AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

Thomas J Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

osler@rowan.edu

Chapter 8

The Laplace Transform

CHAPTER 8

THE LAPLACE TRANSFORM

8.1 Introduction to integral transforms

In this Chapter we are going to study an integral transform known as the Laplace transform. It is defined by the relation

$$(1) \qquad \mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \quad .$$

Here $F(t)$ is the given function of the real variable t, and the integral (1) converts it into its Laplace transform $f(s)$, a function of the complex variable s. As an example we have

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{s - a} \quad .$$

Throughout this Chapter we will use upper case letters to denote the given function (such as F, G, H) and the corresponding lower case letters to denote the transform (f, g, h).

Every integral transform of use has a corresponding inversion integral. The inversion integral permits us to find the original function $F(t)$ from its transform $f(s)$. The inversion integral for the Laplace transform (1) is

$$(2) \qquad F(t) = \mathcal{L}^{-1}\{f(s)\} = \frac{1}{2\pi i} \int_\Gamma e^{ts} f(s)\, ds$$

where Γ is a straight line parallel to the imaginary axis in the complex s-plane having all the singularities of $f(s)$ to its left.

Other important integral transforms and their corresponding inversion integrals are:

Exponential Fourier Transform

$$f(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\, dt , \qquad F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega} f(\omega)\, dt$$

Fourier Sine Transform

$$f(\omega) = \int_{0}^{\infty} F(t) \sin \omega t\, dt , \qquad F(t) = \frac{2}{\pi}\int_{0}^{\infty} f(\omega) \sin \omega t\, dt$$

Fourier Cosine Transform

$$f(\omega) = \int_{0}^{\infty} F(t) \cos \omega t\, dt , \qquad F(t) = \frac{2}{\pi}\int_{0}^{\infty} f(\omega) \cos \omega t\, dt$$

Mellin Transform

$$f(s) = \int_{0}^{\infty} F(t)\, t^{s-1}\, dt , \qquad F(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\, t^{-s}\, dt$$

Integral transforms are important in applied mathematics. Problems from the real world are often reduced to the solution of ordinary, partial, or integral equations. These equations in the variable t are then reduced to simpler equations in the variable s by means of the appropriate integral transform. The new equation in s is then solved, and the desired solution in the variable t is obtained through the inversion integral.

As an example of this process, consider the differential equation

$$Y''(t) + 4\,Y(t) = 2\,e^{2t}$$

subject to the initial conditions

$$Y(0) = 0\,, \qquad Y'(0) = 1\;.$$

We will see later that taking the Laplace transform (1) of both sides of this differential equation yields the simple algebraic equation

$$s^2\,y(s) - 1 + 4\,y(s) = s^{-2}$$

in which $y(s)$ is the Laplace transform of $Y(t)$. Solving this equation for $y(s)$ gives

$$y(s) = \frac{s}{(s-2)(s^2+4)}\;. \qquad \frac{s^2+1}{s^2(s^2+4)}$$

The inversion integral (2) converts this last expression into the desired solution $Y(t) = [\,e^{2t} + \sin 2t - \cos 2t\,]/4$.

We have only outlined the method by which integral transforms assist us in solving difficult equations. It the next sections, we will examine the process in greater detail. Our main interest will center on the inversion integral and its solution, because it is an important application of the methods of contour integration from Chapter 6.

8.2 The Laplace Transform

The Laplace transform of the function $F(t)$ is defined to be the function $f(s)$ given by

$$(1) \quad \mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt .$$

We note in particular the following:

1. The given function $F(t)$, is a function of the real variable t, where $0 < t < \infty$.
2. The Laplace transform is a function of the complex variable s .
3. The given function is denoted by an upper case letter, while its transform is denoted by the corresponding lower case letter.
4. The integral (1) defining the Laplace transform is an improper integral. When we investigate this integral for convergence later, we will see that it is defined for all complex numbers s in some right-half plane $\gamma < \mathrm{Re}(s)$, where γ is a constant.

