AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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CHAPTER 5

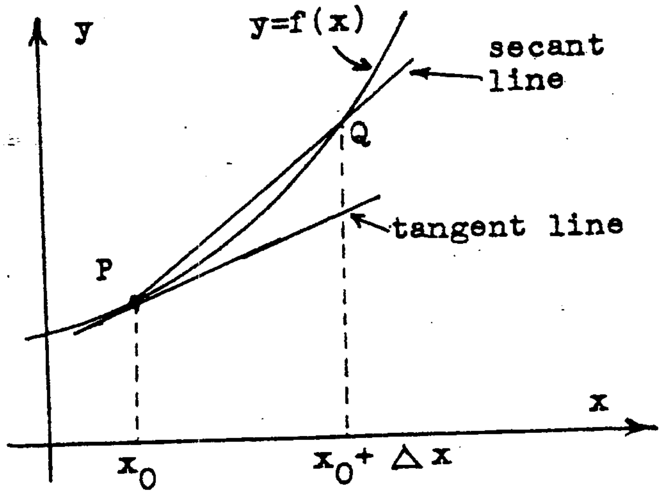
DIFFERENTIATION

In the previous chapter we differentiated functions such as z^p, e^z, sin z, log z, etc., without giving serious attention to the fundamental meaning of differentiation with respect to a complex variable. We simply wrote down the same formulas for derivatives that we learned in the real calculus, and hoped that they would remain true in the new complex calculus. Since our analytic functions are "natural", we have grown to expect that formulas learned previously for real variables carry over into the complex case without change. In this chapter we return to the concept of differentiation and investigate its fundamental and precise meaning. After finding a suitable exact "definition" for the derivative of a function of a complex variable, we will see why derivatives of a complex function which is analytic look essentially like their real counterparts. We will also encounter new expressions which could serve as new definitions for the analyticity of a function. Finally, we will see that this investigation into the basic meaning of differentiation leads us to new insights into the nature of analytic functions.

5.1 The definition of differentiation

What is the real meaning of $f'(z)$ when z is a complex variable ? Let us first review the meaning of the derivative learned in the real calculus. There we defined the derivative of $f(x)$ at x_0 to be

$$(1)\quad f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} ,$$



We think of the secant line drawn through the fixed point P and the moving point Q shown in the figure. As Q approaches P, the slope of the secant line approaches the slope of the tangent line at P. Of course, the slope of this tangent line is the derivative given by the limit in (1). Now the point Q can approach P from two different directions:

(i) from the right (Δx is positive), and

(ii) from the left (Δx is negative).

For $f'(x_0)$ to exist, we require that the limit (1) give the same value in both cases (i) and (ii).

EXAMPLE 1

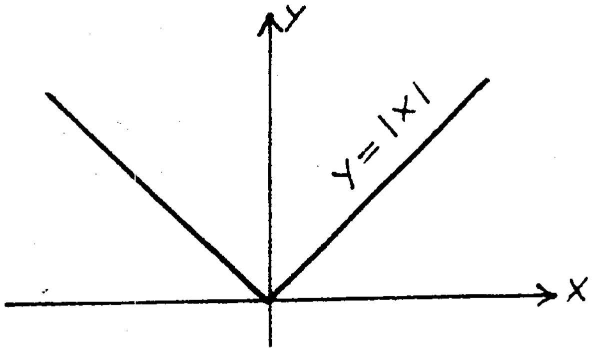
Is the function $f(x) = |x|$ differentiable at $x_0 = 0$?

Solution

For $\Delta x > 0$, we have

$$\lim_{\substack{\Delta x\to 0\\ \Delta x>0}} \frac{f(\Delta x+0)-f(0)}{\Delta x} =$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta x>0}} \frac{|\Delta x+0| - |0|}{\Delta x} = 1 ,$$



and for $\Delta x < 0$ we have

$$\lim_{\substack{\Delta x\to 0\\ \Delta x<0}} \frac{f(\Delta x+0)-f(0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{|\Delta x|}{\Delta x} = -1 .$$

since we get +1 for the limit from the right and -1 for the limit from the left, we see that $f'(0)$ does not exist. It is also very easy to see that $f'(0)$ does not exist just by glancing at the graph of $y = |x|$. We see that just to the right of $x=0$ the slope of the graph is +1, while just to the left of $x=0$ it is -1 . Since these two slopes are not equal, the derivative does not exist at $x=0$.

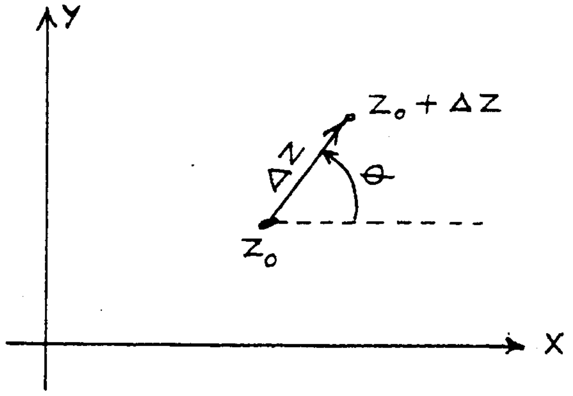
Problem

1. Does the derivative of the function $f(x) = |\sin x|$ exist at the point $x_0 = \pi$?

How should we define $f'(z)$ when f is a complex valued function of a complex variable ? The analogous expression to (1) is

$$(2) \qquad f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z} .$$

Since z is a complex variable, $\triangle z$ can approach zero from many different directions determined by the angle θ shown in the figure, In (1), $\triangle x$ could approach zero from only two directions, the right and the left. In (2), however, $\triangle z$ can approach zero from infinitely many different directions. It is natural to require that the limit given by (2) be only one value, regardless of the direction θ in which $\triangle z$ approaches zero. We select this idea as our precise definition and state it as follows:



DEFINITION OF THE DERIVATIVE

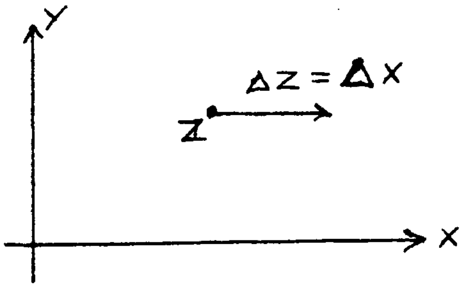
We say that the derivative of the complex valued function $f(z)$ of the complex variable z exists at the point z_0 if and only if the limit given by (2) exists. This limit must give only one value, regardless of the manner in which Δz approaches zero.

Example 2

Does the derivative of the function $f(z) = \sin x + i\,(x + \sin y)$ exist at any point ?

