AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

Thomas J Osler
Mathematics Department
Rowan University
Glassboro NJ 08028

osler@rowan.edu

(1)

CHAPTER 5

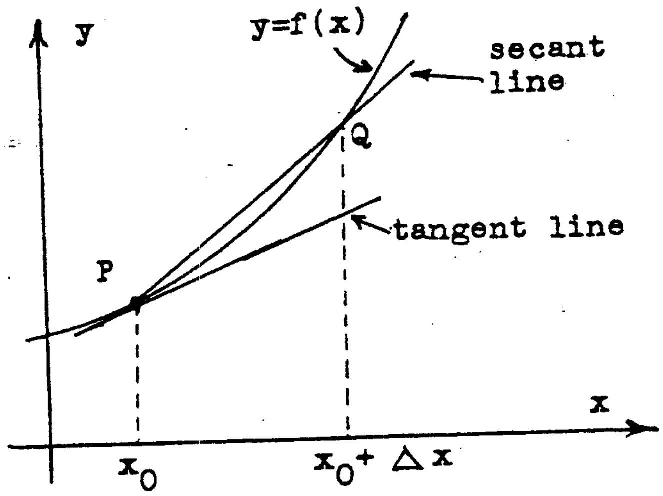
DIFFERENTIATION

In the previous chapter we differentiated functions such as z^p, e^z, sin z, log z, etc., without giving serious attention to the fundamental meaning of differentiation with respect to a complex variable. We simply wrote down the same formulas for derivatives that we learned in the real calculus, and hoped that they would remain true in the new complex calculus. Since our analytic functions are "natural", we have grown to expect that formulas learned previously for real variables carry over into the complex case without change. In this chapter we return to the concept of differentiation and investigate its fundamental and precise meaning. After finding a suitable exact "definition" for the derivative of a function of a complex variable, we will see why derivatives of a complex function which is analytic look essentially like their real counterparts. We will also encounter new expressions which could serve as new definitions for the analyticity of a function. Finally, we will see that this investigation into the basic meaning of differentiation leads us to new insights into the nature of analytic functions.

5.1 The definition of differentiation

What is the real meaning of $f'(z)$ when z is a complex variable ? Let us first review the meaning of the derivative learned in the real calculus. There we defined the derivative of $f(x)$ at x_0 to be

$$(1) \quad f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} ,$$



We think of the secant line drawn through the fixed point P and the moving point Q shown in the figure. As Q approaches P, the slope of the secant line approaches the slope of the tangent line at P. Of course, the slope of this tangent line is the derivative given by the limit in (1). Now the point Q can approach P from two different directions:

(i) from the right (Δx is positive), and

(ii) from the left (Δx is negative).

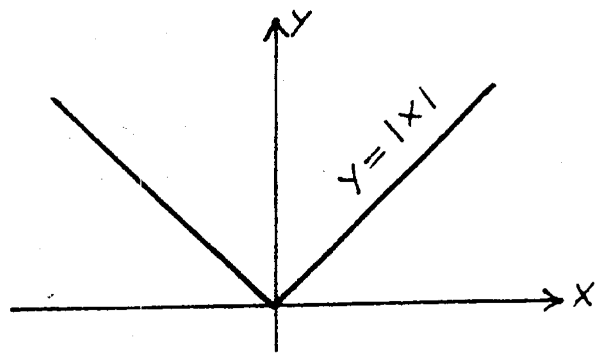
For $f'(x_0)$ to exist, we require that the limit (1) give the same value in both cases (i) and (ii).

EXAMPLE 1

Is the function $f(x) = |x|$ differentiable at $x_0 = 0$?

Solution

For $\Delta x > 0$, we have



$$\lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(\Delta x + 0) - f(0)}{\Delta x} =$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{|\Delta x + 0| - |0|}{\Delta x} = 1 ,$$

and for $\Delta x < 0$ we have

$$\lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{f(\Delta x + 0) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} = -1 .$$

since we get +1 for the limit from the right and -1 for the limit from the left, we see that $f'(0)$ does not exist. It is also very easy to see that $f'(0)$ does not exist just by glancing at the graph of $y = |x|$. We see that just to the right of $x=0$ the slope of the graph is +1, while just to the left of $x=0$ it is -1. Since these two slopes are not equal, the derivative does not exist at $x=0$.

