AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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CHAPTER 3

SOME GENERAL PROPERTIES OF ANALYTIC FUNCTIONS

In this chapter we reflect upon general properties of the functions we have studied in detail in Chapter 2. By examining both analytical and graphic representations of these functions, we can anticipate properties that are likely to be true for many of the functions that we have not yet studied, but that we will meet in the future. While these general properties are sometimes quite difficult to derive in a rigorous manner, we will see that it is often easy to guess or conjecture their nature. Thus we will learn some rather far reaching results concerning general functions through simple inductive reasoning.

3.1 Regular and singular points

In Chapter 2 we were careful to obtain "natural" definitions for our functions. We have not given a precise meaning to the term "natural". Yet, we sense intuitively that these definitions which arose through the manipulation of formulas are "natural" since the mathematics itself seemed to suggest them. These definitions for $\sin z$, $\log z$, e^z, etc. did not come from the arbitrary fancy of our imaginations. Natural functions of a complex variable are usually termed " analytic functions" or " regular functions" by mathematicians. All the functions we studied in Chapter 2 were "analytic functions".

Consider now the function

(1) $$w = \frac{z^3 + 2z}{(z - 3)(z + i)^2}.$$

For all values of z, with the exception of the two peculiar values $z = 3$ and $z = -i$, this function is easily computed, and the behavior of the function near these points is quite "nice". These "nice" values of z are called "regular" or "analytic" points for this function. The peculiar values $z = 3$ and $-i$ are called "singular points. Notice that as z approaches either 3 or $-i$, $|w|$ approaches infinity.

<u>Example 1</u>

Find the singular points of the function $w = \cot z$.

<u>Solution</u>

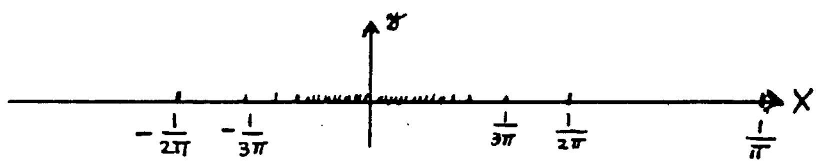
Since $\cot z = \cos z/\sin z$, and since $\sin z$ is zero only for $z = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, (see Figures 2.7 and 2.8), we see that cot z has $z = n\pi$ as its singular points. All other values of z are called "regular" or "analytic" points for cot z.

<u>Example 2</u>

Find the singular points of the function $w = \cot(1/z)$.

<u>Solution</u>

We saw in Example 1 that cot z is singular at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, thus $\cot(1/z)$ is singular for $z = 1/n\pi$. Since $\cot(1/z)$ is not defined at $z = 0$, this point is also a singular point. Notice that $z \neq 0$ is a "limit point" of the singular points. Any circle, however small centered at $z = 0$ contains an infinity of singular points.



Singular points $1/n\pi$ crowd into $z=0$ from both sides on the x-axis making $z=0$ a "limit point" of singularities.

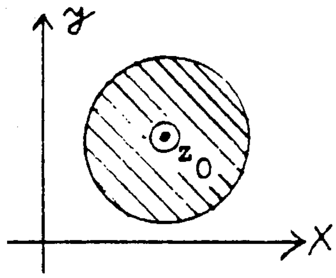
Example 3

Find the singular points of the function $w = \sqrt{z}$.

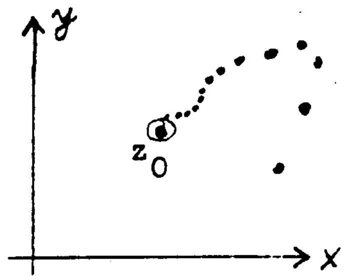
Solution

There are no values of z at which division by zero identifies singularities, yet we do know that $\sqrt{z}$ behaves in a very unusual manner at the point $z = 0$. This is the branch point for this function, and branch points are always singular points.

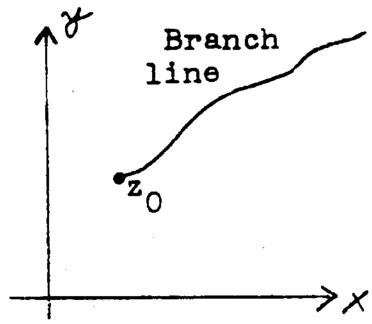
A singular point is said to be "isolated" if it is completely surrounded by regular points. More precisely, an isolated singular point is at the center of some circle all of whose points are regular (except the center itself). The singular points of cot z (Example 1) are isolated. All singularities except $z = 0$ of $\cot(1/z)$ (Example 2) are isolated. The function $w = \sqrt{z}$ has a non-isolated singularity at $z = 0$ because a branch line of discontinuities emerges from this point. (Example 3).



Isolated Singularity at $z = z_0$
The point $z = z_0$ is the center of a circle all of whose points are regular except z_0.



Non-isolated singular point at $z = z_0$.
The point $z = z_0$ is a "limit point of singularities"



Non-isolated singular point at $z=z_0$.
The point $z=z_0$ is a branch point.

Example 4

Determine the singular points of the function

$$w = \frac{e^{1/z}}{z^2 + 1}\sqrt{\frac{z + 2}{z - 2}}$$

and classify them as isolated or non-isolated singularities.

Solution

Since the function $e^{1/z}$ approaches infinity as z approaches zero along the positive real axis, z = 0 is a singular point. Because $e^{1/z}$ is single valued and well defined at all values of z surrounding the origin, z = 0 is and isolated singular point. Since $z^2 + 1 = (z+i)(z-i)$ we see that both i and -i are also isolated singular points. The function "square root" has branch points at those values of z which make the expression under the symbol $\sqrt{}$ zero or infinity. Thus the points z = 2 and z = -2 are branch points and are non-isolated singularities.

It is sometimes useful think of $z = \infty$ as a point. The usual values of z form what we call the "finite z-plane", and when we add the point at infinity we speak of the "extended z-plane". To examine the nature of a function $w = f(z)$ at the point at infinity, we replace z by $1/\zeta$ and examine the nature of $w = f(1/\zeta)$ at $\zeta = 0$.

Example 5

Determine the nature of the function

$$(2) \qquad w = \frac{z^2}{z + 1}$$

at infinity.

