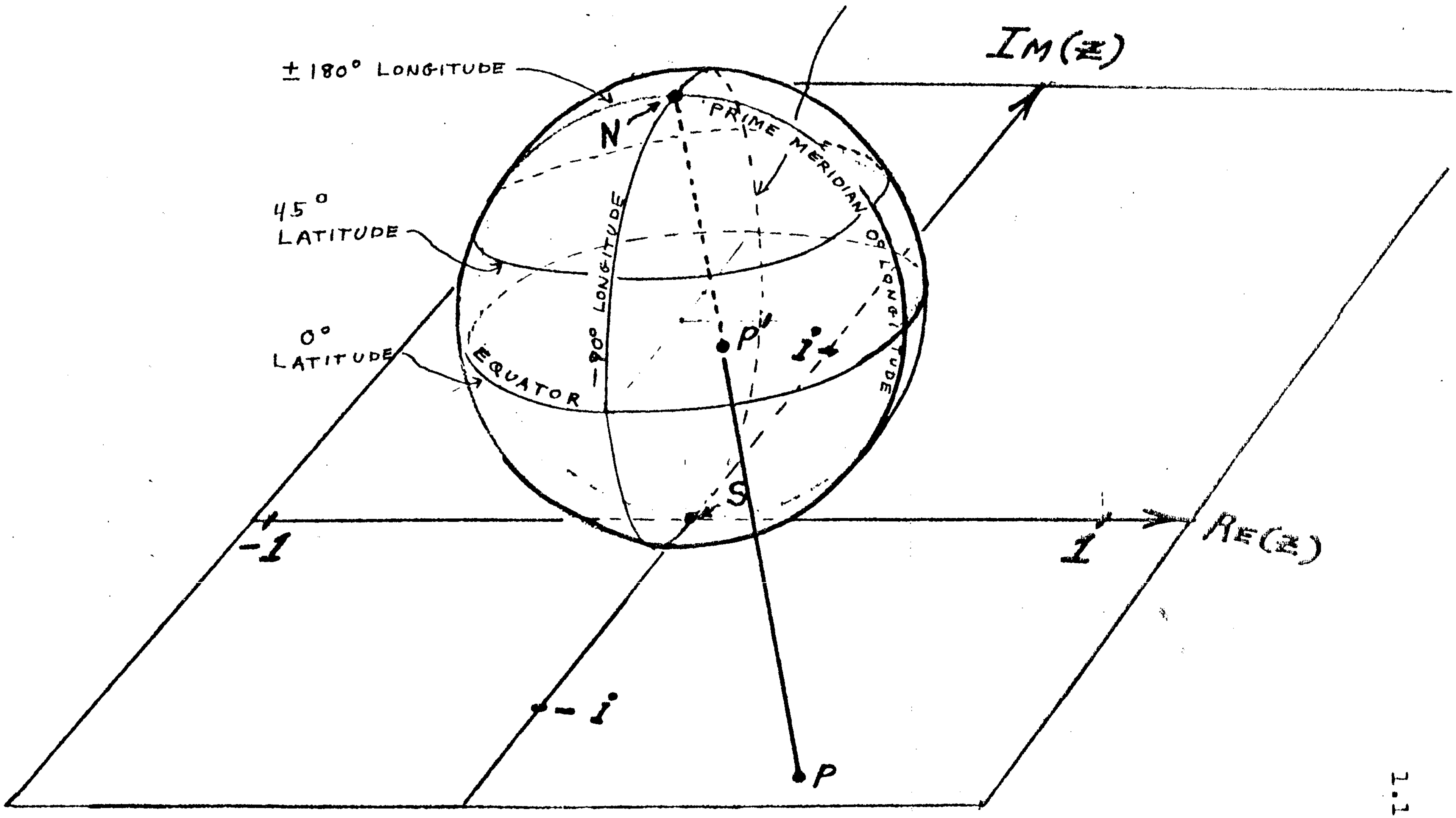
(1) A straight line is drawn from the north pole of the sphere "N" to the point P in the plane.

(2) The point P′ to which P is mapped is the point where the above straight line intersects the surface of the sphere.

The mapping of points P on the plane to points P′ on the sphere is called a stereographic projection. The sphere itself is called the Riemann sphere of the sphere of complex numbers.

To identify points on the Riemann sphere, we will use the terminology used to identify points on a globe of the earth. In Figure 1 we see the north pole "N" and the south pole "S". We also see the equator and circles of longitude and circles of latitude. An examination of Figure 1 reveals the following mapping of points from the plane to the sphere:

<u>Figure 2</u> The Riemann sphere of numbers and the stereographic projection



Point on the Plane	Point on the Sphere
$z = 0$	South pole "S"
$z = 1$	Equator at 0° longitude
$z = i$	Equator at 90° longitude
$z = -1$	Equator at $\pm 180^\circ$ longitude
$z = -i$	Equator at -90° longitude

If we ignore the north pole "N", we see that the stereographic projection defines a one to one mapping of points on the plane with points on the sphere.

Notice that no point on the complex z-plane maps onto the north pole of the Riemann sphere. However, we do notice that points near N correspond to points on the z-plane very very far from the origin. If we introduce a new (improper) point $z = \infty$, it seems reasonable to identify the north pole with this point. We call the ordinary complex plane (without $z = \infty$) the finite plane. When we add the point $z = \infty$ to the complex plane we call it the extended plane or the closed plane. Thus we see that the concept of a point at infinity takes on a very tangible form on the Riemann sphere since it corresponds to a well defined point, the north pole. The Riemann sphere is very convenient for visualizing phenomena involving large z because these points occur near the north pole.