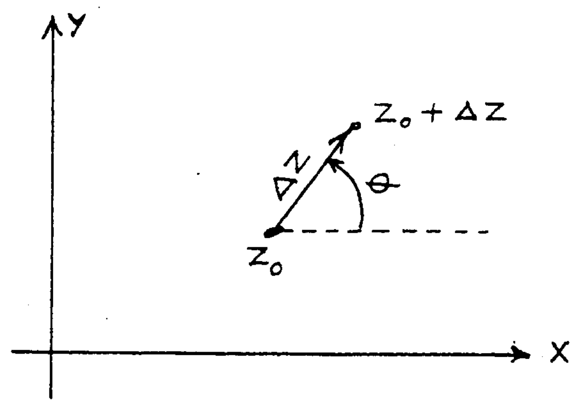
Problem

1. Does the derivative of the function $f(x) = |\sin x|$ exist at the point $x_0 = \pi$?

How should we define $f'(z)$ when f is a complex valued function of a complex variable ? The analogous expression to (1) is

$$(2) \qquad f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} .$$

Since z is a complex variable, Δz can approach zero from many different directions determined by the angle θ shown in the figure, In (1), Δx could approach zero from only two directions, the right and the left. In (2), however, Δz can approach zero from infinitely many different directions.



It is natural to require that the limit given by (2) be only one value, regardless of the direction θ in which Δz approaches zero. We select this idea as our precise definition and state it as follows:

DEFINITION OF THE DERIVATIVE

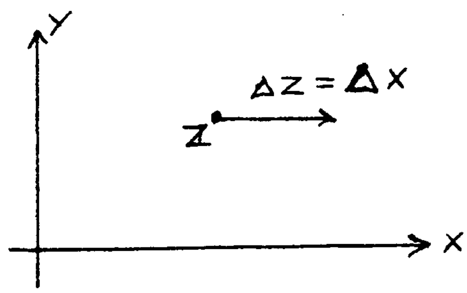
We say that the derivative of the complex valued function $f(z)$ of the complex variable z exists at the point z_0 if and only if the limit given by (2) exists. This limit must give only one value, regardless of the manner in which Δz approaches zero.

Example 2

Does the derivative of the function $f(z) = \sin x + i\,(x + \sin y)$ exist at any point ?

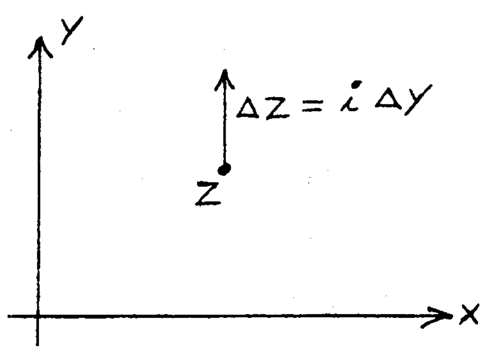
Solution

Consider first the case in which Δz approaches zero with $\theta = 0$. Now $\Delta z = \Delta x$, and the limit given by (2) is simply the "partial derivative with respect to x", $\frac{\partial f(z)}{\partial x}$, given by



$$(3)\quad \frac{\partial f(z)}{\partial x} = \frac{\partial \sin x}{\partial x} + i\,\frac{\partial (x + \sin y)}{\partial x} = \cos x + i\ .$$

Next let Δz approach zero from the direction $\theta = \pi/2$. Now $\Delta z = i\Delta y$, and the limit given by (2) is the "partial derivative with respect to iy", $\frac{\partial f(z)}{i\,\partial y}$, given by



$$\frac{\partial f(z)}{i\,\partial y} = \frac{\partial \sin x}{i\,\partial y} + i\,\frac{\partial (\sin y + x)}{i\,\partial y}$$

$$(4)\qquad = 0 + \cos y\ .$$

Since (3) does not equal (4) for any $z = x + iy$, $f'(z)$ does not exist at any point.