Solution

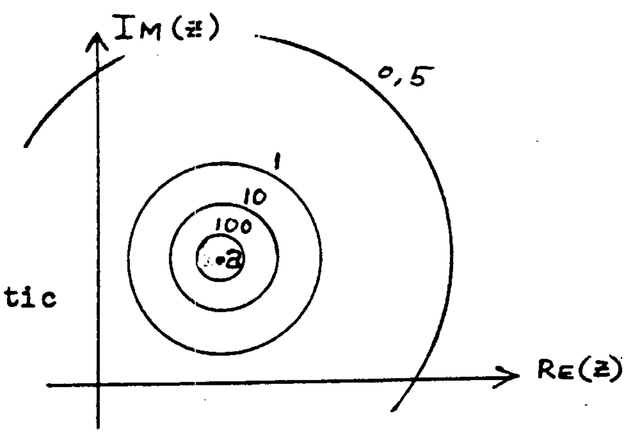
Replacing z by $1/\zeta$ we get

$$w = \frac{1/\zeta^2}{1/\zeta + 1} = \frac{1}{\zeta + \zeta^2} = \frac{1}{\zeta(\zeta+1)} .$$

Since this function of ζ has an isolated singular point at $\zeta = 0$, we say that our original function has an isolated singularity at infinity.

When an isolated singularity of a function is due to division by the factor $(z-a)^n$, ($n = 1,2,3,...$), we call the singularity at $z = a$ " a pole of order n". Poles of order one are usually called "simple poles". For example, the function given by (1) has a simple pole at $z = 3$ and a pole of order two at $z = -i$. The function given by (2) has simple poles at $z = -1$ and ∞.

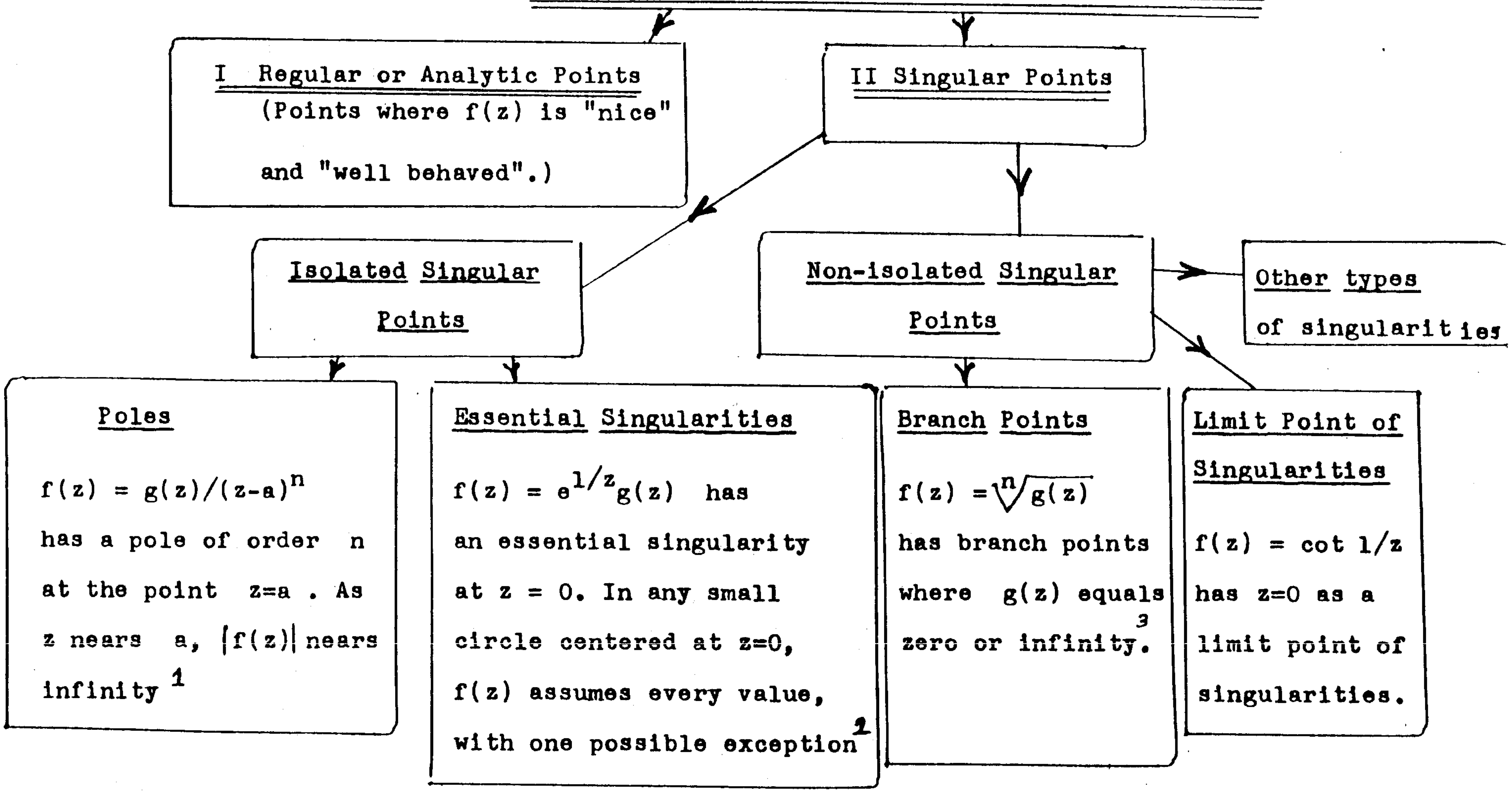
We have just seen that a pole at the point $z = a$ is an isolated singularity due to division by the factor $(z-a)^n$. Since $(z-a)^n$ approaches zero from every direction as z approaches a ; the function itself approaches infinity from every direction as z approaches a . The diagram shows the lines of constant modulus plotted over the z-plane. Note that as we get near $z = a$, these level lines approach circles centered at $z = a$. This is characteristic of a pole at the point $z = a$. Another graphic example is seen in Figure 2.4 which shows the simple pole at z=0 of the function $w = 1/z$.