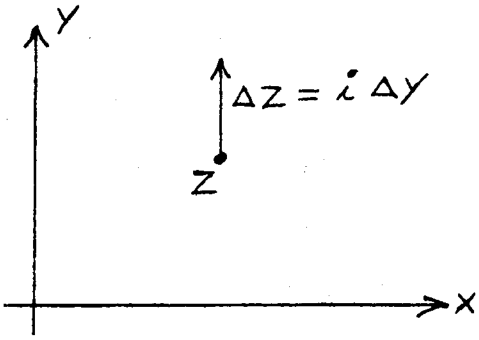
Solution

Consider first the case in which Δz approaches zero with $\theta = 0$. Now $\Delta z = \Delta x$, and the limit given by (2) is simply the "partial derivative with respect to x", $\frac{\partial f(z)}{\partial x}$, given by



$$(3)\quad \frac{\partial f(z)}{\partial x} = \frac{\partial \sin x}{\partial x} + i\,\frac{\partial (x + \sin y)}{\partial x} = \cos x + i\ .$$

Next let Δz approach zero from the direction $\theta = \pi/2$. Now $\Delta z = i\Delta y$, and the limit given by (2) is the "partial derivative with respect to iy", $\frac{\partial f(z)}{i\,\partial y}$, given by



$$\frac{\partial f(z)}{i\,\partial y} = \frac{\partial \sin x}{i\,\partial y} + i\,\frac{\partial (\sin y + x)}{i\,\partial y}$$

$$(4)\qquad = 0 + \cos y\ .$$

Since (3) does not equal (4) for any $z = x + iy$, $f'(z)$ does not exist at any point.

Example 3

Does the derivative of $f(z) = z^2$ exist at any point ?

Solution

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \frac{2z\Delta z + (\Delta z)^2}{\Delta z}$$

$$= 2z + \Delta z \quad .$$

Therefore, the limit given by (2) will be $f'(z) = 2z$ regardless of the manner in which Δz approaches zero. Thus $f'(z)$ exists for all z.

Problems

For each of the following functions, determine the points in the complex z-plane at which the derivative exists.

2. $f(z) = x^2 + y + ix$.
3. $f(z) = \bar{z} = x - iy$.
4. $f(z) = z^2 + 2z$.
5. $f(z) = z^3$.

5.2 A second definition of analyticity

Looking back over the examples and problems just solved, we see that the derivative exists in the sense that the limit (2) of the previous section is independent of the manner in which Δz approaches zero for the functions z^2 , $z^2 + 2z$, and z^3, and the derivatives obtained are the ones we would expect from the elementary calculus. Notice that these functions are all analytic. The functions for which the derivative does not exist were all artificially constructed by adding together a real function with i times another real function. These artificial functions are of course, not analytic.

Conjecture 5.1

In section 4.8 we gave a definition of the analyticity of a function based on the possibility of expanding the function in a convergent Taylor's series. Conjecture a second definition based on the experience just gained. (See Appendix II.)

In a rigorous development of a mathematical theory, we would wish to prove that the two definitions stated thus far are "equivalent". That is, we would show that a function $f(z)$ which satisfies the requirements of either definition, also satisfies those of the other definition. We will not show the equivalence of these definitions here.

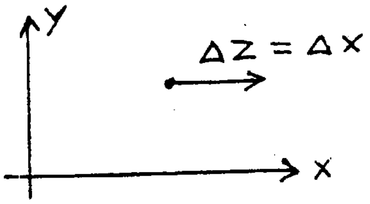
Problem

6. Using the new definition of analyticity introduced in this section, show that $(z-a)^n$, where $n = 0, 1, 2, \ldots$, is analytic for all z.

5.3 Cauchy-Riemann equations

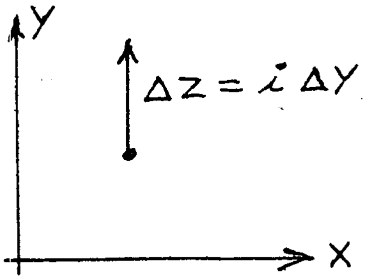
In the previous section, we saw how to characterize an analytic function in terms of its derivative. There we saw that if R is an open subset of the complex z-plane, and that if $f'(z)$ exists at all points of R, then $f(z)$ is analytic on R. In this section we give yet another characterization of the analytic function. Here we assume that $f(z)$ is given in terms of its real and imaginary parts, $f(z) = u(x,y) + iv(x,y)$, and we will show how to determine if $f(z)$ is analytic by looking at the partial derivatives of $u(x,y)$ and $v(x,y)$.

Let $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are realfunctions. Let R be some open subset of the complex z-plane, and let $f(z)$ be analytic on R. This means that $f'(z)$ can be computed at each z in R by allowing $\Delta z = \Delta x + i\Delta y$ to approach zero in any manner. If we let $\Delta z = \Delta x$, then $\frac{d}{dz} = \frac{\partial}{\partial x}$ and we have



(1) $$f'(z) = u_x + i\,v_x\ .$$

If we let Δz approach zero such that $\Delta z = i\Delta y$, then $\frac{d}{dz} = \frac{\partial}{i\partial y}$ and we have



(2) $$f'(z) = \frac{\partial u}{i\partial y} + i\,\frac{\partial v}{i\partial y}$$
$$= v_y - i\,u_y\ .$$

Equating the real and imaginary parts of (1) and (2) we get:

CAUCHY - RIEMANN EQUATIONS

(3) $$u_x = v_y$$

(4) $$u_y = -\,v_x$$

Even though we have allowed Δz to approach zero in only two different directions, the Cauchy - Riemann equations can be used to characterize an analytic function. Here is a Theorem which could itself be used as a third precise definition of analyticity:

Theorem

Let R be an open subset of the complex z-plane, and let $f(z) = u(x,y) + iv(x,y)$ be defined on R. Let u_x, v_x, u_y, and v_y be continuous on R, and let the Cauchy-Riemann equations be satisfied on R . Then $f(z)$ is anlytic on R.

Example 1

Using the above Theorem, show that $e^z = e^x \cos y + i\, e^x \sin y$ is an analytic function for all z .

Solution

Here $u = e^x \cos y$, and $v = e^x \sin y$. We must show that

(i) u_x , u_y , v_x , v_y , are continuous for all z, and
(ii) $u_x = v_y$ and $u_y = - v_x$ for all z.

Now

$$u_x = e^x \cos y, \qquad u_y = - e^x \sin y$$

$$v_x = e^x \sin y, \qquad v_y = e^x \cos y .$$

Clearly (i) and (ii) are satisfied, and therefore e^z is analytic for all z .

Problems

Show that the following functions are analytic on some open subset R of the complex z-pane, and specify R.

7. $\sin z = \sin x \cosh y + i \cos x \sinh y$
8. $z^{-1} = (x - iy)/(x^2 + y^2)$
9. $z^2 = x^2 - y^2 + 2xyi$.

Example 2

Let $u(x,y) = 4xy$ be the real part of an analytic function. Find the corresponding imaginary part $v(x,y)$.

Solution

From (3) we have

$$\frac{\partial\, 4xy}{\partial x} = v_y$$

$$4y = v_y ,$$

Integrating with respect to y we have

$$\int 4y \, dy = \int v_y \, dy$$

(5) $$2y^2 + g(x) = v ,$$

where $g(x)$ is only a function of x. Using (4) we get

$$\frac{\partial 4xy}{\partial y} = - \frac{\partial (2y^2 + g(x))}{\partial x}$$

$$4x = - g'(x) .$$

Integrating this last equation gives

(6) $$- 2x^2 + c = g(x)$$

where c is any real constant. Thus from (5) and (6) we have

$$v(x,y) = 2y^2 - 2x^2 + c .$$

Problems

The following are the real parts of analytic functions. Find the imaginary parts. These pairs , u, v, are called conjugate functions.