Example 3

Does the derivative of $f(z) = z^2$ exist at any point ?

Solution

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \frac{2z\Delta z + (\Delta z)^2}{\Delta z}$$

$$= 2z + \Delta z \quad .$$

Therefore, the limit given by (2) will be $f'(z) = 2z$ regardless of the manner in which Δz approaches zero. Thus $f'(z)$ exists for all z.

Problems

For each of the following functions, determine the points in the complex z-plane at which the derivative exists.

2. $f(z) = x^2 + y + ix$.
3. $f(z) = \bar{z} = x - iy$.
4. $f(z) = z^2 + 2z$.
5. $f(z) = z^3$.

5.2 A second definition of analyticity

Looking back over the examples and problems just solved, we see that the derivative exists in the sense that the limit (2) of the previous section is independent of the manner in which Δz approaches zero for the functions z^2 , $z^2 + 2z$, and z^3, and the derivatives obtained are the ones we would expect from the elementary calculus. Notice that these functions are all analytic. The functions for which the derivative does not exist were all artificially constructed by adding together a real function with i times another real function. These artificial functions are of course, not analytic.

Conjecture 5.1

In section 4.8 we gave a definition of the analyticity of a function based on the possibility of expanding the function in a convergent Taylor's series. Conjecture a second definition based on the experience just gained. (See Appendix II.)

In a rigorous development of a mathematical theory, we would wish to prove that the two definitions stated thus far are "equivalent". That is, we would show that a function $f(z)$ which satisfies the requirements of either definition, also satisfies those of the other definition. We will not show the equivalence of these definitions here.

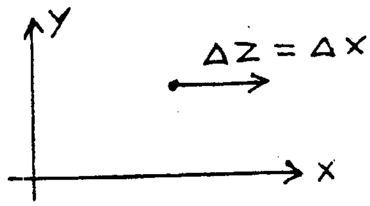
Problem

6. Using the new definition of analyticity introduced in this section, show that $(z-a)^n$, where $n = 0, 1, 2, \dots$, is analytic for all z.

5.3 Cauchy-Riemann equations

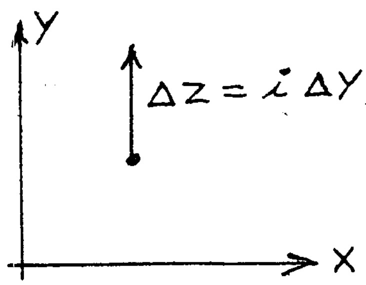
In the previous section, we saw how to characterize an analytic function in terms of its derivative. There we saw that if R is an open subset of the complex z-plane, and that if $f'(z)$ exists at all points of R, then $f(z)$ is analytic on R. In this section we give yet another characterization of the analytic function. Here we assume that $f(z)$ is given in terms of its real and imaginary parts, $f(z) = u(x,y) + iv(x,y)$, and we will show how to determine if $f(z)$ is analytic by looking at the partial derivatives of $u(x,y)$ and $v(x,y)$.

Let $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are realfunctions. Let R be some open subset of the complex z-plane, and let $f(z)$ be analytic on R. This means that $f'(z)$ can be computed at each z in R by allowing $\Delta z = \Delta x + i\Delta y$ to approach zero in any manner. If we let $\Delta z = \Delta x$, then $\frac{d}{dz} = \frac{\partial}{\partial x}$ and we have



(1) $f'(z) = u_x + i\, v_x$.

If we let Δz approach zero such that $\Delta z = i\Delta y$, then $\frac{d}{dz} = \frac{\partial}{i\partial y}$ and we have



(2) $f'(z) = \frac{\partial u}{i\partial y} + i\,\frac{\partial v}{i\partial y}$

$= v_y - i\, u_y$.