Table 1 shows a few Laplace transforms of elementary functions. We can always use the defining integral (1) to find $f(s)$ from a given $F(t)$, and sometimes we can use a Table of Laplace transforms.

Example 1

Find the Laplace transform of the function $F(t) = 2e^{2t} + 3$.

First Solution

Using the defining relation (1) we have

$$\mathcal{L}\{2\,e^{2t} + 3\} = \int_0^{\infty} e^{-st}\,[\,2\,e^{2t} + 3\,]\,dt$$

$$= 2\int_0^{\infty} e^{(2-s)t}\,dt + 3\int_0^{\infty} e^{-st}\,dt$$

$$= \frac{2\,e^{(2-s)t}}{2-s} + \frac{3\,e^{-st}}{-s}\Bigg|_{t=0}^{\infty}$$

$$= \frac{2}{s-2} + \frac{3}{s}\,.$$

Second Solution

We will now use Table 1. Since the operator $\mathcal{L}$ is defined by the integral (1), it is a linear operator. Therefore

$$\mathcal{L}\{2\,e^{2t} + 3\} = 2\mathcal{L}\{e^{2t}\} + 3\mathcal{L}\{1\}$$

Now $\mathcal{L}\{e^{2t}\}$ and $\mathcal{L}\{1\}$ are given as items 3 and 1 in Table 1. Therefore

$$\mathcal{L}\{2\,e^{2t} + 3\} = 2\,[\,\tfrac{1}{s-2}\,] + 3\,[\,\tfrac{1}{s}\,]$$

$$= \frac{2}{s-2} + \frac{3}{s}$$

Example 2

Using the defining integral (1), find the Laplace transform of $F(t) = t^p$, where $Re(p) > -1$.

Solution

Euler's integral from Chapter 7 is

$$\Gamma(p+1) = \int_0^{\infty} e^{-u} u^p \, du ,$$

where we require that $-1 < \mathrm{Re}(p)$ so that this integral will converge. Set $u = ts$ and get entry 2 of Table 1 at once.

Problems

1. Use Table 1 to find the Laplace transforms of the following functions

(a) $2t^3 + 4t + 3$, (b) $2 \sin 3t + 4 \cos 3t$, (c) $3 \sin(t + \frac{\pi}{4})$,

(d) $4 e^{2t} \cos t + e^{-t} \sinh 3t$, (e) $e^{4t} t^{1/2}$.

2. Verify items 3, 4 and 5 in Table 1. (Hint: Write sine and cosine in exponential form.)

The defining integral (1) is an improper integral because of the upper limit infinity. When does it converge ? We must restrict the growth of $F(t)$ as t nears infinity. In particular, $F(t)$ must not grow so fast as to overcome the rapidly diminishing term e^{-st} in the integrand of (1). Suppose $F(t) = M e^{\gamma t}$ (with γ real), then (1) becomes

$$\mathcal{L}\{M e^{\gamma t}\} = M \int_0^{\infty} e^{-(s-\gamma)t} \, dt - \left. \frac{M e^{-(s-\gamma)t}}{s-\gamma} \right|_{t=0}^{\infty}$$

As t nears infinity, $e^{-(s-\gamma)t}$ will tend to zero only if $\mathrm{Re}(s-\gamma)$ is positive. In other words we require that the complex variable s be restricted to the half-plane $\mathrm{Re}(s) > \gamma$. In this case we have