A curve on the complex z-plane now maps onto a curve on the surface of the Riemann sphere. It is easy to see that the circle $|z| = 1$ in the plane maps onto the equator of the sphere. The positive real axis maps onto the semi-circle of longitude 0°. The positive imaginary axis maps onto the semi-circle of longitude 90° . An examination of Figure 2 reveals that any straight line in the plane maps onto a circle through the north pole of the sphere. Suppose we call a straight line in the z-plane a <u>circle of infinite radius</u> . With this terminology, we have the following property of the stereographic projection:

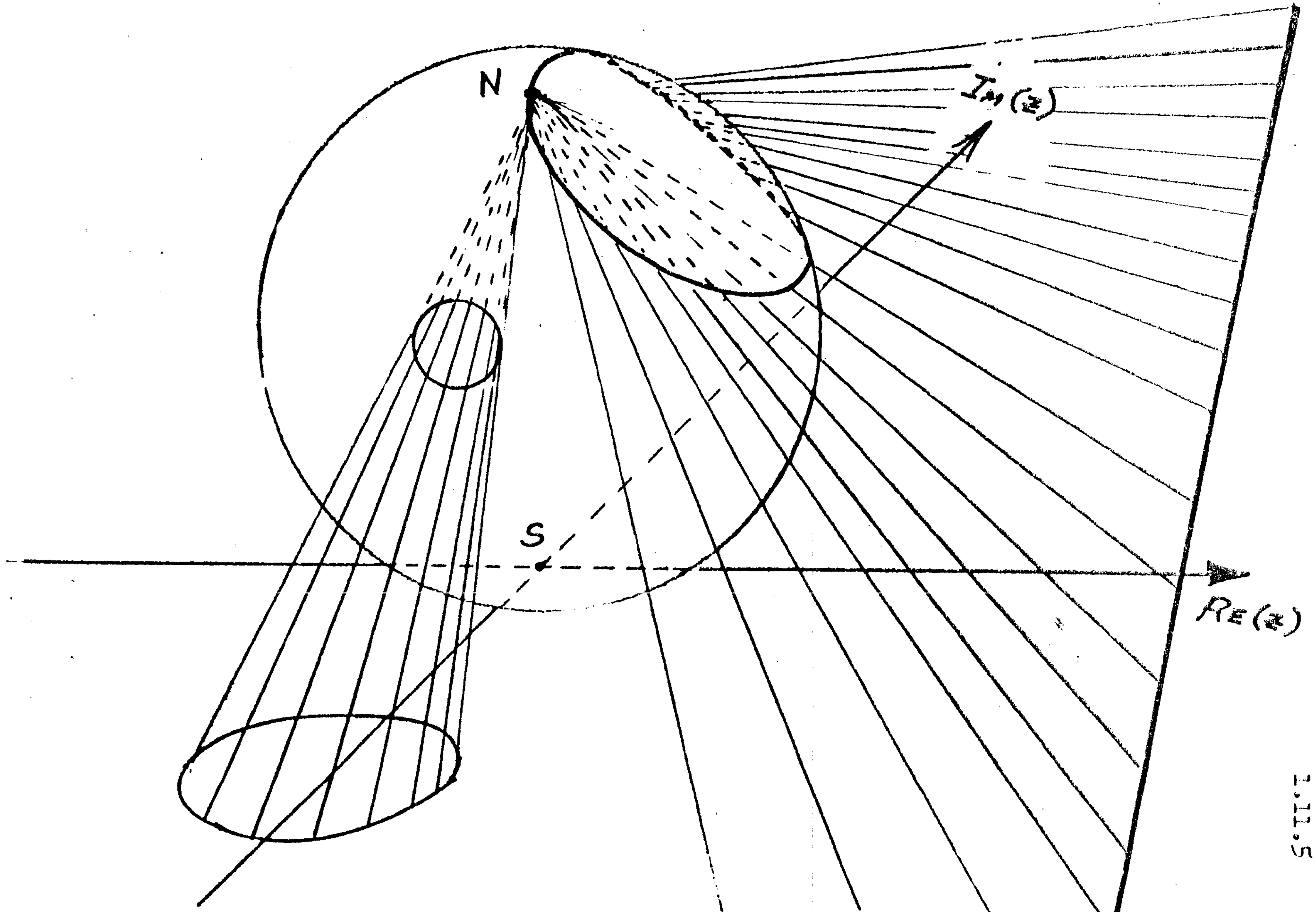
> Every circle on the plane maps onto a circle on the sphere
>
> and
>
> every circle on the sphere maps onto a circle on the plane.

This property of the mapping is summarized by saying that it is <u>circular</u> . We will not prove the circular property in this section. The student can prove this property by following the steps outlined in supplementary problem 1.8.

<u>Problem</u>

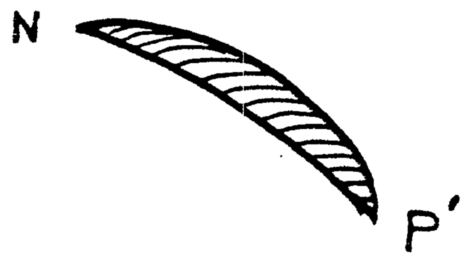
17. Look at Figure 1 and convince yourself that
    - (a) A circle $|z| = R$ maps onto a circle of latitude on the sphere.
    - (b) Two parallel lines on the plane map onto two circles on the sphere tangent at the north pole.
    - (c) The region $|z| < 1$ maps onto the southern hemisphere.
    - (d) The region $|z| > 1$ maps onto the northern hemisphere.
    - (e) The half-plane $Im(z) > 0$ maps onto the hemisphere having longitude between 0° and 180° .