10. $u = 3x + 5y$
11. $u = y^2 - x^2 + 2x$
12. $u = e^{3y} \sin 3x$

5.4 Harmonic functions

The equation

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} = 0$$

is called Laplace's equation. Any solution $U(x,y)$ of Laplace's equation is called a "harmonic function". Some important physical quantities are described by harmonic functions. These include temperature, gravitational potential, electrostatic potential, velocity potential of an ideal fluid, etc. . We will investigate some of these physical applications later in this book.

Both the real and the imaginary parts of an analytic function are harmonic. That is, for an analytic function described by $f(z) = u(x,y) + iv(x,y)$, both u and v satisfy Laplace's equation. To see this, we start with the Cauchy-Riemann equations

(1) $u_x = v_y$

(2) $u_y = - v_x$.

Taking the partial derivative of (1) with respect to x and of (2) with respect to y we get

$$\frac{\partial u_x}{\partial x} = \frac{\partial v_y}{\partial x} \Longrightarrow u_{xx} = v_{yx}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial - u_y}{\partial y} \Longrightarrow v_{xy} = - u_{yy} .$$

Since $v_{yx} = v_{xy}$, these last two equations imply that $u_{xx} = -u_{yy}$, and thus u is harmonic. To see that v is harmonic, take $\frac{\partial}{\partial y}$ of (1) and $\frac{\partial}{\partial x}$ of (2) and the result follows at once.

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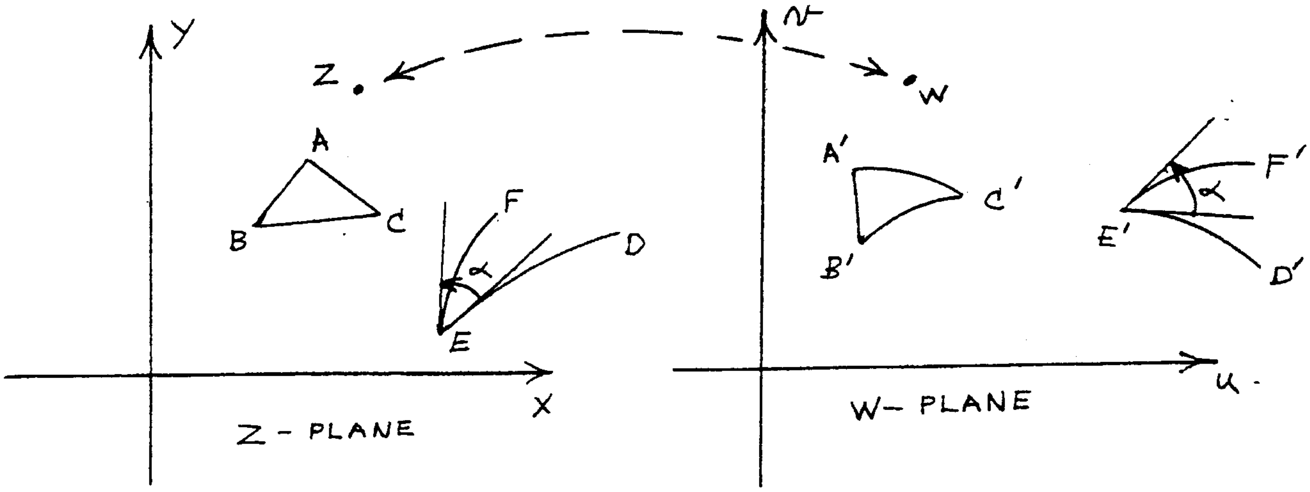
Problems

Show that the following functions are harmonic.

13. $2xy$, 14. $x^2 - y^2$, 15. $e^x \cos y$, 16. $\sin x \cosh y$.

5.5 Conformal mapping

In Chapter 2 we examined different graphic means of visualizing an analytic function $w = f(z)$, where $w = u + iv$ and $z = x + iy$. One method, that of mapping, was to visualize that to each point in the z-plane, there corresponds a point in the w-plane. In this way regions in the z-plane are mapped into corresponding regions on the w-plane as shown in the figure. Let us assume here that our mapping is "one to one", that is, to each point z there corresponds only one point w, and to each point w there corresponds only one point z . This means that $f(z)$ and $f^{-1}(z)$ are single-valued functions. If these are by nature multiple-valued, we assume that they have been suitably restricted to appropriate single-valued branches.

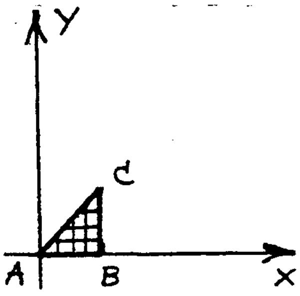


Example 1

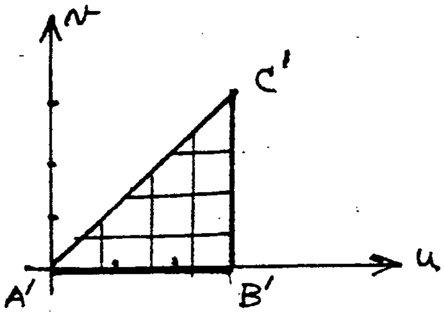
Discuss the mapping properties of the function $w = 3z$.

Solution

Writing $z = re^{i\theta}$ and $w = \rho e^{i\phi}$ in polar form we have $\rho = 3r$ and $\theta = \phi$. Thus the mapping is a pure magnification by the factor 3 .



Z - PLANE



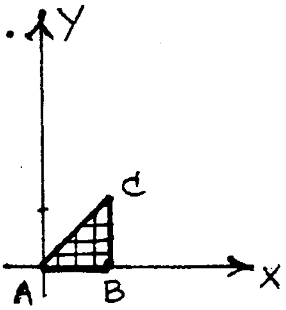
W- PLANE

Example 2

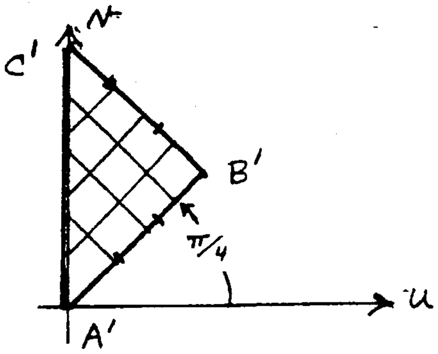
Discuss the mapping properties of the function $w = 3e^{i\pi/4}\ z$.

Solution

Writing z and w in polar form again we have $\rho e^{i\phi} = 3r\ e^{i\pi/4}\ e^{i\theta}$ and thus $\rho = 3r$ and $\phi = \theta + \pi/4$. This mapping is a magnification by the factor plus a rotation of $\pi/4$ radians about the origin.