TABLE 1

A SHORT LIST OF LAPLACE TRANSFORMS

No.	Function $F(t)$	Laplace transform $f(s)$
1.	1	s^{-1}
2.	$t^p \quad p > -1$	$p!\, s^{-p-1}$ when $p = 1, 2, 3, \ldots$ $\Gamma(p+1)\, s^{-p-1}$ otherwise
3.	e^{at}	$(s - a)^{-1}$
4.	$\sin at$	$\frac{a}{s^2 + a^2}$
5.	$\cos at$	$\frac{s}{s^2 + a^2}$
6.	$\sinh at$	$\frac{a}{s^2 - a^2}$
7.	$\cosh at$	$\frac{s}{s^2 - a^2}$
8.	$e^{bt}\sin at$	$\frac{a}{(s-b)^2 + a^2}$
9.	$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2 + a^2}$
10.	$e^{bt}\sinh at$	$\frac{a}{(s-b)^2 - a^2}$
11.	$e^{bt}\cosh at$	$\frac{s-b}{(s-b)^2 - a^2}$
12.	$e^{bt}\, t^p,\ p > -1$	$p!\,(s-b)^{-p-1} \quad p = 0, 1, 2, \ldots$ $\Gamma(p+1)\,(s-b)^{-p-1}$ otherwise

(2) $\mathcal{L}\{Me^{\gamma t}\} = \frac{M}{s-\gamma}$ for $Re(s) > \gamma$.

This last relation leads us to suspect that if $F(t)$ grows no faster than $M e^{\gamma t}$, for some constants M and γ, (and is otherwise well behaved), then the defining integral (1) should converge. We thus define

Functions of Exponential Order

The function $F(t)$ is said to be of exponential order if there exist real constants M and γ such that

$$|F(t)| \leq M e^{\gamma t}$$

for all t greater than some value t_o .

Example 3

Which of the following functions are of exponential order, and determine an appropriate constant γ .

(a) $3t + 2$, (b) $t^3 - \sin t$, (c) $e^{2t+1} + 5t$, (d) e^{t^2} , (e) t^t .

Solution

Since polynomials tend to infinity slower than $e^{\gamma t}$ for any positive γ, both (a) and (b) are of exponential order, with γ any positive real constant.

The function (c) is of exponential order since the term e^{2t} dominates all others. The constant γ is 2 or any number greater.

Since both e^{t^2} and t^t grow much more rapidly than $e^{\gamma t}$ for any γ , (d) and (e) are not of exponential order.

Problem

3. Which of the following are of exponential order,? Determine the exponential constant γ.

(a) $4\, e^{-4t} + 3t$, (b) $t^{-t} + t^2$, (c) $\sinh t$, (d) $3\, e^{5t}$, (e) $e^{\log t}$, (f) e^{t^4}.

Relation (2) suggests that when $F(t)$ is of exponential order, the integral (1) converges for all s in the right half-plane $Re(s) > \gamma$. It also suggests that $f(s)$ tends to zero like M/s or faster for large s in that half-plane. The following theorem gives precise results:

Theorem 1

Let $F(t)$ and $F'(t)$ be continuous and of exponential order

$$|F(t)| \leq M\, e^{\gamma t}$$

$$|F'(t)| \leq M\, e^{\gamma t}$$

for all $t > 0$. Then the Laplace transform

$$f(s) = \int_0^{\infty} e^{-st}\, F(t)\, dt$$

exists for $Re(s) > \gamma$ and $|f(s)| < \frac{M'}{|s|}$ in this half-plane, for some constant M'.

Problem

4. Observe that each $F(t)$ in Table 1 is of exponential order and that the corresponding Laplace transforms $f(s)$ tend to zero at least as fast as M'/s for s in some right half-plane.

8.3 The inversion integral

We now consider the problem of finding the function $F(t)$ when its Laplace transform $f(s)$ is given. We call $F(t)$ the inverse Laplace transform of $f(s)$ and write

$$F(t) = \mathcal{L}^{-1}\{f(s)\} .$$

In certain simple cases, we can use Table 1 to find inverse Laplace transforms.

Example 1

Find $\mathcal{L}^{-1}\left\{4\,s^{-6} + \frac{6s}{s^2+4}\right\}$.