<u>Figure 2</u> The circularity of the stereographic projection

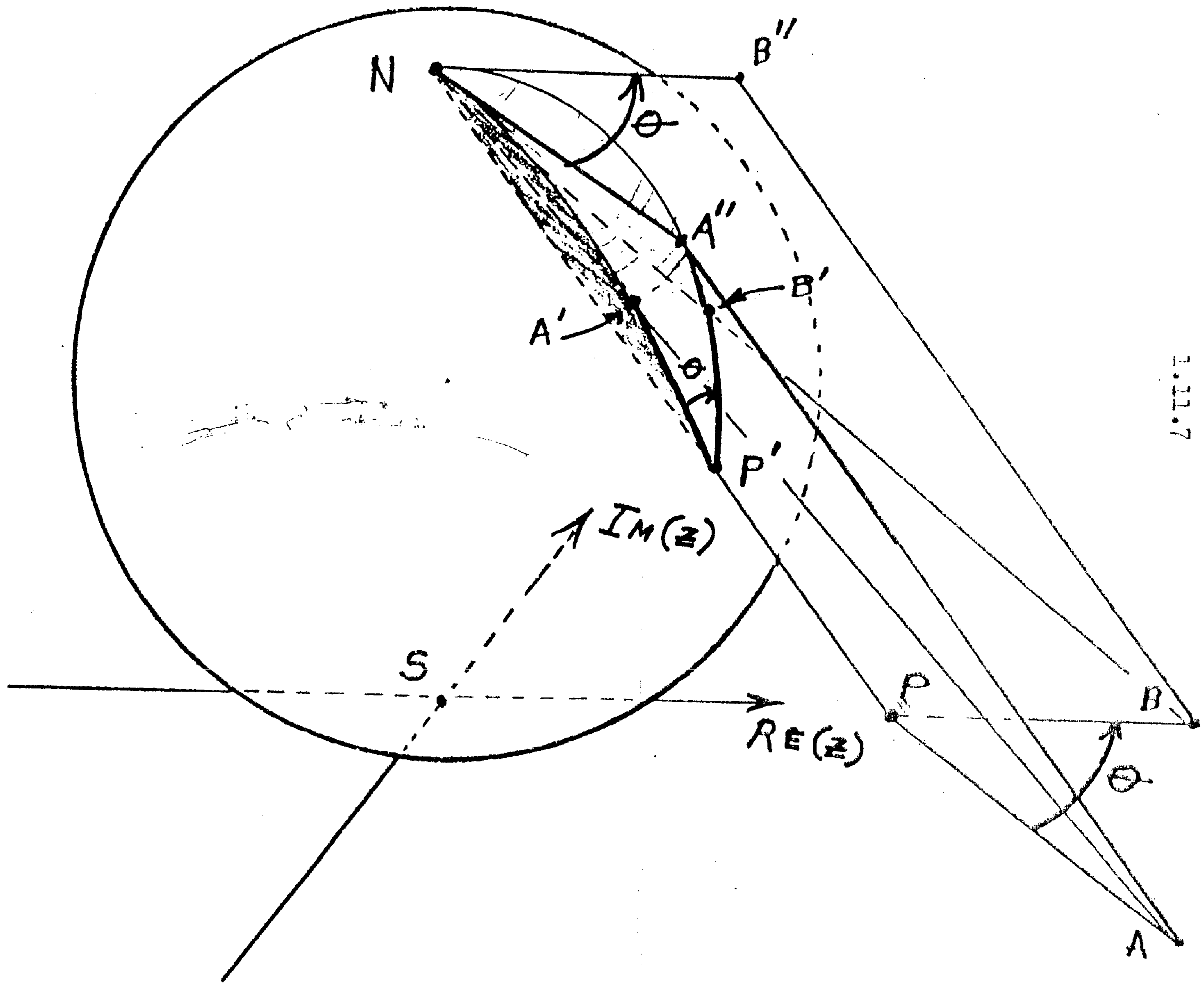


Another important property of the stereographic projection is that it preserves angles. That is to say, angles on the plane map into equal angles on the sphere. We say that the mapping is <u>isogonal</u>. In Figure 3 we see the angle θ at the point P in the plane. This angle is defined by the straight lines AP and BP. The lines AP and BP map onto the arcs $A'P'$ and $B'P'$ on the surface of the sphere. The mapping is isogonal if the angle $A'P'B'$ is also θ.

It is easy to prove the isogonality of the mapping. First note that the sector APB in the plane (with the lines extended to infinity) maps onto a slice of the sphere which resembles a slice of the skin of an orange. Note that the angle at N must equal the angle at P'. But the angle at N is clearly θ since the tangents $B''N$ and $A''N$ are parallel to BP and AP. Thus the mapping is isogonal.



<u>Figure 3</u> The isogonality of the stereographic projection.



<u>Review problems for Chapter 1</u>

1. Describe the regions in the z-plane and state if they are open or closed. (a) $2 \leq \mathrm{Re}(z)$, (b) $\{z \mid -\pi/4 < \arg(z) \leq \pi/4$, and $|z| < \pi\}$, (c) $2 / |z-1-i| < 1$; (d) $2 < |z-2| < 4$, (e) $|z-1-i| < |z|$.

2. Let $z = 1 + \sqrt{3}\,i$, $w = 3i$, $\zeta = 2\sqrt{3} - 2i$. (a) Express z, w, and ζ in polar form. (b) What is $|zw\zeta|$? (c) What is $\arg(zw\zeta)$?, (d) What is $\arg(\overline{zw\zeta})$?

3. Prove that $e^{i\theta} e^{i\omega} = e^{i(\theta+\omega)}$.

4. Find all values of z such that $z^4 = -81$. Express the results in Cartesian form.

SUPPLEMENTARY PROBLEMS

Note that all supplementary problems are identified by

CHAPTER - SECTION - PROBLEM NUMBER

Thus the item identified by 1.3.12 is the twelfth problem for section 3 of chapter one.

1.1.1 Find $z + w$, $z - w$, zw and z/w for each of the following pairs of complex numbers and express the result in the form $x + iy$. (a) $z = 4-4i$, $w = -8-8i$; (b) $z = 2+2\sqrt{3}\,i$, $w = 2-2\sqrt{3}\,i$; (c) $z = 3+4i$, $w = 4-3i$; (d) $z = 3-i$, $w = 2+3i$

1.1.2 Compute $(-1 + \sqrt{3}\,i)^3$. Ans. 8 .

1.1.3 Compute $(-1 - \sqrt{3}\,i)^6$. Ans. 64 .

1.1.4 Compute $(1 + i)^8$

1.1.5 Compute $(2 - 2i)^4$

1.2.1 Find $Re(z)$, $Im(z)$, $|z|$, $arg(z)$ and $\overline{z}$ for each of the following complex numbers: (a) $z = 3$, (b) $z = 4i$, (c) $z = 3-3i$, (d) $z = \sqrt{3} - i$, (e) $z = 4 - 3i$.

1.2.2 (a) Show that $z + \overline{z} = 2\,Re(z)$.

(b) Show that $z - \overline{z} = 2i\,Im(z)$.

1.2.3 Show that $|z|^2 = z\,\overline{z}$

1.3.1 Add the numbers given in problem 1.1.1 vectorially.

1.3.2 Find the sum of the following numbers algebraically and vectorially: 3, $2-3i$, $4i$, $-6+i$, $2-2i$.

1.3.3 What conditions should be imposed on z_1 and z_2 so that $|z_1 + z_2| = |z_1| + |z_2|$?

1.3.4 What conditions should be imposed on z_1 and z_2 so that $|z_1 - z_2| = |z_1| - |z_2|$? the vectors share the same direction and $z_1 \geq z_2$

1.3.5 Give a geometric argument to demonstrate the identity

$$|z_1 - z_2| \geq \big||z_1| - |z_2|\big| .$$

1.4.1 Describe the following regions on the complex plane, and decide if they are (i) open, (ii) closed, (iii) neither open nor closed . (a) $\mathrm{Im}(z) < -1$; (b) $\mathrm{Re}(z) \geq 2$; (c) $1 \leq \mathrm{Re}(z) < \pi$; (d) $\pi < \arg(z) \leq 3\pi/2$; (e) $|z| < 5$; (f) $|z-2| \leq 2$; (g) $3 \leq |z| \leq 4$; (h) $2 < |z-2| < 4$; (i) $\{z \mid 3 \leq |z| \leq 4,\ \arg(z) < \pi/4\}$.