Z - PLANE



W- PLANE

Example 3

Discuss the mapping properties of the function

$w = 3\ e^{i\pi/4}\ \textcircled{z} + (2+i)$.

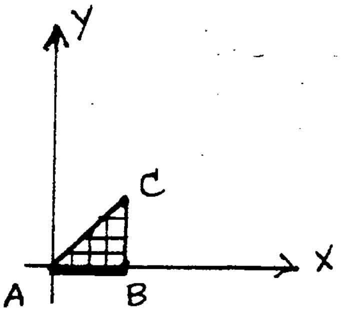
Solution

We must now add the vector $2 + i$ to each point mapped in the previous example. This means that we have

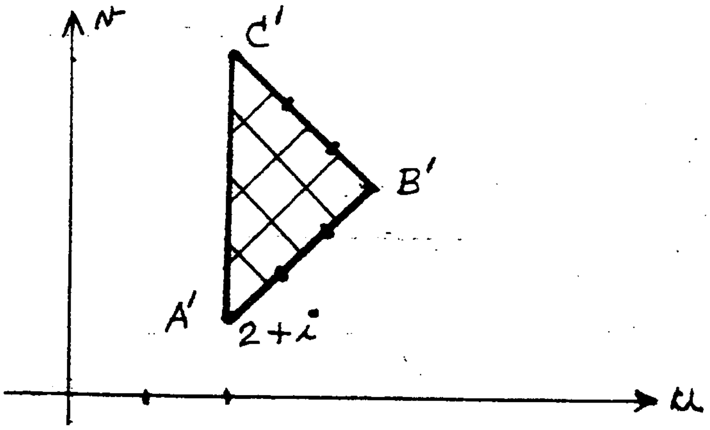
(i) first a magnification by the factor 3,

(ii) followed by a rotation about the origin through the angle $\pi/4$ radians, and finally

(iii) a translation by the vector $2+i$.



Z- PLANE



W- PLANE

Problem

17. Discuss the mapping properties of the function $w = (-2+2i)z + 3$.

Observe that since a mapping by the function $w = az + b$ consists of a magnification followed by a rotation and then a translation, it must be angle preserving. For example, the angles CAB in the z-plane in the above figures always mapped into the equal angles C'A'B' in the w-plane. A mapping that is angle preserving is called "conformal". If the direction of the angle is also preserved, it is called "sense preserving". Notice that a positive angle in the above figures always maps into a positive angle.

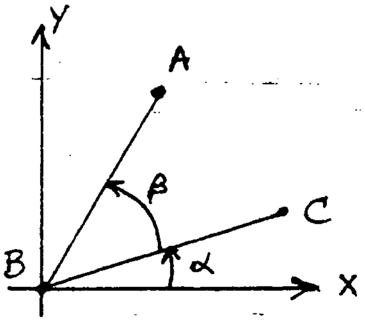
Thus we see that a mapping through the function $w = az + b$ is both conformal and sense preserving . It consists of a magnification by the factor $|a|$, a rotation through the angle $\arg(a)$ followed by a translation by the vector b.

Example 4

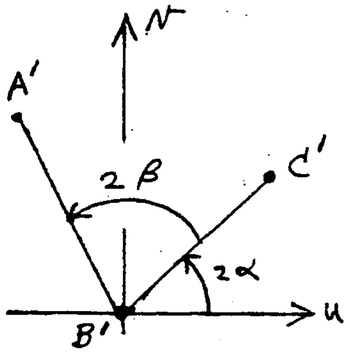
Show that the mapping described by $w = z^2$ is not conformal at $z = 0$, while it is sense preserving.

Solution

In polar form the function becomes $\rho e^{i\phi} = r^2 e^{i2\theta}$. Thus we see that $\phi = 2\theta$, which means that angles are doubled at the origin.



Z-PLANE



W-PLANE

Problems

18. Is the mapping described by $w = z^3 + 2$ conformal at $z = 0$? Is it sense preserving ?

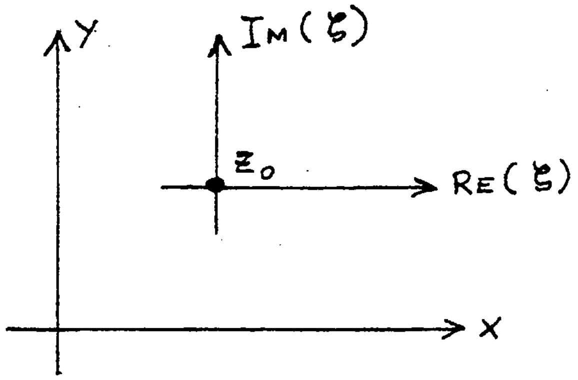
19. Is the mapping described by $w = \bar{z} = x - iy$ conformal at $z = 0$? Is it sense preserving ?

Now consider a general analytic function $w = f(z)$. When is $f(z)$ conformal and sense preserving ? We shall eventually see the answer to this question. Suppose $z = z_0$ is a point of analyticity of $f(z)$. Let us now look at a very small region of the z-plane surrounding the point z_0 and consider its mapping onto the w-plane through $f(z)$. (We imagine that we are looking through a microscope at z_0 .) Since $f(z)$ is analytic at z_0 we can expand it in a Taylor's series about z_0 , and since $z-z_0$ is so very small, we can write

$$(1) \quad w = f(z) \approx f(z_0) + f'(z_0)(z-z_0) \qquad (f'(z_0) \neq 0)$$

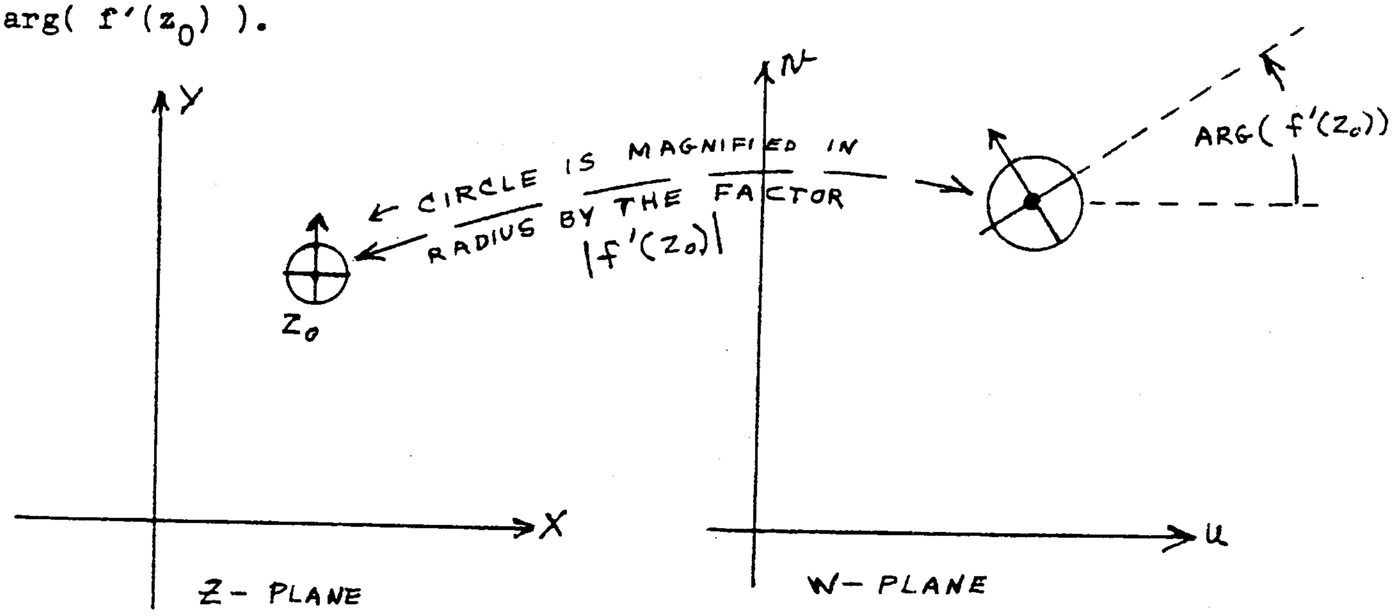
where we have neglected higher powers of $z-z_0$ because they are assumed to be too small to be of significance. We have also assumed that $f'(z_0) \neq 0$. Now draw new coordinate axes with