There is another type of isolated singularity, which unlike the pole, approaches many different values as z approaches the singular point. This type of isolated singularity is called an "essential singularity". Figure 3.1 shows the contour plot for the function $w = e^{1/z}$. This function has an essential singularity at $z = 0$. Examine this graphic representation carefully. Notice that every curve of the form ρ = constant passes through the origin. (Contrast this with the pole in which these curves surround the singularity rather pass throgh it.) Also every curve of the form ϕ = constant passes through the origin. The only value of ρ which is not seen is $\rho = 0$. Thus we see the following remarkable property of this function: "Inside every circle, however small, centered at $z = 0$, the function $w = e^{1/z}$ assumes every value except $w = 0$! "

There are other types of singularities besides those described above, but these are the most important, and the most often encountered. We will have much more to say about singularities in the future, but for now, we summarize the notions just introduced.

TYPES OF POINTS FOR AN ANALYTIC FUNCTION $w = f(z)$



1. $g(z)$ regular and not zero at $z=0$.
2. $g(z)$ regulat at $z = 0$.
3. It is assumed that factors such as $(z-a)^n$ have been removed from under the radical.

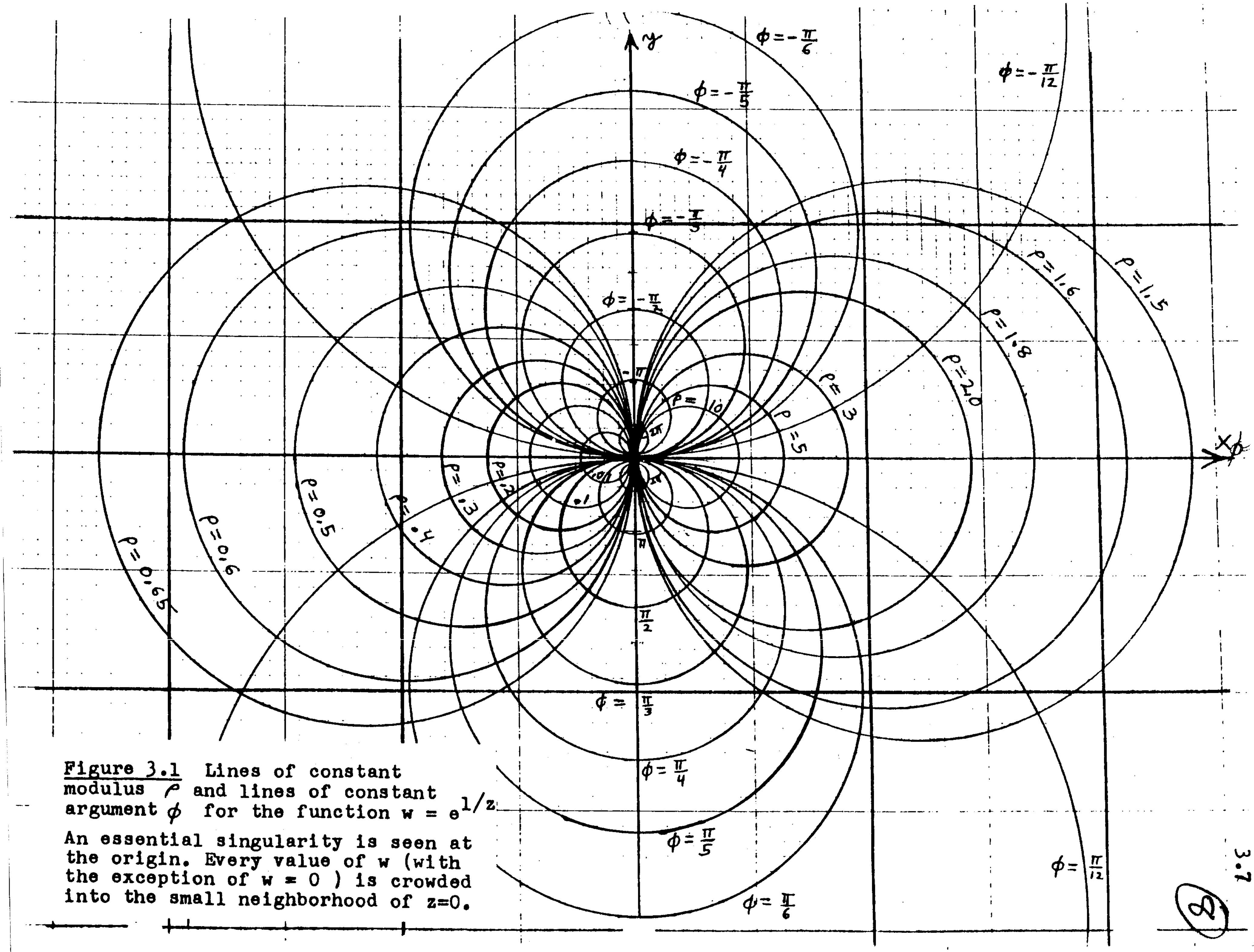


Figure 3.1 Lines of constant modulus ρ and lines of constant argument ϕ for the function $w = e^{1/z}$

An essential singularity is seen at the origin. Every value of w (with the exception of w = 0) is crowded into the small neighborhood of z=0.

(9)

Problem

1. For each function, locate the singular points in the extended complex z-plane. Determine if the singularities are (a) isolated or (b) non-isolated. If the singularities are isolated, are they poles (give order) or are they essential singularities ? If the singularity is non-isolated, is it a branch point or a limit point of singularities?

(i) $z^2 + z + 1$, (ii) $\sin z$, (iii) $\dfrac{z^2 + 2z + 1}{z^2 + 4}$,

(iv) $\dfrac{z^3}{z^2 + 4z + 4}$, (v) $(z^3 + 3z^2 + 3z + 1)^{-1}$,

(vi) $\dfrac{z - 2}{z^2 - 4}$, (vii) e^z , (viii) $\csc z$, (ix) $\sqrt{z^2 + 1}$,

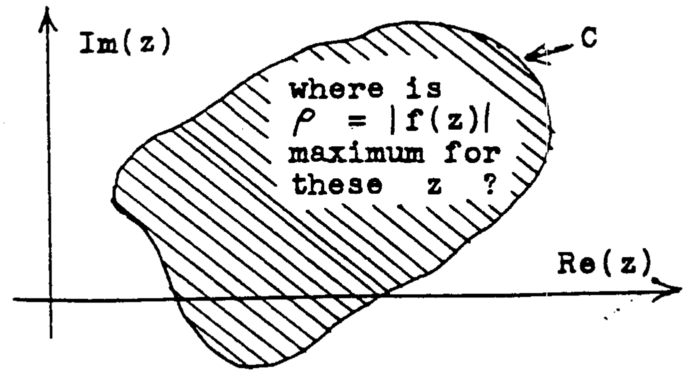
(x) $\exp(1/(z-1))$, (xi) $\sqrt{\dfrac{z+1}{z+3}}$, (xii) $\log z$,

(xiii) $\log(z^2 - 1)$.