1.4.2 Describe the following regions on the complex plane: (a) $|3+3z| < 6$; (b) $|2z-4| \leq 2$; (c) $8 < |4-4z| < 12$; (d) $|z| = |z-1|$; (e) $\left|\frac{z+i}{z-i}\right| < 1$; (f) $\left|\frac{z-2i}{z+2}\right| \geq 1$.

1.4.3 Show that the equation $|z+1| + |z-1| = 2\sqrt{2}$ defines the ellipse $x^2 + 2y^2 = 2$.

1.4.4 Show that $|z+i| - |z-i| = 2\sqrt{2}$ defines a hyperbola, and write its equation in terms of x and y .

1.4.5 Show that $|z| = 2|z-1|$ defines a circle and write its equation in terms of x and y .

1.5.1 Express the following numbers in the form $r e^{i\theta}$ with $r > 0$ and with $0 \le \theta < 2\pi$: (a) $-3i$; (b) $1+i$; (c) $-1 - \sqrt{3}\, i$; (d) -3 .

1.5.2 Express the following numbers in the form $x + iy$: (a) $3\, e^{\pi i}$; (b) $2\, e^{\pi i/2}$; (c) $7\, e^{7\pi i}$; (d) $-2\, e^{-\pi i/4}$; (e) $4\, e^{\pi i/6}$; (f) $5\, e^{341\pi i}$.

1.6.1 Using the numbers given in problem 1.1.1, find zw and z/w vectorially .

1.6.2 Show that $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$. (De Moivre's Theorem)

1.6.3 Show that $\cos 5\theta = 16 \cos^5\theta - 20 \cos^3\theta + 5 \cos\theta$ and that $\sin 5\theta = \sin\theta\, [\, 16\cos^4\theta - 12 \cos^2\theta + 1\,]$.

1.7.1 Solve the following equations and express the results in the form $x+iy$: (a) $z^2 + 4i = 0$; (b) $z^2 + 1 = 0$; (c) $z^3 + 8 = 0$; (d) $z^4 + 16 = 0$; (e) $z^6 - 64 = 0$; (f) $z^4 + 1 - i = 0$. Answers (a) $\pm\sqrt{2}\,(-1+i)$; (b) $\pm i$; (d) $\pm\sqrt{2}\,(1+i)$, $\pm\sqrt{2}\,(-1+i)$.

1.7.2 Find the square roots of $5 + 12i$.

1.7.3 Find all the roots of the equation $z^4 + z^2 + 1 = 0$. Ans. $(1 \pm \sqrt{3}\,)/2$, $(-1 \pm \sqrt{3}\,)/2$.

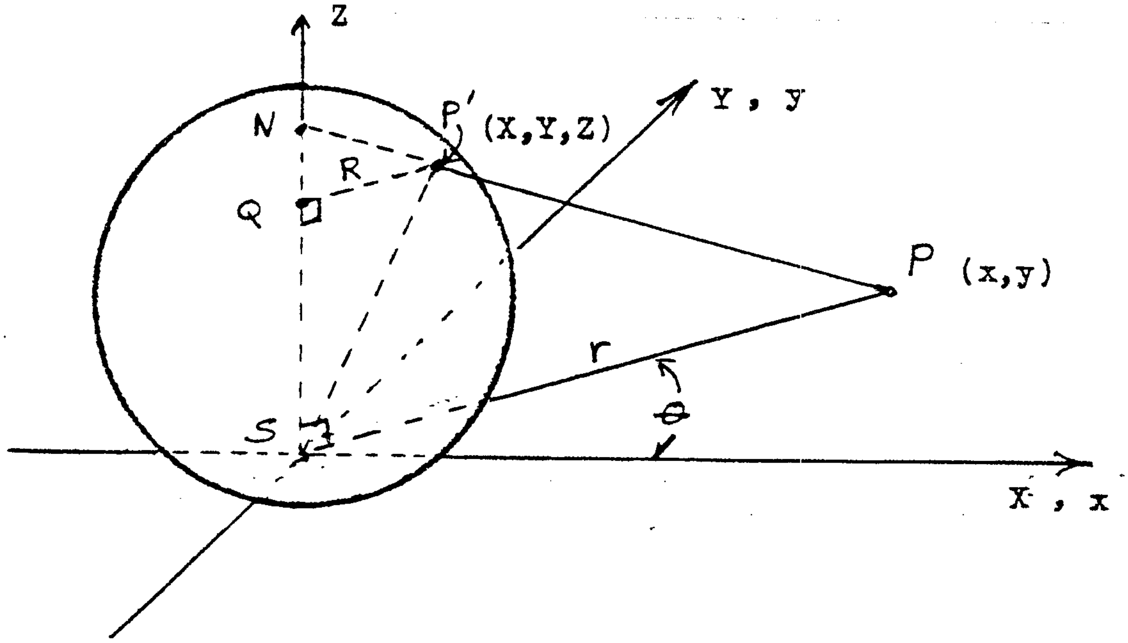
1.8.1 Argue that the straight line $x = 1$ maps onto a circle through the north pole and tangent to the equator at the point of 0° longitude.

1.8.2 Let C be a circle in the z-plane that intersects the unit circle $|z| = 1$ at diametrically opposite points. Argue that C maps onto a great circle on the Riemann sphere. (A great circle is the largest possible circle, i.e. a circle of diameter one.)

1.8.3 Describe the image of the circle of latitude 30°.

Ans. $|z| = \sqrt{3}$.

1.8.4 In this problem, the student will prove that the mapping effected by our stereographic projection from the plane to the sphere is circular, that is, it preserves circles.



In the figure, (x,y) refers to the point P in the plane, and (X,Y,Z) refers to its image P′ on the sphere. The length of the line segment QP′ is denoted by R, and the length of the line segment SP is denoted by r.

(a) Use similar triangles to show that $rR = Z$ and $r(1-Z) = R$.

(b) Use the results of (a) to show that

$$R = \frac{r}{1+r^2} \quad \text{and} \quad Z = \frac{r^2}{1+r^2} .$$

(c) Since $X = R \cos\theta$ and $Y = R \sin\theta$, and since $x = r\cos\theta$ and $y = r\sin\theta$, use (b) to show that

$$X = \frac{x}{1 + x^2 + y^2}, \quad Y = \frac{y}{1 + x^2 + y^2}, \quad Z = \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

These equations give the coordinates of a point on the sphere when the coordinates (x,y) of the corresponding point on the plane are known.