with origin at $z = z_0$ as shown in the figure. This new set of axes will describe the ζ variable given by the equation $\zeta = z - z_0$. Now (1) becomes



$$(2) \qquad w = f(z) \approx f'(z_0)\,\zeta + f(z_0) .$$

This relation (2) describes a magnification by the factor $|f'(z_0)|$ followed by a rotation through the angle $\arg(f'(z_0))$. There is also a translation by the vector $f(z_0)$. We see that a small circle centered at z_0 maps into a circle in the w-plane magnified in radius by the factor $|f'(z_0)|$ and rotated through the angle $\arg(f'(z_0))$.



The above intuitive investigation of the mapping of a small region surrounding the point z_0 suggests that $w = f(z)$ provides both a conformal and sense preserving map at $z = z_0$ if

(i) $f(z)$ is analytic at z_0, and

(ii) $f'(z_0) \neq 0$.

If $f'(z_0) = 0$, then (1) is replaced by

$$(3) \quad w = f(z) \approx f(z_0) + f^{(N)}(z_0)(z-z_0)^N ,$$

where N is the order of the first non-zero derivative at z_0 of $f(z)$. The relation (3) does not give a conformal mapping because it multiplies angles at $z = z_0$ by the factor N.

Example 5

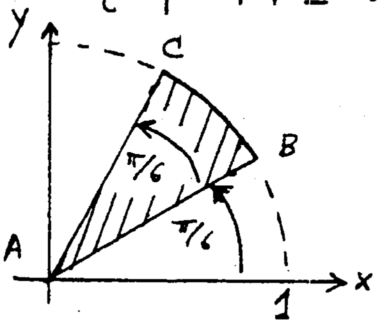
Where is the mapping described by the function $w = \cos z$ conformal and sense preserving.

Solution

Since $w = \cos z$ is analytic for all z and since $\frac{dw}{dz} = -\sin z$ is zero only for $z = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, we see that $\cos z$ describes a conformal and sense preserving map at all points z except the integral multiples of π.

Problem

20. Where is the function $w = z^6 + 2$ conformal and sense preserving ? Describe the mapping of the sector $\{ z \mid |z| \leq 1, \ \pi/6 \leq \arg(z) \leq \pi/3 \}$ into the w-plane and comment on the mapping of the angles at A, B and C.



We have previously given three equivalent characterizations of the analyticity of a function $w = f(z)$ in terms of

(i) the Taylor's series,

(ii) the existence of $f'(z)$ on an open set, and

(iii) the Cauchy-Riemann equations satisfied on an open set.

A rigorous theory of analytic functions could be constructed from any one of the above three characterizations. We can now give a fourth characterization in terms of the mapping properties described in this section. The following theorem is true.

Theorem

Let a one to one mapping of the open set R in the z-plane onto the open set S on the w-plane be described by the function $f(z) = u(x,y) + iv(x,y)$. Assume that u_x , u_y , v_x and v_y are continuous throughout R. Then $f(z)$ is analytic at each point of R if and only if it is both conformal and sense preserving at each point of R .

Example 6

Let R be the open circle $|z| < 1$. The function $f(z) = z^2$ is analytic at each point of R, yet it is not conformal at $z = 0$ (since it has a derivative equal to zero at $z=0$). Does this contradict the above theorem ?

Solution

No, for the mapping of the region $|z| < 1$ onto the w-plane by $w = z^2$ is not one to one. Both the points z and $-z$ map onto the same point in the w-plane.

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Review Problems for Chapter 5

1. Use the Cauchy-Riemann equations to show that the function $w = e^z/z$ is analytic for all z except $z = 0$.

2. Find the conjugate harmonic function to $e^x(x \cos y - y \sin y)$.

3. By comparing derivatives in the r direction with those in the θ direction, derive the polar form of the Cauchy-Riemann equations:

$$r\, u_r = v_\theta \,, \qquad u_\theta = - r\, v_r \;.$$

4. Use the results of the previous problem to show that the polar form of Laplace's equation is satisfied by u and v.

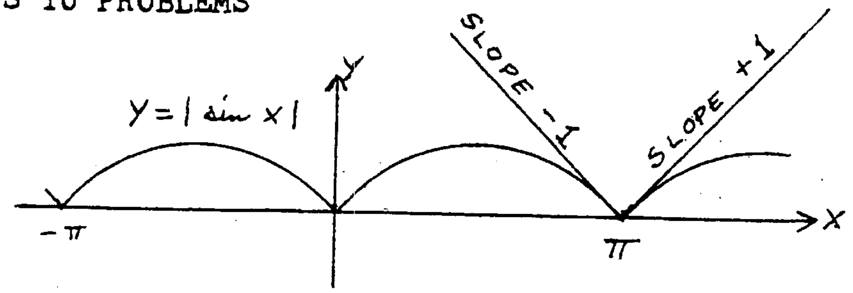
$$U_{rr} + r^{-1} U_r + r^{-2} U_{\theta\theta} = 0.$$

5. Where is the mapping described by $w = z^3 + 6z^2 + 3z - 1$ conformal and sense preserving.

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 5

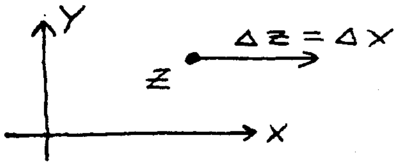


1/ No, because the limit from the right of

$$\lim_{\Delta x \to 0} \frac{|\sin(\pi + \Delta x)| - |\sin \pi|}{\Delta x}$$

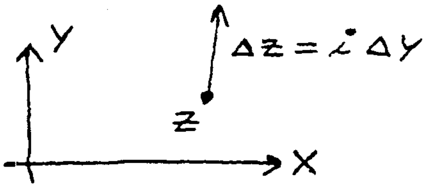
is +1 while the limit from the left is −1. Both limits must agree for the derivative to exist at $x = \pi$.