Solution

The operator $\mathcal{L}^{-1}$ is linear and we have

$$F(t) = \mathcal{L}^{-1}\left\{4\,s^{-6} + \frac{6s}{s^2+4}\right\}$$

$$= 4\mathcal{L}^{-1}\{s^{-6}\} + 6\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} .$$

From Table 1 we see that $\mathcal{L}^{-1}\{5!\cdot s^{-6}\} = t^5$. Therefore

$$\mathcal{L}^{-1}\{s^{-6}\} = \frac{t^5}{5!} = \frac{t^5}{120} .$$

We also have from Table 1

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t .$$

Therefore

$$F(t) = 4\left[\frac{t^5}{120}\right] + 6\left[\cos 2t\right]$$

$$= \frac{t^5}{30} + 6\cos 2t \ .$$

Example 2

Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\}$.

Solution

We do not immediately see this $f(s)$ in our table. However if we use partial fractions techniques which we learned in previous courses we can write

(1) $$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \ .$$

To find the constant A we multiply (1) by s

$$\frac{1}{s-2} = A + \frac{Bs}{s-2}$$

and set $s = 0$ to get $A = -1/2$. To find B we multiply (1) by $(s-2)$

$$\frac{1}{s} = \frac{A(s-2)}{s} + B$$

and set $s = 2$ to get $B = 1/2$. Now (1) becomes

$$\frac{1}{s(s-2)} = \frac{-1/2}{s} + \frac{1/2}{s-2} \ .$$

From Table 1 we have $\mathcal{L}^{-1}\{1/s\} = 1$ and $\mathcal{L}^{-1}\{1/(s-2)\} = e^{2t}$. Therefore

$$F(t) = -1/2 + e^{2t}/2 .$$

Problem

5. Use Table 1 to find the inverse Laplace transforms of

(a) $\frac{5}{s+3}$, (b) $4\,(s-2)^{-3}$, (c) $6\,s^{-3} + s^{-5}$,

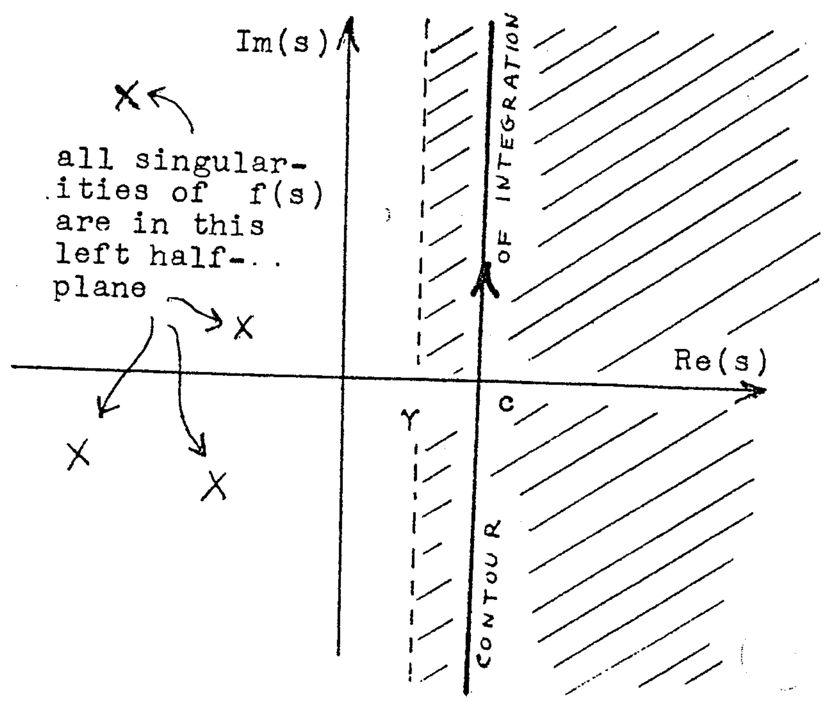
(d) $\frac{2}{(s+1)(s+2)}$, (e) $\frac{1}{s^2-4}$, (f) $\frac{1}{s^2(s^2+1)}$.