(d) Use the results of (c) to show that

$$x^2 + y^2 = \frac{Z}{1 - Z}, \quad x = \frac{X}{1 - Z}, \quad y = \frac{Y}{1 - Z}.$$

These equations give the coordinates of a point on the plane when the coordinates (X,Y,Z) of the corresponding point on the sphere are known.

(e) The equation of a circle on the complex plane is

$A(x^2 + y^2) + Bx + Cy + D = 0$. Use the results of (d) to show that this circle maps to $AZ + BX + CY + D(1-Z) = 0$ which is the equation of a plane. This plane cuts the Riemann sphere to form a circle. Thus the stereographic projection preserves circles !

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 1:

1/ (a) $z + w = 4 + (\sqrt{3} - 3)i$, (b) $z - w = 2 - (\sqrt{3} + 3)i$,

(c) $zw = 3 + 3\sqrt{3} + 3(\sqrt{3} - 1)i$,

(d) $$\frac{3-3i}{1+\sqrt{3}i} \cdot \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{3(1-\sqrt{3}) - 3(1+\sqrt{3})i}{1+3}$$

$$= \frac{3}{4}(1-\sqrt{3}) - \frac{3}{4}(1+\sqrt{3})i$$

2/ (a) $$\overline{z+w} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i$$

$$\bar{z} + \bar{w} = a - bi + c - di = (a+c) - (b+d)i$$

THUS $\overline{z+w} = \bar{z} + \bar{w}$

(c) $$\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac - bd) + (bc + ad)i}$$

$$= (ac - bd) - (ad + bc)i$$

$$\bar{z} \cdot \bar{w} = \overline{a+bi} \cdot \overline{c+di} = (a-bi)(c-di)$$

$$= (ac - bd) + (-ad - bc)i$$

THUS $\overline{zw} = \bar{z} \cdot \bar{w}$.

(d) $$\overline{\left(\frac{z}{w}\right)} = \overline{\left(\frac{a+bi}{c+di}\right)} = \overline{\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}} = \overline{\frac{(ac+bd) + (bc-ad)i}{c^2+d^2}}$$

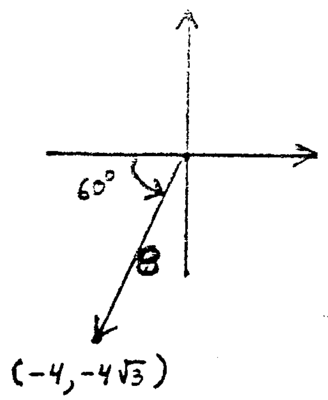
$$= \frac{ac+bd}{c^2+d^2} + \frac{ad-bc}{c^2+d^2}i,$$ AND

$$\frac{\bar{z}}{\bar{w}} = \frac{a-bi}{c-di} = \frac{a-bi}{c-di} \cdot \frac{c+di}{c+di} = \frac{(ac+bd) + (ad-bc)i}{c^2+d^2}$$

THUS $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

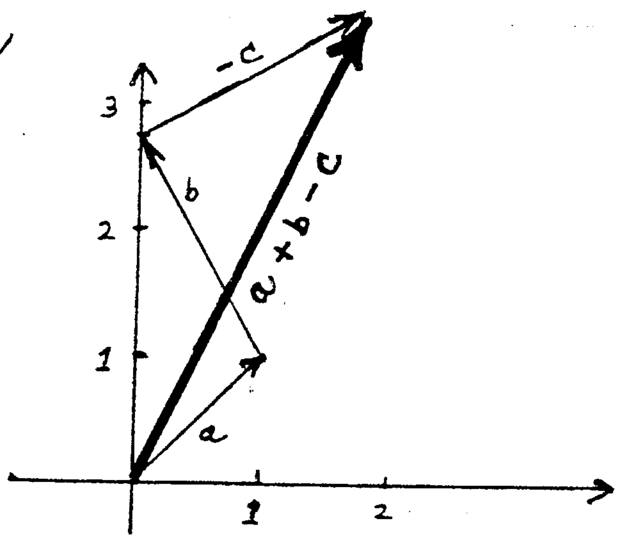
3/ (a) -4, (b) $-4\sqrt{3}$, (c) $\sqrt{(-4)^2+(-4\sqrt{3})^2} = 8$.

(d)

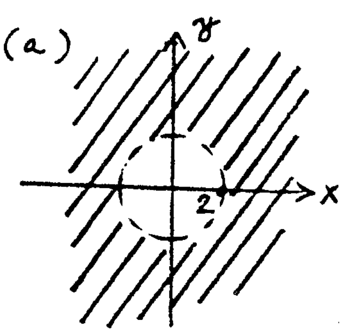


$$\arg(-4-4\sqrt{3}\,i) = \frac{4\pi}{3} + 2\pi n,$$
$$n = 0, \pm 1, \pm 2, \cdots.$$

4/

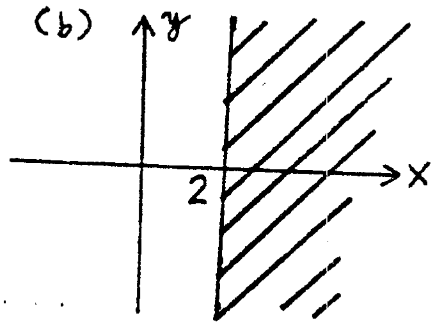


5/ (a)



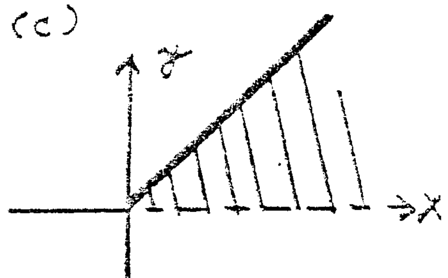
OPEN SET

(b)



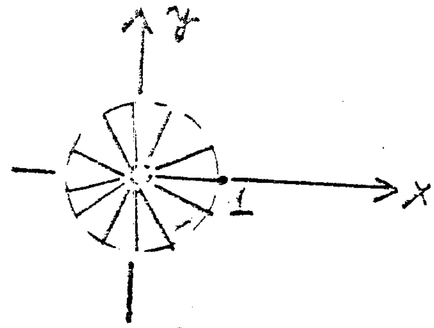
CLOSED SET

5/ (c)