2/ If we let $\Delta z = \Delta x$, then $\frac{d}{dz} = \frac{\partial}{\partial x}$ and we have



$$\frac{d}{dz}(x^2 + y + ix) = \frac{\partial}{\partial x}(x^2 + y + ix) = 2x + i,$$

While if we let $\Delta z = i\Delta y$, then $\frac{d}{dz} = \frac{\partial}{i\partial Y}$ and we have



$$\frac{d}{dz}(x^2 + y + ix) = \frac{\partial}{i\partial y}(x^2 + y + ix) = \frac{1}{i} = -i.$$

Since these values are not in agreement for any z, $f'(z)$ never exists.

3/ As in the previous problem, for $\Delta z = \Delta x$ we have $\frac{d}{dz}(x - iy) = \frac{d}{dx}(x - iy) = 1$, while for $\Delta z = i\Delta y$ we have $\frac{d}{dz}(x - iy) = \frac{\partial}{i\partial y}(x - iy) = -1$. Thus $\bar{z}$ has no derivative for any z.

20

4/
$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{(z+\Delta z)^2+2(z+\Delta z)-z^2-2z}{\Delta z}$$

$$= \frac{z^2+2z\Delta z+(\Delta z)^2+2z+2\Delta z-z^2-2z}{\Delta z}$$

$= 2z+2+\Delta z$, and the limit of this last expression as $\Delta z \to 0$ in any manner whatsoever is always the one value $2z+2$. Thus $f'(z)$ exists for all z.

5/
$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{(z+\Delta z)^3-z^3}{\Delta z}$$

$$= \frac{z^3+3z^2\Delta z+3z(\Delta z)^2+(\Delta z)^3-z^3}{\Delta z}$$

$= 3z^2+3z\Delta z+(\Delta z)^2$, and the limit of this expression as $\Delta z \to 0$ is $3z^2$. Thus $f'(z)$ exists for all z.

6/ If we can show that

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z-a)^n-(z-a)^n}{\Delta z}$$

exists and is independent of the manner in which $\Delta z \to 0$, then we know that $f(z)$ is analytic for all z.

Now $(z+\Delta z-a)^n = [(z-a)+\Delta z]^n =$

$$(z-a)^n + n(z-a)^{n-1}\Delta z + \frac{n}{1}\frac{n-2}{2}(z-a)^{n-2}(\Delta z)^2$$

$$+ \dots + (\Delta z)^n.$$

THUS THE LIMIT OF INTEREST IS

$$\lim_{\Delta z \to 0} n(z-a)^{n-1} + \left(\begin{matrix}\text{terms having } \Delta z \\ \text{as a factor}\end{matrix}\right)$$

$= n(z-a)^{n-1}$. Thus the function $(z-a)^n$ is analytic for all z.

7/ Let R be the z-plane itself.

$u = \sin x \cosh y$ and $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $v_y = \cos x \cosh y$

Now u_x, u_y, v_x and v_y are continuous for all x and y. Also $u_x = v_y$ and $u_y = -v_x$. Thus $\sin z$ is analytic for all z.

8/ Let R be the set $0 < |z|$.

$u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$

$u_x = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$ $\qquad$ $v_x = \dfrac{2xy}{(x^2+y^2)^2}$

$u_y = \dfrac{-2xy}{(x^2+y^2)^2}$ $\qquad$ $v_y = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

Now u_x, u_y, v_x and v_y are continuous for all x, y except when both x and y are zero. Thus $z=0$ is excluded. Since $u_x = v_y$ and $u_y = -v_x$, $\frac{1}{z}$ is analytic for all z except $z=0$.

9/ Let R be the z-plane itself.

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$u_x = 2x$ $\qquad$ $v_x = 2y$

$u_y = -2y$ $\qquad$ $v_y = 2x$

Since u_x, u_y, v_x and v_y are continuous for all x, y and since the Cauchy-Riemann equations are satisfied for all x, y, $f(z) = z^2$ is analytic for all z.

10/ $\dfrac{\partial u}{\partial x} = 3 = \dfrac{\partial v}{\partial y} \implies v = 3y + g(x)$

$\dfrac{\partial v}{\partial x} = g'(x) = -\dfrac{\partial u}{\partial y} = -5 \implies g(x) = -5x + c$

Thus $\boxed{v = 3y - 5x + c}$

11/ $\frac{\partial u}{\partial x} = -2x+2 = \frac{\partial v}{\partial y} \Rightarrow v = \int -2x+2 \, dy$

$v = -2xy + 2y + g(x)$,

$\frac{\partial v}{\partial x} = -2y + g'(x) = -\frac{\partial u}{\partial y} = -2y$

Thus $g'(x) = 0$, and $g(x) = c$.

$$\boxed{v = -2xy + 2y + c}$$

12/ $\frac{\partial u}{\partial x} = 3e^{3y}\cos 3x = \frac{\partial v}{\partial y} \Rightarrow v = \int 3e^{3y}\cos 3x \, dy$

$v = e^{3y}\cos 3x + g(x)$

$\frac{\partial v}{\partial x} = -3e^{3y}\sin 3x + g'(x) = -\frac{\partial u}{\partial y} = -3e^{3y}\sin 3x$.

Thus $g'(x) = 0$, and $g(x) = c$.

$$\boxed{v = e^{3y}\cos 3x + c}$$

13/ $u = 2xy$, $u_{xx} = 0$, $u_{yy} = 0$, Thus $u_{xx} + u_{yy} = 0$ and $2xy$ is harmonic. Notice that $2xy$ is the imaginary part of z^2.

14/ $u = x^2 - y^2$ and $u_{xx} = 2$, while $u_{yy} = -2$, Thus $u_{xx} + u_{yy} = 0$. Notice that u is the real part of z^2.

15/ $u = e^x \cos y$ and $u_{xx} = e^x \cos y$, while $u_{yy} = -e^x \cos y$. Thus $u_{xx} + u_{yy} = 0$, Note that u is the real part of e^z.

16/ $u = \sin x \cosh y$ and $u_{xx} = -\sin x \cosh y$ while $u_{yy} = \sin x \cosh y$, Thus $u_{xx} + u_{yy} = 0$. Note that u is the real part of $\sin z$.

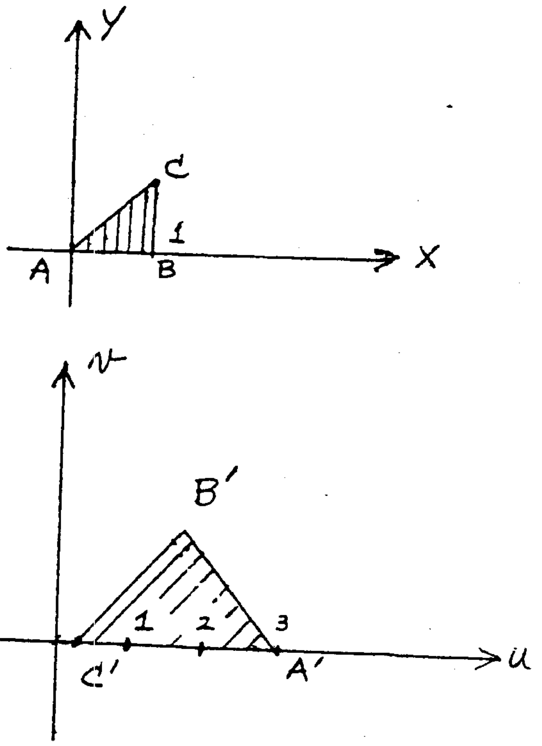
17/ Since $-2 + 2i = 2\sqrt{2}\, e^{i\frac{3\pi}{4}}$, $w = 2\sqrt{2}\, e^{i\frac{3\pi}{4}} z + 3$.