While our notions of singular and regular points have not been introduced with mathematical precision, nevertheless, we have been able to identify and classify them for many important functions. As we continue our study, we will gain a clearer picture of the nature of regular and singular points.

3.2 The maximum modulus theorem

In the elementary calculus, considerable effort was extended to locate the maximum values of functions. In the calculus we are developing here in the complex plane, there is an important theorem which gives us information concerning where to find the value of z which makes the modulus $\rho = |f(z)|$ a maximum. In particular, we are concerned only with values of z inside or on a simple closed curve C, (a curve is <u>simple</u> if it does not interesect itself), and we ask where is $\rho = |f(z)|$ a maximum for these values of z.



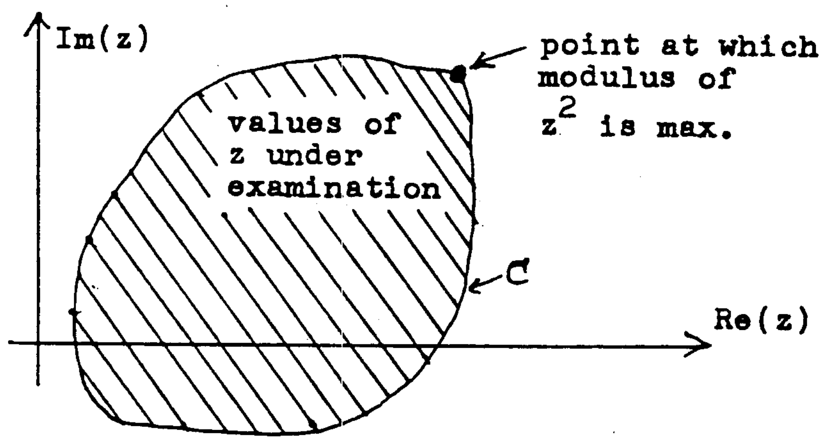
We will now examine several special examples in an attempt to discover information concerning the location of the"maximum modulus" .

<u>Example</u>

For any closed curve drawn on the complex plane, determine the value (or values) of z, inside or on the curve, at which the function $f(z) = z^2$ assumes its maximum modulus.

<u>Solution</u>

The modulus of z^2 is $|z^2| = |r^2 e^{2\theta i}| = r^2$. The expression r^2 is greatest when the point z is furthest from the origin. Thus the maximum modulus occurs on the curve itself, at the point (or points) which are the greatest distance from the origin.



Problem

2. Draw a simple closed curve on the complex z-plane. For each function f(z) given, determine the value (or values) of z (either inside or on the curve) at which the modulus of the function becomes a maximum. For each function considered, give a statement as to where the maximum modulus is located for all points z inside or on an arbitrarily given closed curve.

(a) $4z^3$, (b) e^z (look at Figures 2.5 and 2.6), (c) 1/z (Figure 2.4), (d) sin z (Figure 2.7), (e) $e^{1/z}$ (Figure 3.1).

Conjecture 3.1

Given a general function w = f(z), and a simple closed curve C on the complex plane. After making suitable restrictions on the function f(z), where will the maximum modulus of f(z) occur, when we consider all values of z inside or on C ? (See Appendix 2 for a discussion of the answer.)

3.3 The argument principle

In the previous section we examined a theorem concerning the modulus of an analytic function. In this section we will examine several examples of the behavior of the argument of a function. From these special cases we can conjecture a general theorem known as the " Argument Principle".

The "Argument Principle" that we will eventually discover relates the following three quantities:

Z(f;C) = The number of zeros of the function f(z) inside the simple closed curve C . Each zero is counted as often as its order.

$P(f;C)$ = The number of poles of the function $f(z)$ inside the curve C. Each pole is counted as often as its order.

$A(f;C)$ = The net change in the argument of $f(z)$ divided by 2π as we traverse the curve C once in the positive sense.

Example

Let $f(z)$ be the function $w = z^2$. Let the curve C be the circle of radius one centered at the origin in the z-plane. Determine $Z(f;C)$, $P(f;C)$, and $A(f;C)$.

Solution

Since $f(z) = z^2$ has a zero of order two at $z = 0$, and no other zeros inside C, $Z(f;C) = 2$. Since $f(z) = z^2$ has no poles inside C, $P(f;C) = 0$. Look at Figure 2.2 and watch the argument ϕ as we traverse C once, starting and ending at $z = -1$. We traverse C in the positive (counterclockwise) sense. We note that ϕ varies continuously from -2π to 2π. Thus

$$A(f;C) = \frac{\text{net change in } \phi}{2\pi} = \frac{2\pi - (-2\pi)}{2\pi} = \frac{4\pi}{2\pi} = 2.$$

We see that $A(f;C)$ is the net number of (positive) revolutions made by the vector w as we traverse the curve C once in the positive sense.

Problems

3. Consider the function $w = \tan z$ shown in Figure 3.2 . Determine the values of $Z(\tan z; C)$, $P(\tan z; C)$, and $A(\tan z; C)$ for each of the following curves. Note that $\tan z$ has simple poles where z equals $\ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$, and simple zeros at z equals $\ldots, -\pi, 0, \pi, 2\pi, \ldots$.