Equating the real and imaginary parts of (1) and (2) we get:

CAUCHY - RIEMANN EQUATIONS

(3) $u_x = v_y$

(4) $u_y = - v_x$

Even though we have allowed Δz to approach zero in only two different directions, the Cauchy - Riemann equations can be used to characterize an analytic function. Here is a Theorem which could itself be used as a third precise definition of analyticity:

Theorem

Let R be an open subset of the complex z-plane, and let $f(z) = u(x,y) + iv(x,y)$ be defined on R . Let u_x , v_x , u_y , and v_y be continuous on R, and let the Cauchy-Riemann equations be satisfied on R . Then f(z) is anlytic on R.

Example 1

Using the above Theorem, show that $e^z = e^x \cos y + i\, e^x \sin y$ is an analytic function for all z .

Solution

Here $u = e^x \cos y$, and $v = e^x \sin y$. We must show that

(i) u_x , u_y , v_x , v_y , are continuous for all z, and

(ii) $u_x = v_y$ and $u_y = - v_x$ for all z.

Now

$$u_x = e^x \cos y, \qquad u_y = - e^x \sin y$$

$$v_x = e^x \sin y, \qquad v_y = e^x \cos y .$$

Clearly (i) and (ii) are satisfied, and therefore e^z is analytic for all z .

Problems

Show that the following functions are analytic on some open subset R of the complex z-pane, and specify R.

7. $\sin z = \sin x \cosh y + i \cos x \sinh y$

8. $z^{-1} = (x - iy)/(x^2 + y^2)$

9. $z^2 = x^2 - y^2 + 2xyi$.

Example 2

Let $u(x,y) = 4xy$ be the real part of an analytic function. Find the corresponding imaginary part $v(x,y)$.

Solution

From (3) we have

$$\frac{\partial\, 4xy}{\partial x} = v_y$$

$$4y = v_y \ ,$$

Integrating with respect to y we have

$$\int 4y \, dy = \int v_y \, dy$$

(5) $$2y^2 + g(x) = v ,$$

where $g(x)$ is only a function of x. Using (4) we get

$$\frac{\partial 4xy}{\partial y} = - \frac{\partial (2y^2 + g(x))}{\partial x}$$

$$4x = - g'(x) .$$

Integrating this last equation gives

(6) $$- 2x^2 + c = g(x)$$

where c is any real constant. Thus from (5) and (6) we have

$$v(x,y) = 2y^2 - 2x^2 + c .$$

Problems

The following are the real parts of analytic functions. Find the imaginary parts. These pairs , u, v, are called conjugate functions.

10. $u = 3x + 5y$
11. $u = y^2 - x^2 + 2x$
12. $u = e^{3y} \sin 3x$

5.4 Harmonic functions

The equation

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} = 0$$

is called Laplace's equation. Any solution $U(x,y)$ of Laplace's equation is called a "harmonic function". Some important physical quantities are described by harmonic functions. These include temperature, gravitational potential, electrostatic potential, velocity potential of an ideal fluid, etc. . We will investigate some of these physical applications later in this book.

Both the real and the imaginary parts of an analytic function are harmonic. That is, for an analytic function described by $f(z) = u(x,y) + iv(x,y)$, both u and v satisfy Laplace's equation. To see this, we start with the Cauchy-Riemann equations

(1) $u_x = v_y$

(2) $u_y = - v_x$.

Taking the partial derivative of (1) with respect to x and of (2) with respect to y we get

$$\frac{\partial u_x}{\partial x} = \frac{\partial v_y}{\partial x} \implies u_{xx} = v_{yx}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial - u_y}{\partial y} \implies v_{xy} = - u_{yy} .$$

Since $v_{yx} = v_{xy}$, these last two equations imply that $u_{xx} = -u_{yy}$, and thus u is harmonic. To see that v is harmonic, take $\frac{\partial}{\partial y}$ of (1) and $\frac{\partial}{\partial x}$ of (2) and the result follows at once.