The desired mapping is achieved in three steps:

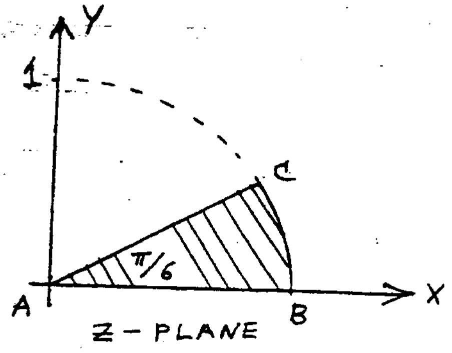
(a) A magnification by the factor $2\sqrt{2}$.

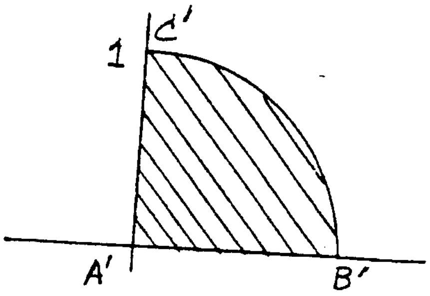
(b) A rotation about $z = 0$ through 135°.

(c) A translation of 3 units to the right.



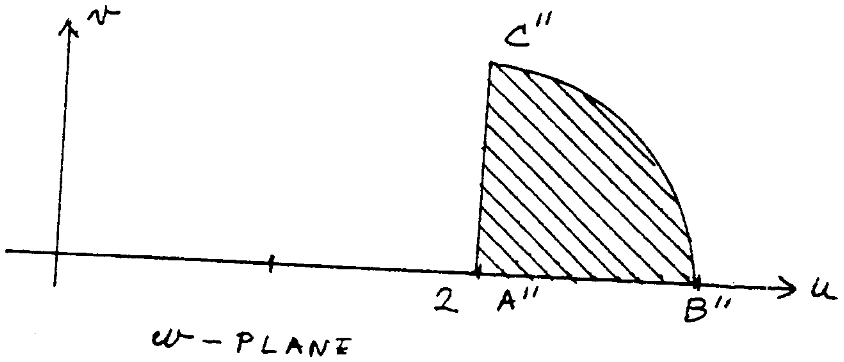
18/ $w = z^3 + 2 = r^3 e^{i3\theta} + 2$. Now consider the mapping of the figure shown, where we pay particular attention to the 30° at the origin. The mapping z^3 produces





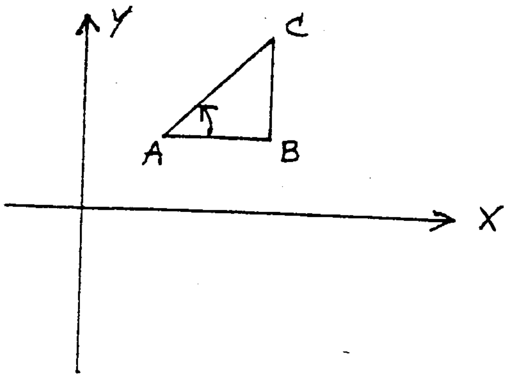
$w' = z^3$

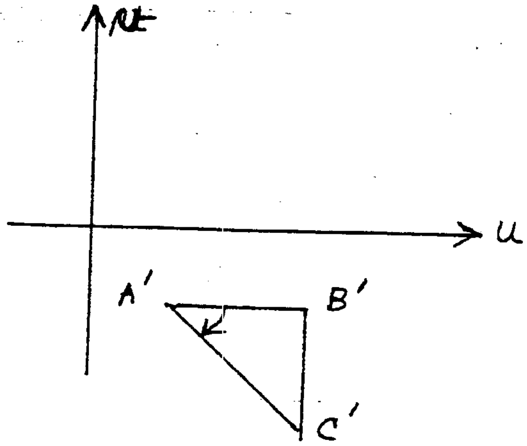
The addition of +2 simply translates this second figure to produce the final mapping shown in the third figure.



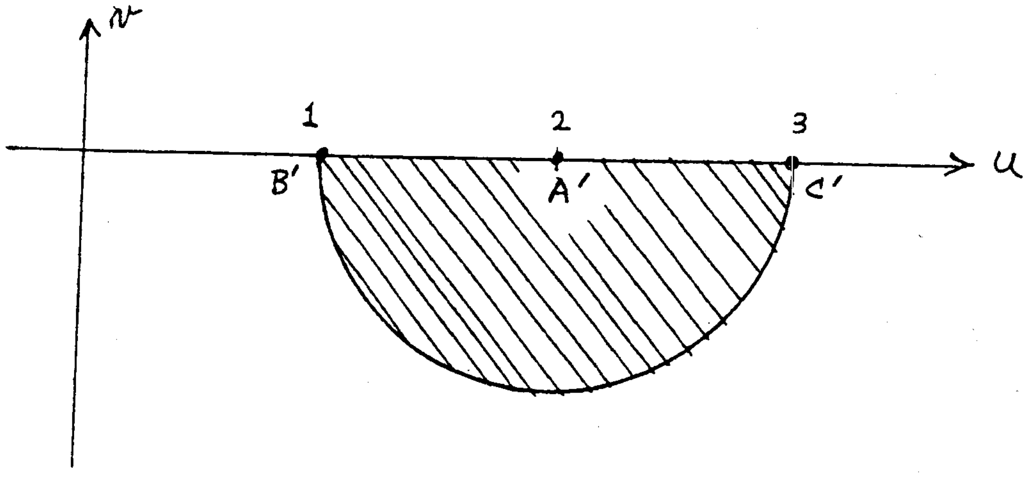
The mapping triples angles at $z = 0$, and thus it is not conformal there. However, it is sense preserving at $z = 0$.

19/ The mapping $\bar{z}$ simply reflects any figure through the x-axis. Thus the mapping is conformal, but not sense preserving.





20/ $\frac{dw}{dz} = 6z^5$, and thus $\frac{dw}{dz} = 0$ for $z = 0$. Therefore w is conformal and sense preserving except at $z = 0$ where angles are multiplied by the factor 6.