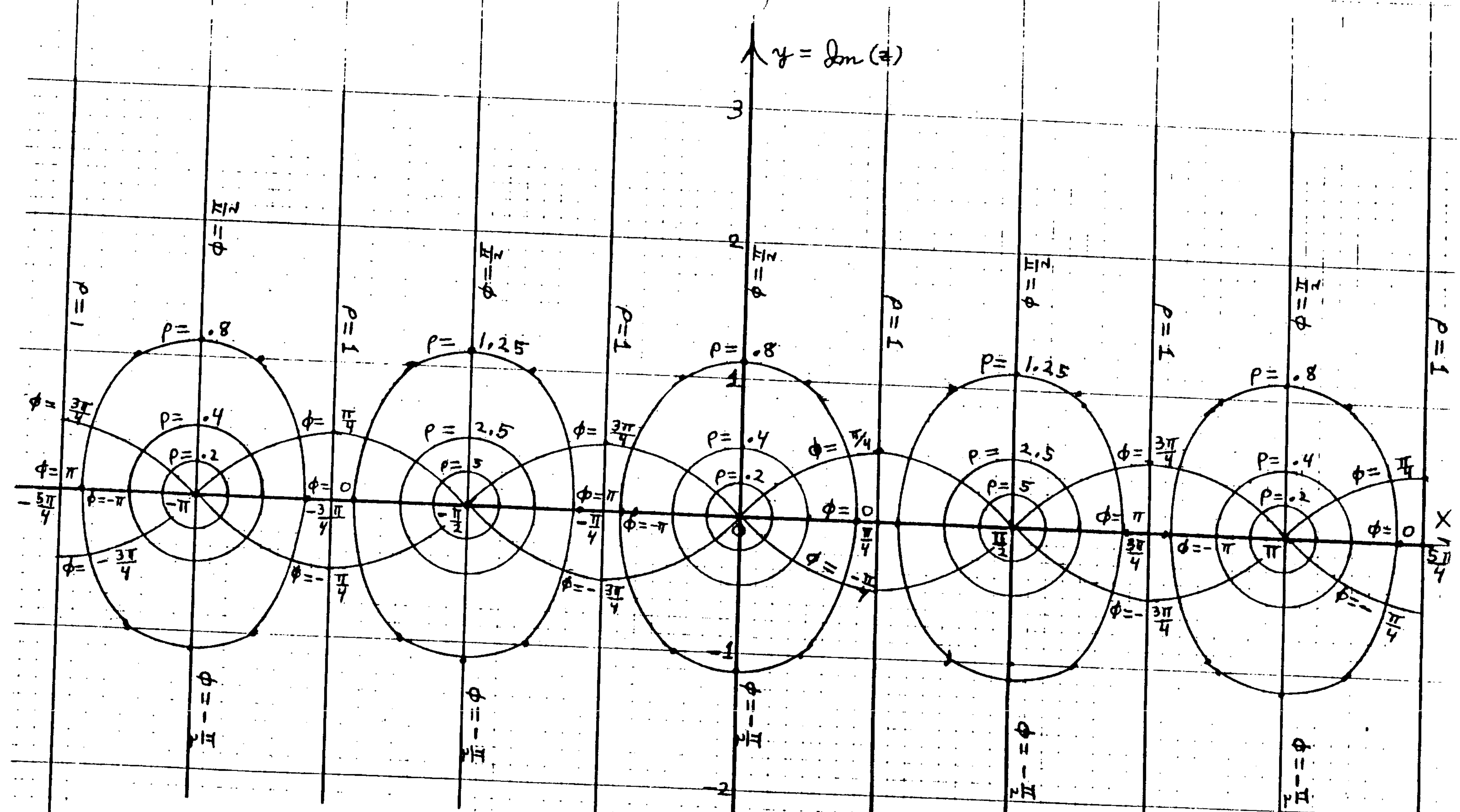


Figure 3.2 Lines of constant modulus and lines of constant argument for the function $w = \tan z$. Note the zeros at $z = n\pi$, and the poles at $z = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The function has period π.

(a) The rectangle having verticies at i, $\pi/2 + i$, $\pi/2 + 2i$, and $2i$; (b) $|z| = 1$; (c) $|z| = 3$; (d) $|z| = 4$.

4. Consider the function $w = z^{-2}$. Determine $Z(z^{-2}; C)$, $P(z^{-2};C)$ and $A(z^{-2};C)$ for (a) the curve $|z| = 1$ and (b) the curve $|z - 1 - i| = 1$.

Conjecture 3.2

From the experience just gathered, conjecture a theorem relating the three quantities $Z(f;C)$, $P(f;C)$ and $A(f;C)$. (See Appendix 2 for the result.)

3.4 Rouché's Theorem

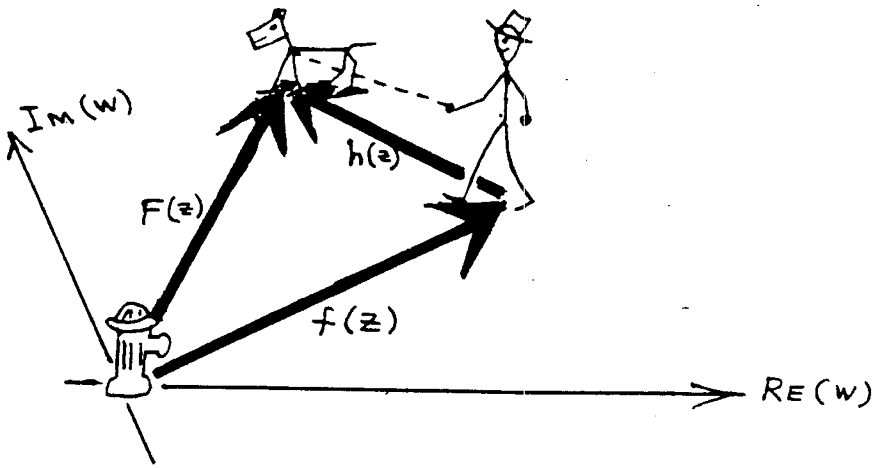
We now state a theorem which allows us to determine the number of zeros of certain functions located inside a closed curve. It is called "Rouché's Theorem". If we assume that the argument principle studied in the previous section is true, then we can give a rigorous mathematical proof for Rouché's theorem without difficulty. Thus the statement and proof of the theorem are given.

Rouché's Theorem :

Let C be a simple closed curve on the z-plane. Let $f(z)$ and $h(z)$ be analytic functions at all points inside and on the curve C. Assume also that $f(z)$ has no zeros on C, and that $|h(z)| < |f(z)|$ for each z on the curve C. Then the functions $f(z)$ and $F(z) = f(z) + h(z)$ have the same number of zeros inside C.

Proof:

We know from the argument principle that $f(z)$ and $F(z) = f(z) + h(z)$ have the same number of zeros inside C if the vector f circles the origin the same number of times as does F when z traverses C. Think of the origin as a fire-hydrant, and think



of the vector $f(z)$ as giving the location of a man who walks around the hydrant. The man holds a dog on a leash which is described by the vector $h(z)$. It is clear that if the man holds the leash short enough, then the dog circles the hydrant the same number of times as does the man (i.e. $F(z)$ makes the same number of revolutions as does $f(z)$). How short must the leash be for this purpose ? Short enough so that the dog cannot reach the hydrant. But the hypothesis assures this situation since $|h(z)| < |f(z)|$ (length of the leash is less than the distance of the man from the hydrant). Thus the theorem is proved.