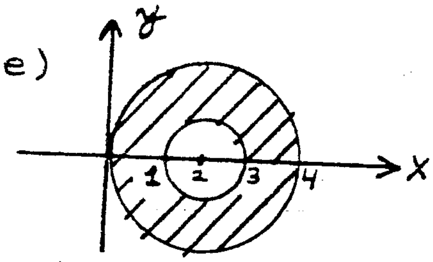
NOT OPEN OR CLOSED

(d)

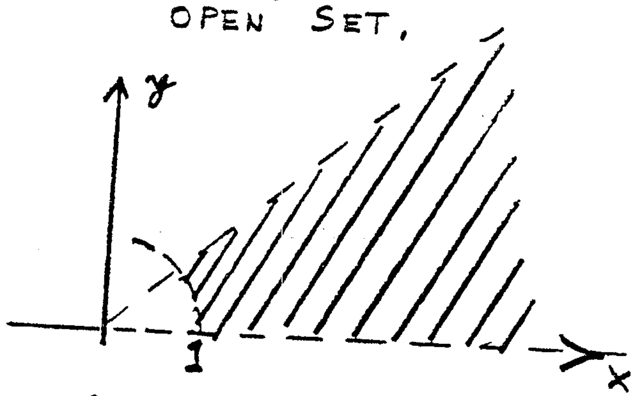


OPEN SET.

(e)



CLOSED SET

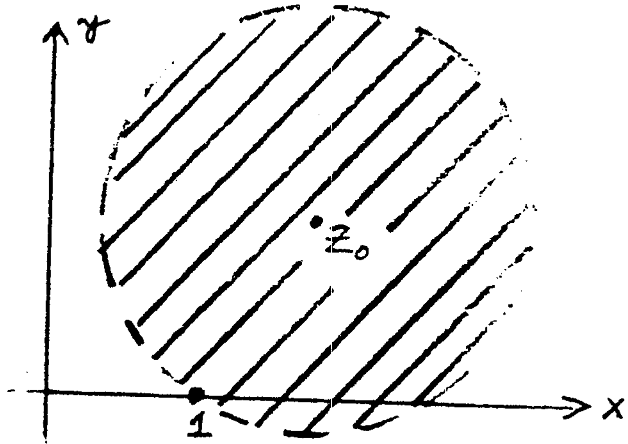


OPEN SET

6/ (a) $|z-1|>2$, open set, exterior of the circle centered at $z=1$ having radius 2.

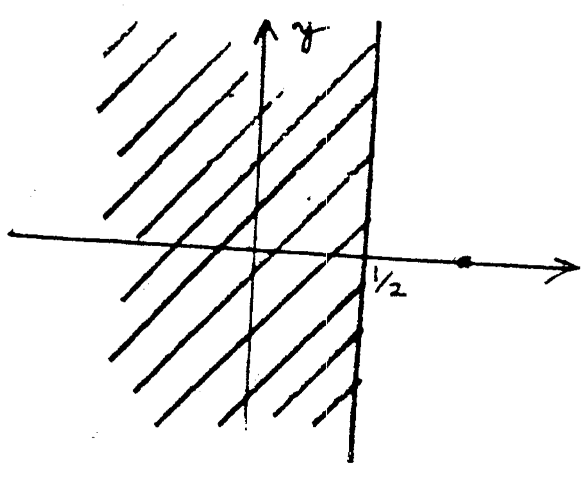
(b) $|z-z_0| < |1-z_0|$

Distance from z to z_0 is less than the distance from 1 to z_0



(c) $|z| \leq |z-1|$

The distance from z to 0 is less than or equal to the distance from z to 1.



7/ (a) $5e^{i\frac{\pi}{2}}$, (b) $7\sqrt{2}\,e^{i\frac{3\pi}{4}}$, (c) $8e^{i\frac{4\pi}{3}}$, (d) $e^{i\pi}$

8/ (a) -2, (b) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, (c) $\frac{7\sqrt{3}}{2} - \frac{7}{2}i$,

(d) $-i$

9/ $z = 3\sqrt{2}\,e^{-i\frac{\pi}{4}}$ AND $w = 2e^{i\frac{\pi}{3}}$, THUS

$zw = 6\sqrt{2}\,e^{i\frac{\pi}{12}}$ AND $\frac{z}{w} = \frac{3}{\sqrt{2}}\,e^{-i\frac{7\pi}{12}}$.

10/ $(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$

$$e^{i2\theta} = (\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta\, i.$$

SINCE $e^{i2\theta} = \cos 2\theta + i\sin 2\theta$ (EULER'S FORMULA), ON COMPARING THE LAST TWO EXPRESSIONS AND EQUATING REAL AND IMAGINARY PARTS WE GET

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{AND} \quad \sin 2\theta = 2\sin\theta\cos\theta$$

11/ $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$

$$e^{i3\theta} = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$$

(1) $$e^{i3\theta} = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).$$

BUT

(2) $e^{i3\theta} = \cos 3\theta + i \sin 3\theta$.

COMPARING (1) AND (2) WE SEE THAT

$\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta$ AND

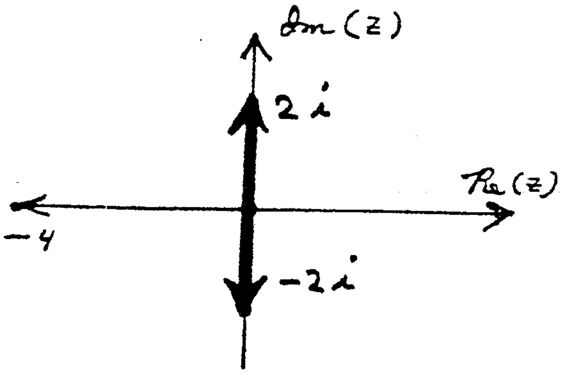
$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$,