How can we find inverse Laplace transforms when $f(s)$ cannot be found in a table ? For this purpose, we have the

Inversion Integral

(2) $$F(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\, f(s)\, ds$$

The contour of integration is shown in the figure as a vertical straight line with all singularities of $f(s)$ on its left. We will first use (2) to find several inverse Laplace transforms. At the close of this section, we will suggest why (2) is true.



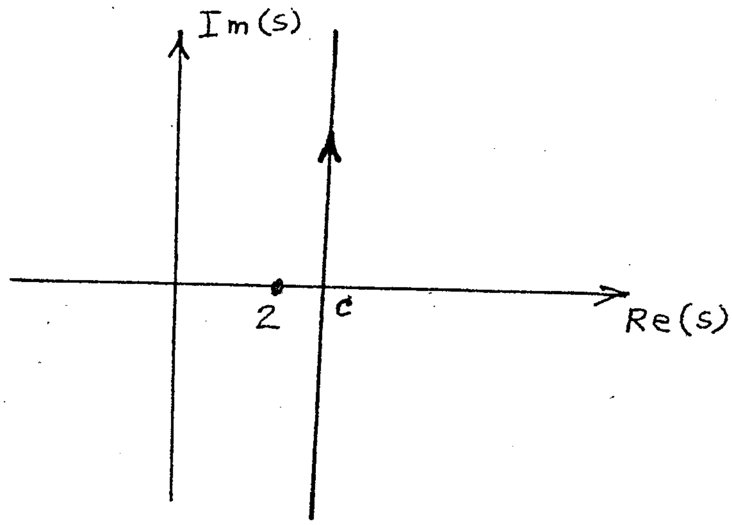
Example 1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$ using the inversion integral (2).

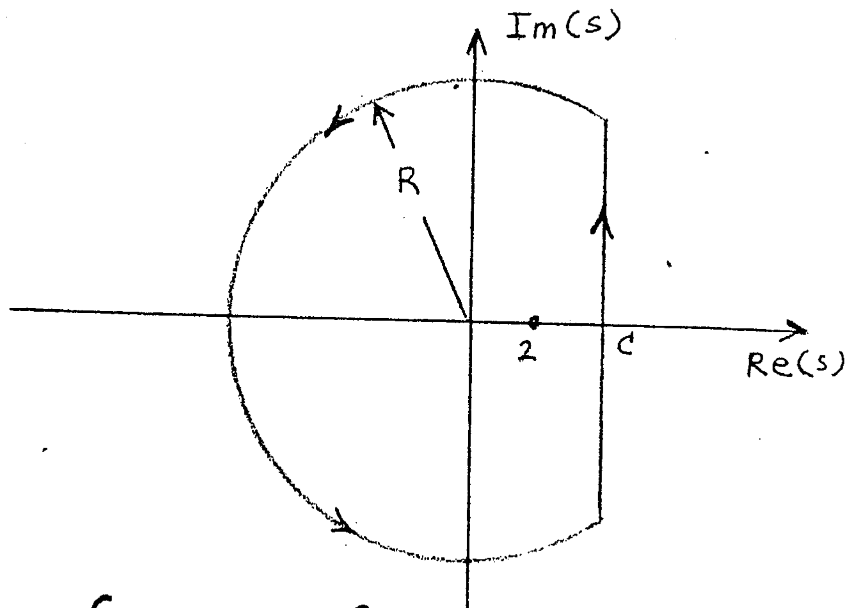
Solution

We have

$$F(t) = \frac{1}{2\pi i}\int_{\uparrow} \frac{e^{ts}\,ds}{(s-2)^2}$$



where the contour of integration is shown as the infinite vertical straight line to the right of the singularity at $s = 2$. Consider now the same integral over the closed contour consisting of two parts: (i) A finite portion of the previous straight line contour; and (ii) the circular arc in the left half-plane. We see that