We can sometimes use Rouché's theorem to determine the number of zeros of a given function in a specific region.

Example

How many zeros does $z^4 + 2z + 5$ contain in the region $1 < |z| < 2$?

Solution

First let C be the circle $|z| = 2$, let $f(z) = z^4$ and let

$h(z) = 2z + 5$. On the curve C, $|h(z)| = |2z + 5| \leq |2z| + |5| \leq 4+5 = 9$. But $|f(z)| = z^4 = 16$ on the curve $|z| = 2$, and thus the hypothesis of Rouché's Theorem is satisfied. Since $f(z) = z^4$ has four zeros inside $|z| = 2$, Rouche's Theorem tells us that $f(z) + h(z) = z^4 + 2z + 5$ also has four zeros inside $|z| = 2$.

Next let C be $|z| = 1$, let $f(z) = 5$ and let $h(z) = z^4 + 2z$. On $|z| = 1$, $|h(z)| = |z^4 + 2z| \leq |z^4| + |2z| \leq 1+2 = 3$. But $|f(z)| = 5$ on $|z| = 1$, and thus the hypothesis of Rouché's Theorem is satisfied. Since $f(z)$ has no zeros inside $|z| = 1$, neither does $z^4 + 2z + 5$. Thus $z^4 + 2z + 5$ has four zeros in the region $1 < |z| < 2$.

<u>Problems:</u>

5. How many roots does $z^3 + z + 5 = 0$ have between the circles $|z| = 1$ and $|z| = 2$?

6. How many roots does $z^4 + z^3 + 30 = 0$ have in the anulus $2 < |z| < 3$?

7. Let $a > e$. Show that $az^n + e^z = 0$ has n roots in $|z| < 1$.

8. Show that all roots of $z^4 + z^3 + 1 = 0$ are inside the circle $|z| < 3/2$.

9. Show that every polynomial equation of degree n

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0$$ has n complex roots.

(Fundamental Theorem of Algebra)

Review problems for Chapter 3

1. Classify the singularities of $f(z) = \sqrt{z^2 - 1}\; e^{1/z} \sec z$.

2. Let C be the circle $|z| = 1$. Which of the following functions satisfy the hypothesis of the Maximum Modulus Theorem ?
(a) $\sin z$, (b) $\exp(z^2)$, (c) $\exp(z^{-2})$, (d) $\csc z$,
(e) $\sqrt{4z^2 - 1}$, (f) $(4z - 1)^{-1}$.

3. Minimum Modulus Theorem:

If $f(z)$ is analytic inside and on the simple closed curve C, and $f(z) \neq 0$ inside or on C, then $|f(z)|$ assumes its minimum value on C.

Let C be an arbitrary simple closed curve, and let $f(z) = e^z$. Locate the minimum modulus, and thereby verify the Minimum Modulus Theorem for this function.

4. Prove the Minimum Modulus Theorem. (Hint: Consider the function $1/f(z)$ and apply the Maximum Modulus Theorem.)

5. Determine a value for R such that all the roots of $z^3 + iz + 4 = 0$ are inside the circle $|z| = R$.

(18)

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 3

1/(i) z^2+z+1 has no singularities in the finite plane. at ∞, replace z by $\frac{1}{\zeta}$ and get

$\frac{1}{\zeta^2}+\frac{1}{\zeta}+1 = \frac{1+\zeta+\zeta^2}{\zeta^2}$ which has an isolated singularity at $\zeta=0$ which is a pole of second order. Thus z^2+z+1 has an isolated singularity at infinity which is a second order pole.

(ii) $\sin z$ has no singularities in the finite plane. at $z=\infty$, replace z by $\frac{1}{\zeta}$ and get $\frac{1}{2i}\left[e^{i\frac{1}{\zeta}}-e^{-i\frac{1}{\zeta}}\right]$

Each of these exponentials has an essential singularity at $\zeta=0$. Thus $\sin z$ has an isolated singularity at $z=\infty$ which is an essential singularity.

(iii) $\frac{z^2+2z+1}{z^2+4} = \frac{(z+1)^2}{(z+2i)(z-2i)}$ has isolated singularities at $z=-2i$ and $z=2i$ which are simple poles. At $z=\infty$ we set $z=\frac{1}{\zeta}$ and get

$\frac{\zeta^{-2}+2\zeta^{-1}+1}{\zeta^{-2}+4} = \frac{1+2\zeta+\zeta^2}{1+4\zeta^2}$, which has a regular point at $\zeta=0$. Thus, the only singularities of $\frac{z^2+2z+1}{z^2+4}$ in the extended z-plane are isolated singularities at $z=\pm 2i$ which are simple poles.

1/(iv) $\frac{z^3}{z^2+4z+4} = \frac{z^3}{(z+2)^2}$ has a

pole of order two at $z=-2$, At infinity

$$\frac{\zeta^{-3}}{\zeta^{-2}+4\zeta^{-1}+4} = \frac{1}{\zeta+4\zeta^2+4\zeta^3} = \frac{1}{\zeta(4\zeta^2+4\zeta+1)}$$

which has a simple pole at $\zeta=0$. Thus $\frac{z^3}{z^2+4z+4}$ has an isolated singularity at $z=-2$ and at $z=\infty$. At $z=-2$ the singularity is a pole of order two, at $z=\infty$ its a simple pole.

(v) $(z^3+3z^2+3z+1)^{-1} = (z+1)^{-3}$ which has only a pole of order three at $z=-1$ (isolated singularity). Set $z=\frac{1}{\zeta}$ and get $(\zeta^{-1}+1)^{-3} = \frac{\zeta^3}{(\zeta+1)^3}$. Thus $\frac{1}{(z+1)^3}$ is regular at infinity.

(vii) e^z has only an essential singularity at infinity (isolated singularity).

(viii) $\csc z = \frac{1}{\sin z}$, and since $\sin z$ has simple zeros at $z=n\pi$, $(n=0,\pm1,\pm2,\dots)$, $\csc z$ has simple poles at $z=n\pi$ (isolated singularities). Since $\csc\frac{1}{\zeta}$ has poles at $\zeta=\frac{1}{n\pi}$, $\zeta=0$ is a limit point of singularities, Thus $z=\infty$ is a non-isolated singularity (limit point of singularities).