Solutions to Review Problems from Chapter 5

1/ $$\frac{e^z}{z} = \frac{e^x\cos y + i\, e^x \sin y}{x+iy}$$

$$= \frac{(e^x\cos y + i\, e^x\sin y)}{(x+iy)}\,\frac{(x-iy)}{(x-iy)}$$

$$= \frac{(xe^x\cos y + y\,e^x\sin y) + i\,(xe^x\sin y - y\,e^x\cos y)}{x^2+y^2}$$

Thus

$$u = \frac{xe^x\cos y + y\,e^x\sin y}{x^2+y^2} \quad \text{and } v = \frac{xe^x\sin y - y\,e^x\cos y}{x^2+y^2}$$

$$u_x = \frac{e^x\left[(x^3-x^2+y^2+xy^2)\cos y + (y^3+x^2y-2xy)\sin\right.}{(x^2+y^2)^2}$$

$$= v_y$$

$$v_x = \frac{e^x\left[(2xy-x^2y-y^3)\cos y + (xy^2+y^2-x^2+x^3)\sin y\right]}{(x^2+y^2)^2}$$

$$= -u_y$$

u_x, u_y, v_x and v_y are continuous except at $x=0$ and $y=0$, 2/ $u = e^{x}(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^{x}\{(x+1)\cos y - y\sin y\} = \frac{\partial v}{\partial y}$$

$$v = \int e^{x}(x+1)\cos y - e^{x} y \sin y \, dy$$

$$v = e^{x} x \sin y + e^{x} y \cos y + g(x)$$

$$\frac{\partial u}{\partial y} = e^{x}(-x\sin y - y\cos y - \sin y) = -\frac{\partial v}{\partial x}$$

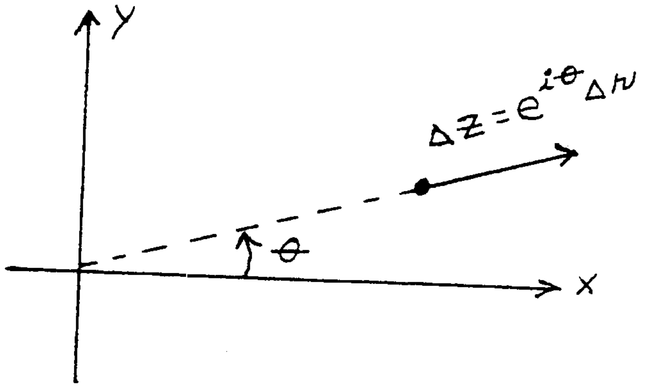
$$= -e^{x}\sin y - x e^{x}\sin y - e^{x}\cos y - g'(x).$$

Thus $g'(x) = 0$ and $g(x) = c$.

$$\boxed{v = e^{x}(x\sin y + y\cos y) + c.}$$

3/

The figure shows Δz taken in the r direction. Here

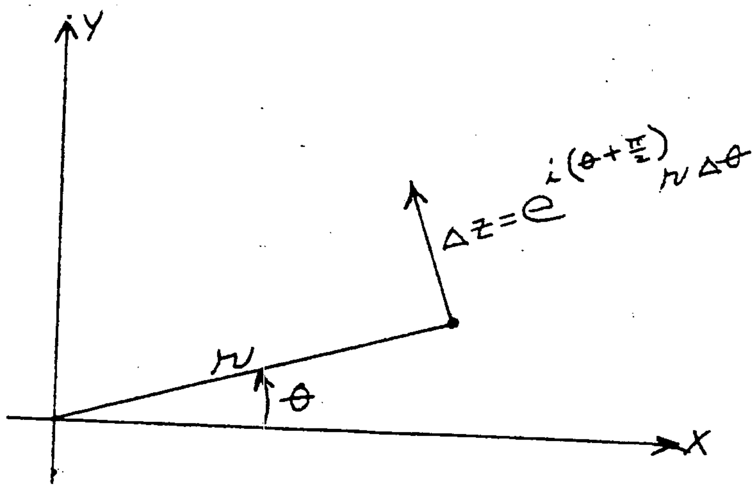


$$\frac{d}{dz} = \frac{\partial}{e^{i\theta}\partial r} = e^{-i\theta}\frac{\partial}{\partial r}.$$

Thus

$$(1) \quad \frac{d(u+iv)}{dz} = e^{-i\theta}\frac{\partial(u+iv)}{\partial r} = e^{-i\theta}(u_r + i v_r).$$

Now the figure shows Δz in the θ direction. Thus



$$\frac{d}{dz} = \frac{\partial}{e^{i(\theta+\frac{\pi}{2})} r\,\partial\theta}$$

$$= e^{-i(\theta+\frac{\pi}{2})} \frac{\partial}{r\,\partial\theta}$$

$$= -i\, e^{-i\theta} \frac{\partial}{r\,\partial\theta}.$$

Therefore

$$(2) \quad \frac{d(u+iv)}{dz} = -i\, e^{-i\theta} \frac{\partial(u+iv)}{r\,\partial\theta} = e^{-i\theta}\left(\frac{1}{r} v_\theta - i\frac{1}{r} u_\theta\right)$$

Equating the real and imaginary parts of (1) and (2) we get

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = \frac{-1}{r} u_\theta .$$

4/ (1) $r u_r = v_\theta$

$$\frac{\partial (r u_r)}{\partial r} = \frac{\partial v_\theta}{\partial r}$$

$$r u_{rr} + u_r = v_{\theta r}$$

(2) $u_\theta = -r v_r$

$$\frac{\partial u_\theta}{\partial \theta} = \frac{\partial(-r v_r)}{\partial \theta}$$

$$u_{\theta\theta} = -r v_{r\theta}$$

$$-\frac{1}{r} u_{\theta\theta} = v_{r\theta}$$

Since $v_{\theta r} = v_{r\theta}$ we have

$$r u_{rr} + u_r = -\frac{1}{r} u_{\theta\theta}.$$

Thus u satisfies the polar form of Laplace's equation, Taking $\frac{\partial}{\partial\theta}$ of (1) and $\frac{\partial}{\partial r}$ of (2) we show that v satisfies the same equation,

5/ This function provides a conformal and sense preserving map at all points except those where $\frac{dw}{dz} = 0$, Now

$$\frac{dw}{dz} = 3z^2 + 12z + 3 = 0$$

$$z^2 + 4z + 1 = 0$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3},$$

Thus the excluded points are $z = -2+\sqrt{3}$ and $z = -2-\sqrt{3}$.

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 5

5.1 A DEFINITON OF ANALYTICITY BASED ON THE EXISTENCE OF $f'(z)$

Let R be an open subset of the complex z-plane. Let $f(z)$ be defined for each point of R, and let the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

exist for each point z in R. The existence of this limit implies that one value only for the limit is obtained regardless of the manner in which Δz approaches zero. Then $f(z)$ is said to be analytic at each point z of the open set R.

Remarks:

It is necessary to have $f'(z)$ existing at each point of some open set. If $f'(z)$ exists at only the point $z=0$, we cannot say that $f(z)$ is analytic at $z=0$. For example, consider the function $f(z) = |z| = \sqrt{x^2 + y^2}$. One can show that $f'(z)$ exists at $z=0$ but does not exist for any other point in the z-plane. We know that the domain of definition of an analytic function can always be enlarged until it is an open set, but this would be impossible for $|z|$ since it has only a derivative at the origin. Thus $|z|$ is not analytic for any z.