(3) $$\frac{1}{2\pi i}\oint_{C} \frac{e^{ts}\,ds}{(s-2)^2} = \int_{\uparrow} + \int_{C}$$

If the integral over the circular arc tends to zero as the radius R grows, we will be able to replace the contour of (2) by this closed contour and then evaluate the integral by the Residue theorem of Chapter 6. The exponential factor in the integrand

$$|e^{ts}| = |e^{t\,Re(s)} \quad e^{t\,Im(s)}|$$

$$= e^{t\,Re(s)}$$

gives no trouble since $t > 0$ and $Re(s) < c$. In fact, over most of the circular arc, $Re(s)$ is negative and this factor is then very small. Thus the integrand $e^{ts}(s-2)^{-2}$ tends to zero as fast as R^{-2} on this circular arc. The length of the contour is less than $2\pi R$. Thus we have

$$\left|\int_{C}\right| < \left\{\begin{matrix}\text{Estimated size of}\\ \text{the integrand}\end{matrix}\right\} \times \left\{\begin{matrix}\text{Length of}\\ \text{the contour}\end{matrix}\right\}$$

$$< \frac{\text{constant}}{R^2} \times 2\pi R$$

$$< \frac{\text{constant}}{R}$$

Thus as R grows, the integral over the circular arc tends to zero. Formula (3) now gives us

$$F(t) = \frac{1}{2\pi i}\int_{\uparrow} \frac{e^{st}\,ds}{(s-2)^2} = \oint$$

The only singularity inside this last integral is a pole of order two at $s = 2$. The Residue theorem gives us

$$F(t) = \text{Res}(\ e^{ts}(s-2)^{-2}\ ,\ 2\ )$$

$$= \left.\frac{d\ e^{ts}}{ds}\right|_{s=2} = \left. t\ e^{ts}\right|_{s=2} = t\ e^{2t}\ .$$

This example suggests that when $f(s)$ has only poles, and no other types of singularities, then

$$(4) \qquad F(t) = \frac{1}{2\pi i}\int_{\uparrow} e^{st}\ f(s)\ ds = \left\{ \begin{array}{l} \text{sum of residues of } e^{st}f(s) \\ \text{at all poles of } f(s) \end{array} \right\}$$

This is certainly true if the integral over the circular part of the contour in the previous example tends to zero. It can be shown that a sufficient condition for the vanishing of this integral is that $f(s)$ tend to zero as rapidly as R^{-k} ($s = R\ e^{i\theta}$) where $k > 0$.

<u>Example 2</u>

Find $\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s^2+1)}\right\}$ using the inversion integral.

<u>Solution</u>

We see that

$$e^{st}f(s) = \frac{e^{st}(2s+1)}{s(s^2+1)} = \frac{e^{st}(2s+1)}{s(s+i)(s-i)}$$

has simple poles at $s = 0$, i and $-i$. Computing residues we have

$$\text{Res}(0) = \left.\frac{e^{st}(2s+1)}{(s+i)(s-i)}\right|_{s=0} = 1$$

$$\text{Res}(i) = \left.\frac{e^{st}(2s+1)}{s(s+i)}\right|_{s=i} = -\frac{(1+2i)e^{it}}{2}$$

$$\text{Res}(-i) = \left.\frac{e^{st}(2s+1)}{s(s-i)}\right|_{s=-i} = -\frac{(1-2i)e^{-it}}{2}$$

Using (4) we add these residues to get

$$F(t) = 1 - \frac{(1+2i)e^{it} + (1-2i)e^{-it}}{2}$$

$$= 1 - \frac{e^{it} + e^{-it}}{2} - i(e^{it} - e^{-it})$$

$$= 1 - \cos t + 2\sin t \ .$$

Problems

Use the inversion integral to find the inverse Laplace transforms of each of the following functions.

6. (a) $1/s$, (b) $(s^2 + 1)^{-1}$.