1/ (ix) $\sqrt{z^2+1}$ has branch point singularities at the values of z that make $z^2+1=0$. These are $z=\pm i$ (non-isolated singularities).

Setting $z=\frac{1}{\zeta}$ we get $\sqrt{\frac{1}{\zeta^2}+1} = \frac{\sqrt{1+\zeta^2}}{\zeta}$ which has a simple pole at $\zeta=0$. Thus $\sqrt{z^2+1}$ has a simple pole at infinity (isolated singularity).

(x) $e^{\frac{1}{z-1}}$ has an essential singularity (isolated) at $z=1$. Let $z=\frac{1}{\zeta}$ and get $\exp\left(\frac{1}{\zeta^{-1}-1}\right) = \exp\left(\frac{\zeta}{1-\zeta}\right)$ which is regular at $\zeta=0$. Thus $e^{\frac{1}{z-1}}$ is regular at infinity.

(xi) $\sqrt{\frac{z+1}{z+3}}$ has branch points at $z=-1$ and $z=-3$ (non-isolated singularities). Let $z=\zeta^{-1}$ and get $\sqrt{\frac{\zeta^{-1}+1}{\zeta^{-1}+3}} = \sqrt{\frac{1+\zeta}{1+3\zeta}}$ which is regular at $\zeta=0$. Thus $\sqrt{\frac{z+1}{z+3}}$ is regular at $z=\infty$.

1/ (xii) $\log z$ has a branch point at $z=0$ (non-isolated singularity). Since $\log \frac{1}{\zeta} = -\log \zeta$ we see that $\log z$ also has a branch point at infinity.

(xiii) $\log(z^2-1) = \log(z+1) + \log(z-1)$, and thus $\log(z^2-1)$ has branch points at $z = \pm 1$. Since $\log(\zeta^{-2}-1) = \log \frac{1-\zeta^2}{\zeta^2}$ $= -2\log \zeta + \log(1-\zeta^2)$, and since $\log \zeta$ has a branch point at $\zeta = 0$, then $\log(z^2-1)$ has a branch point at $z=\infty$. All three of these singularities are non-isolated.

2/ (a) Since $|4z^3| = 4r^3$, the maximum modulus occurs at that value of z (or values) on the curve itself furthest from the origin.

(b) Since $|e^z| = e^x$, the maximum modulus occurs at that value (or values) of z which are located furthest to the right on the curve itself (i.e. x is largest).

2/(c) There are two cases:

1. If the point $z=0$ is inside or on the curve C, then there is no point at which the maximum is obtained. ($\frac{1}{z}$ is undefined at $z=0$, and $\lim_{z\to 0} \frac{1}{z} = \infty$)

2. If the point $z=0$ is outside the curve C, then the maximum modulus is achieved at that point (or points) on the curve C itself closest to the origin since $|z^{-1}| = r^{-1}$.

(d) The maximum modulus always occurs on the curve C itself.

(e) There are three cases:

1. If $z=0$ is inside the curve C, then the maximum is never obtained since $e^{\frac{1}{z}} \to \infty$ as $z \to 0$ ~~from~~ the right half plane.

2. If $z=0$ is on the curve C itself, then for all values of $z \neq 0$ inside or on C, the maximum is obtained on the boundary C itself if C is entirely in the left half plane $Re(z)<0$. If not, the maximum modulus is never obtained.

3. If $z=0$ is outside C, then the maximum modulus is achieved on C itself.

3/ (a) $Z(\tan z; C) = P(\tan z; C) = A(\tan z; C) = 0$.

(b) $Z(\tan z; C) = 1$, $P(\tan z; C) = 0$, $A(\tan z; C) = 1$

(c) $Z(\tan z; C) = 1$, $P(\tan z; C) = 2$, $A(\tan z; C) = -1$

(d) $Z(\tan z; C) = 3$, $P(\tan z; C) = 2$, $A(\tan z; C) = 1$

4/ (a) $Z(\frac{1}{z^2}; C) = 0$, $P(\frac{1}{z^2}; C) = 2$, Since $\phi = -2\theta$,

$A(\frac{1}{z^2}; C) = -2$,

(b) $Z(\frac{1}{z^2}; C) = 0$, $P(\frac{1}{z^2}; C) = 0$, $A(\frac{1}{z^2}; C) = 0$.

5/ Let $f(z) = z^3$ and $h(z) = z + 5$, and let C be $|z| = 2$ in Rouché's Theorem. On C, $|f(z)| = 2^3 = 8$, and $|h(z)| = |z+5| \leq |z| + 5 = 7$. Thus $|h(z)| < |f(z)|$ on C, and since $f(z)$ has a zero of order 3 inside $|z| = 2$, z^3+z+5 has 3 zeros inside $|z| = 2$.

Let $f(z) = 5$, $h(z) = z^3 + z$, and let C be $|z| = 1$ in Rouché's Theorem. On C, $|f(z)| = 5$, and $|h(z)| = |z^3 + z| \leq |z^3| + |z| = 2$. Thus $|h(z)| < |f(z)|$ on C, and $f(z)$ has no zeros inside C, $z^3 + z + 5$ has no zeros inside $|z| = 1$.

Thus $z^3 + z + 5$ has 3 zeros on $1 < |z| < 2$.

6/ Let $f(z) = z^4$, $h(z) = z^3 + 30$, and C be $|z| = 3$ in Rouché's Theorem. On C, $|f(z)| = 3^4 = 81$, and $|h(z)| = |z^3 + 30| \le |z^3| + 30 = 27 + 30 = 57$. Thus $|h(z)| < |f(z)|$ on C, and since $f(z)$ has a zero of order four inside C, $f(z) + h(z) = z^4 + z^3 + 30$ has four zeros inside $|z| = 3$.

Let $f(z) = 30$ and $h(z) = z^4 + z^3$ and let C be $|z| = 2$ in Rouché's Theorem. On C, $|f(z)| = 30$, and $|h(z)| = |z^4 + z^3| \le |z^4| + |z^3| = 2^4 + 2^3 = 24$. Thus $|h(z)| < |f(z)|$ on $|z| = 2$, and since $f(z)$ has no zeros inside $|z| = 2$, $f(z) + h(z) = z^4 + z^3 + 30$ has no zeros inside $|z| = 2$.