12/ $z^2 = -4$

$z^2 = 4e^{i(\pi + 2\pi n)}$, $n = 0, \pm 1, \dots$

$z = 2e^{i(\frac{\pi}{2} + \pi n)}$

$z = 2i$ AND $-2i$

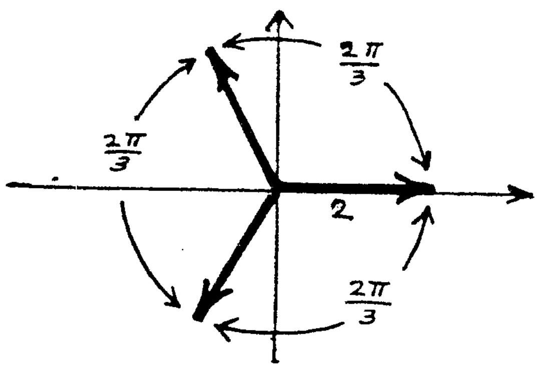


13/ $z^3 = 8$

$z^3 = 8e^{i2\pi n}$, $n = 0, \pm 1, \pm 2, \dots$

$z = 2e^{i\frac{2\pi}{3}n}$

$z = 2$, $-1 + \sqrt{3}i$, AND $-1 - \sqrt{3}i$



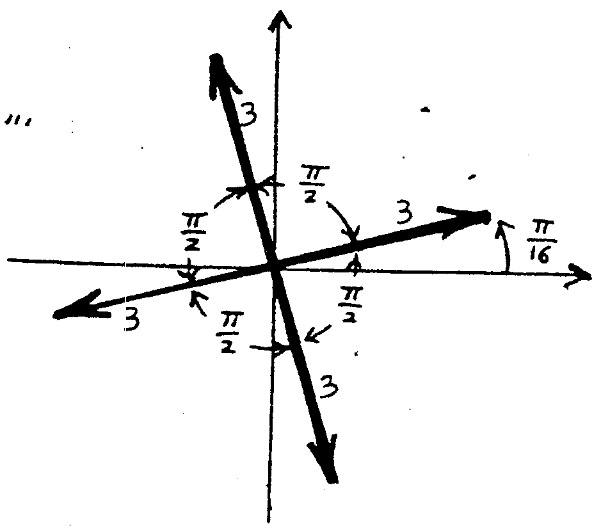
14/ $z^4 = \frac{81}{\sqrt{2}}(1 + i)$

$z^4 = 81e^{i(\frac{\pi}{4} + 2\pi n)}$, $n = 0, \pm 1, \dots$

$z = 3e^{i(\frac{\pi}{16} + \frac{\pi n}{2})}$

$z = 3e^{i\frac{\pi}{16}}$, $3e^{i\frac{17\pi}{16}}$,

$3e^{i\frac{9\pi}{16}}$, $3e^{i\frac{25\pi}{16}}$

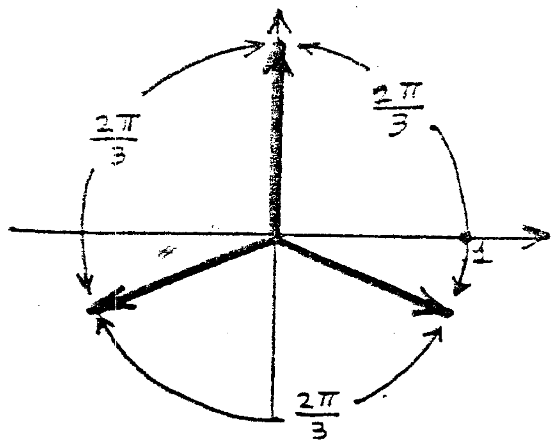


15/ $z^3 = -i$

$$z^3 = e^{\left(\frac{3\pi}{2} + 2\pi n\right)i}, \quad n = 0, \pm 1, \ldots$$

$$z = e^{\left(\frac{\pi}{2} + \frac{2\pi n}{3}\right)i}$$

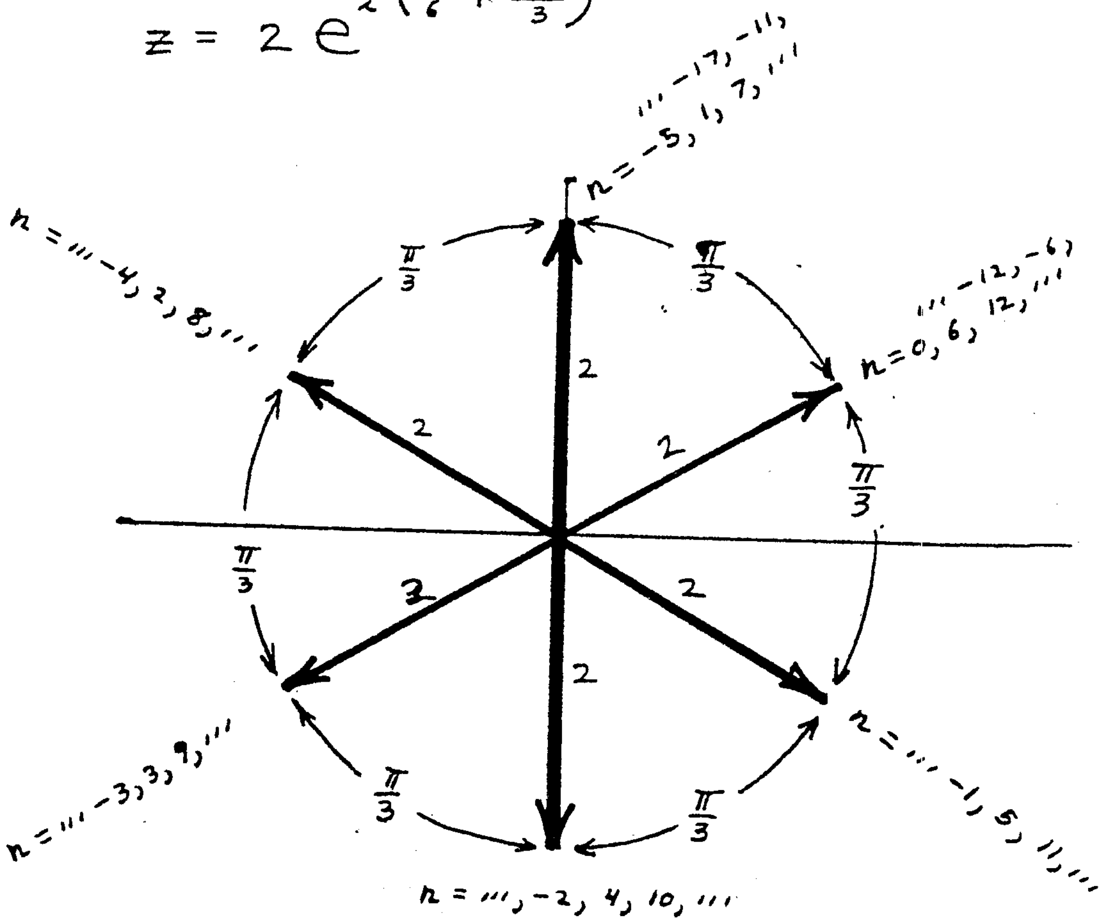
$$z = i, \; -\frac{\sqrt{3}}{2} - \frac{1}{2}i, \; \frac{\sqrt{3}}{2} - \frac{1}{2}i$$