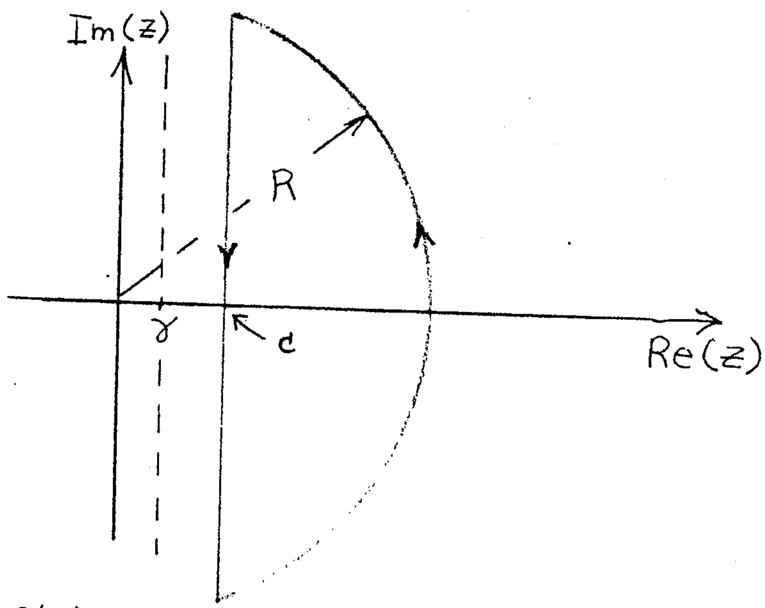
7. $\dfrac{s^2 + 1}{s^2(s^2-1)}$ 8. $\dfrac{1}{s^2(s^2 - 1)}$ 9. $\dfrac{s}{(s-2)(s^2 + 4)}$

10. $\dfrac{s^3 + 4s^2 + 2s + 4}{(s^2 + 1)(s + 2)^2}$ 11. $\dfrac{s^2 - 2s + 3}{(s^2 - 2s + 2)(s^2 + 5s + 6)}$

Now that we have become familiar with the use of the inversion integral (2), we will present an argument that suggests why it is true. Recall from Theorem 1 of the previous section that $f(s)$ is analytic in some right half plane $\gamma < \mathrm{Re}(s)$. We can therefore write, using Cauchy's integral formula

$$f(s) = \frac{1}{2\pi i} \oint_{D} \frac{f(z)\, dz}{z - s}$$

The contour of integration used is shown in the figure where we assume that R is very large, and s is inside this contour. Reversing the direction of this contour we get



$$f(s) = \frac{1}{2\pi i} \oint_{D} \frac{f(z)\, dz}{s - z}$$

$$= \int_{\uparrow} + \int_{\supset}$$

In Theorem 1 of the previous section we saw that $|f(z)|$ behaves like $\frac{M}{|z|}$ for large z. Thus on the circular arc the integrand tends to zero as fast as R^{-2} while the contour length tends to πR. We have

$$\left| \int_{\gamma} \right| < M R^{-2} (\pi R) = M\pi/R$$

and the integral over the arc vanishes as R grows to infinity. Now we have

(5) $$f(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(z)\,dz}{s - z}$$

Apply the operator $\mathcal{L}^{-1}$ to both sides of (5) and get

$$F(t) = \mathcal{L}^{-1}\{f(s)\} = \frac{1}{2\pi i} \mathcal{L}^{-1}\left\{ \int_{\uparrow} \frac{f(z)\,dz}{s - z} \right\}$$

$$= \frac{1}{2\pi i} \int_{\uparrow} f(z) \mathcal{L}^{-1}\left\{ \frac{1}{s - z} \right\} dz$$

$$= \frac{1}{2\pi i} \int_{\uparrow} f(z)\, e^{zt}\, dz \ .$$

We have assumed that we can interchange the operators $\mathcal{L}^{-1}$ and $\int_{\uparrow}$. Replace z by s and we have the inversion integral (2) at once.