Thus $z^4 + z^3 + 30$ has four zeros on $2 < |z| < 3$.

7/ Let $f(z) = a z^n$ and $h(z) = e^z$ and C be $|z| = 1$ in Rouché's Theorem. On C, $|f(z)| = a$, and $|e^z| = e^x \le e$. Thus $|h(z)| < |f(z)|$ on C, and since $f(z)$ has a zero of order n inside $|z| = 1$, $f(z) + h(z) = a z^n + e^z$ has n zeros inside $|z| = 1$.

8/ Let $f(z) = z^4$ and $h(z) = z^3 + 1$ and let C be $|z| = \frac{3}{2}$ in Rouché's Theorem.

On C, $|f(z)| = |z^4| = \frac{3^4}{2^4} = \frac{81}{16}$, and $|h(z)| = |z^3+1| \le |z^3| + 1 = \frac{3^3}{2^3} + 1 = \frac{35}{8}$. Thus $|h(z)| < |f(z)|$ on C, and since $f(z)$ has a zero of order four at $z = 0$, $f(z) + h(z) = z^4 + z^3 + 1$ has four zeros inside $|z| = \frac{3}{2}$.

9/ Let $f(z) = z^n$, and $h(z) = \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_0}{a_n}$, and let C be the circle $|z| = R^2$, where R is some number satisfying

$$\left|\frac{a_0}{a_n}\right| < R$$

$$\vdots$$

$$\left|\frac{a_{n-1}}{a_n}\right| < R$$

$$n < R.$$

On C, $|f(z)| = R^{2n}$, and $|h(z)| = \left|\frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_0}{a_n}\right|$

$$\le \left|\frac{a_{n-1}}{a_n} z^{n-1}\right| + \dots + \left|\frac{a_0}{a_n}\right| < R\,R^{2n-2} + R\,R^{2n-4} + \dots + R$$

$$< R\,R^{2n-2} + R\,R^{2n-2} + \dots + R\,R^{2n-2} = nR^{2n-2} < R^{2n-1}.$$

Thus $|h(z)| < |f(z)|$ on C, and since $f(z)$ has a zero of order n at $z = 0$, $z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_0}{a_n}$ has n zeros inside C. In elementary algebra, we learn that a polynomial of degree n can have no more than n roots.

Solutions to Review Problems from Chapter 3

1/ Since $\sqrt{z^2-1} = \sqrt{(z+1)(z-1)}$, $z=-1$ and $z=1$ are branch points (non-isolated singularities). The factor $e^{1/z}$ gives an essential singularity at $z=0$ (isolated singularity). Since $\sec z = \frac{1}{\cos z}$, and since $\cos z$ has zeros at $z = \frac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \cdots$, we have simple poles at $z = \frac{\pi}{2} + n\pi$ (isolated singularities). To investigate the nature of $f(z)$ at $z=\infty$, set $z = \frac{1}{\zeta}$ and get

$$f\left(\frac{1}{\zeta}\right) = \sqrt{\frac{1-\zeta^2}{\zeta^2}}\, e^{\zeta} \sec\frac{1}{\zeta} = \frac{1}{\zeta}\sqrt{1-\zeta^2}\, e^{\zeta} \sec\frac{1}{\zeta}.$$

Since $\sec\frac{1}{\zeta}$ has poles at $\zeta = \frac{1}{\frac{\pi}{2}+n\pi}$, we see that $\zeta = 0$ is a "limit point of singularities". Thus $z=\infty$ is a "limit point of singularities" and is therefore a non-isolated singularity.

2/ The Maximum Modulus Theorem requires that $f(z)$ have no singularities inside or on $|z|=1$. Only $\sin z$ and e^{z^2} satisfy this requirement. $e^{z^{-2}}$ has a singularity at $z=0$, as does $\csc z$. $\sqrt{4z^2-1}$ has singularities at $z = \pm\frac{1}{4}$, and $\frac{1}{4z-1}$ has a singularity at $z = \frac{1}{4}$.

3/ $|f(z)| = |e^z| = e^x$. The function e^x assumes its minimum value on the curve C at the point or points furthest to the left (i.e. the points on C for which Re (z) is a minimum).

4/ Since f(z) has no zeros inside or on C, and since f(z) is regular inside and on C, then $\frac{1}{f(z)}$ is regular inside and on C. Thus the Maximum Modulus Theorem applies to $\frac{1}{f(z)}$ and $\left|\frac{1}{f(z)}\right| = \frac{1}{|f(z)|}$ assumes its maximum value on C itself (say at $z = z_0$). Since $\frac{1}{|f(z_0)|}$ is a maximum, $|f(z_0)|$ must be a minimum. Thus the Minimum Modulus Theorem is proved.

5/ Let $f(z) = z^3$ and $h(z) = iz + 4$ in Rouché's Theorem. On $|z| = R$, $|f(z)| = R^3$, and $|h(z)| = |iz+4| \leq |iz| + 4 = |z| + 4 = R + 4$. Since we must keep $|h(z)| < |f(z)|$ on $|z| = R$, we require of R only that $R + 4 < R^3$. $R = 2$ satisfies this inequality. Thus $z^3 + iz + 4$ has all three of its zeros inside $|z| = 2$.

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 3

3.1 The Maximum Modulus Theorem

Let $w = f(z)$ be analytic at all points inside and on the simple closed curve C. Then the maximum modulus of $f(z)$ occurs at some point (or points) on the curve C.

Discussion:

If $f(z)$ is a constant, then at every value of z, either inside or on the curve C, $|f(z)|$ is the same. If $f(z)$ is not a constant, then the maximum of $|f(z)|$ occurs only on the curve C.

The case where $f(z)$ has a singularity inside or on C is excluded by the hypothesis since we cannot draw a general conclusion unless f is regular inside and on C.

3.2 The Argument Principle

Let $w = f(z)$ be regular at all points inside and on the simple closed curve C, with the exception of a finte number of poles inside C. Then $A(f;C) = Z(f;C) - P(f;C)$.

Discussion:

Picture the point z traversing the curve C once in the positive (counterclockwise)sense. As z traverses C, the vector $f(z)$ (drawn in the w-plane) rotates. The Argument Principle says that the number of positive rotations minus the number of negative rotations of the vector $w = f(z)$ equals the number of zeros minus the number of poles of $f(z)$ inside C.

We restrict the number of poles inside C to be finite, otherwise $P(f;C)$ is infinite.