16/ $z^6 = -64$

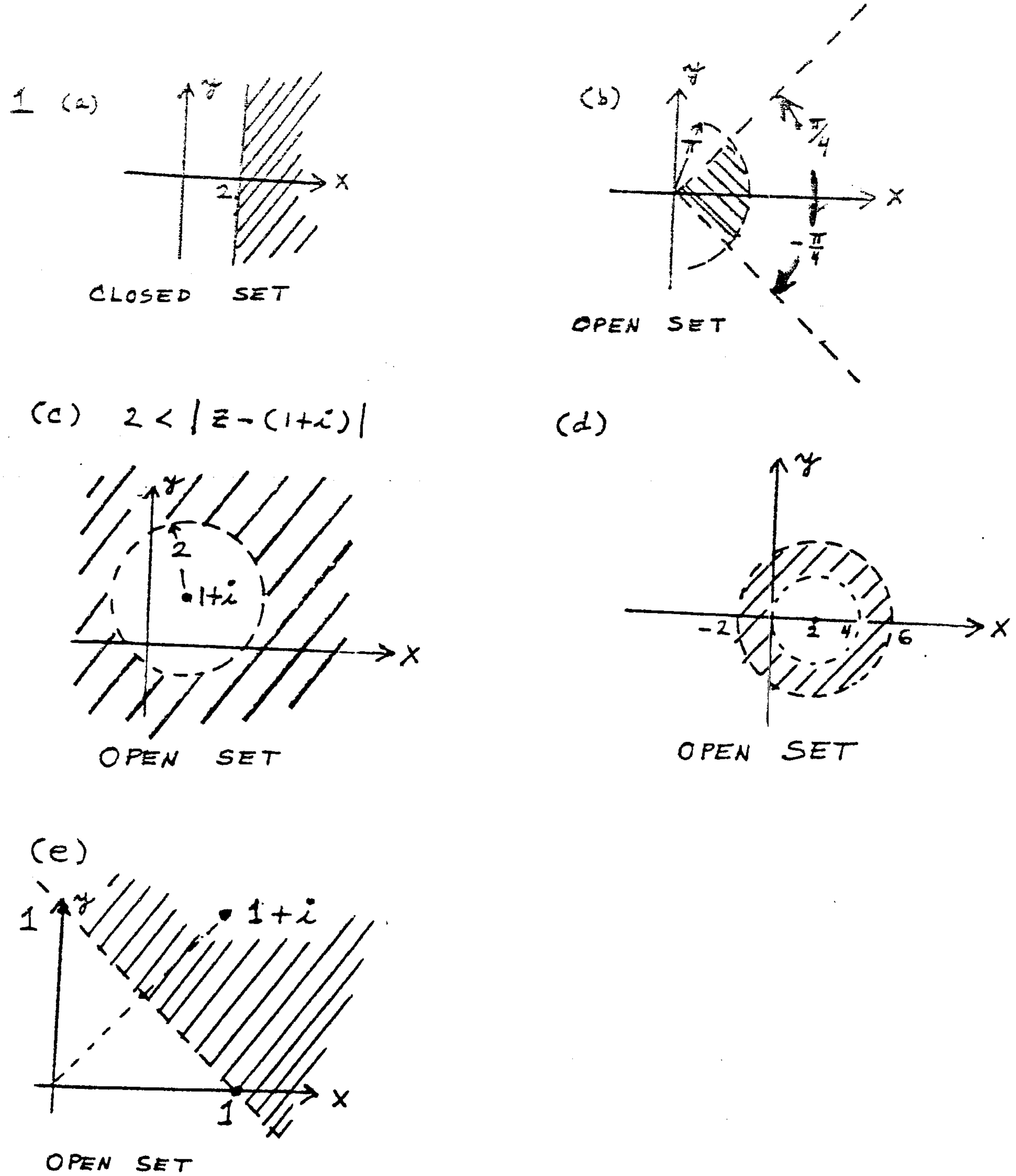
$$z^6 = 64\, e^{i(\pi + 2\pi n)}, \quad n = 0, \pm 1, \ldots$$

$$z = 2\, e^{i\left(\frac{\pi}{6} + \frac{\pi n}{3}\right)}$$



Solutions to Review Problems from Chapter 1

1 (a) (b) (c) $2 < |z-(1+i)|$ (d) (e)



2/ (a) $z = 2e^{i\frac{\pi}{3}}$, $w = 3e^{i\frac{\pi}{2}}$, $\zeta = 4e^{-i\frac{\pi}{6}}$.

WE CAN ALSO, IF WE WISH, ADD $2\pi n i$ TO EACH EXPONENT, WHERE $n = 0, \pm 1, \pm 2, \ldots$.

2/(b) $|zw\zeta| = |z|\cdot|w|\cdot|\zeta| = 2\cdot3\cdot4 = 24$

(c) $\arg(zw\zeta) = \arg(z) + \arg(w) + \arg(\zeta)$

$= \frac{\pi}{3} + \frac{\pi}{2} - \frac{\pi}{6} = \frac{2\pi}{3}$

(d) $\arg(\overline{zw\zeta}) = -\arg(zw\zeta) = -\frac{2\pi}{3}$.

3/ $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{iw} = \cos w + i\sin w$

$e^{i\theta}\cdot e^{iw} = (\cos\theta + i\sin\theta)(\cos w + i\sin w)$

$= (\cos\theta\cos w - \sin\theta\sin w) + i(\sin\theta\cos w + \cos\theta\sin w)$

$= \cos(\theta+w) + i\sin(\theta+w)$

$= e^{i(\theta+w)}$

4/ $-81 = 81e^{i(\pi+2\pi n)}$, WHERE $n = 0, \pm1, \pm2, \ldots$

$z = (81e^{i(\pi+2\pi n)})^{\frac{1}{4}} = 81^{\frac{1}{4}}e^{i(\frac{\pi}{4}+\frac{\pi n}{2})}$

$$= \begin{cases} 3e^{i\frac{\pi}{4}} \text{ (FOR } n=0) = 3\left(\frac{1+i}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i \\ 3e^{i(\frac{\pi}{4}+\frac{\pi}{2})} \text{ (FOR } n=1) = 3e^{i\frac{\pi}{4}}i = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i \\ 3e^{i(\frac{\pi}{4}+\pi)} \text{ (FOR } n=2) = 3e^{i\frac{\pi}{4}}(-1) = -\frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i \\ 3e^{i(\frac{\pi}{4}+\frac{3\pi}{2})} \text{ (FOR } n=3) = 3e^{i\frac{\pi}{4}}(-i) = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i \end{cases}$$