5/(a) $\boxed{5e^{-3t}}$, (b) $\frac{4}{2}e^{2t}t^2 = \boxed{2e^{2t}t^2}$

(c) $\frac{6}{2}t^2 + \frac{t^4}{4!} = \boxed{3t^2 + \frac{t^4}{24}}$

(d) $$\frac{2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{s+2}$$

$$A = \left.\frac{2}{s+2}\right|_{s=-1} = 2 \; ; \quad B = \left.\frac{2}{s+1}\right|_{s=-2} = -2$$

$$F(t) = \boxed{2e^{-t} - 2e^{-2t}}$$

(e) $\boxed{\frac{1}{2}\sinh 2t}$

(f) $$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$F(t) = \boxed{t - \sin t}$$

6/(a) $$F(t) = \frac{1}{2\pi i}\int_{\uparrow} \frac{e^{st}}{s}\,ds = \operatorname{Res}\left(\frac{e^{st}}{s}, 0\right) = \left. e^{st}\right|_{s=0} = \boxed{1}$$

(b) $$\operatorname{Res}\left(\frac{e^{st}}{s^2+1}, i\right) + \operatorname{Res}\left(\frac{e^{st}}{s^2+1}, -i\right) =$$

$$\frac{e^{it}}{2i} - \frac{e^{-it}}{2i} = \boxed{\sin t}$$

7/ $$\operatorname{Res}\left(\frac{s^2+1}{s^2(s^2-1)}e^{st}, 0\right) = \left.\frac{d\left\{\frac{s^2+1}{s^2-1}e^{st}\right\}}{ds}\right|_{s=0} =$$

$$= \left.\frac{(s^2-1)\left[2se^{st} + t(s^2+1)e^{st}\right] - (s^2+1)e^{st}\,2s}{(s^2-1)^2}\right|_{s=0}$$

$$= -t$$

7/ (continued)

$$\operatorname{Res}\left(\frac{s^2+1}{s^2(s^2-1)}e^{st}, 1\right) = \left.\frac{s^2+1}{s^2(s+1)}e^{st}\right|_{s=1} = e^{t}$$

$$\operatorname{Res}\left(\frac{s^2+1}{s^2(s^2-1)}e^{st}, -1\right) = \left.\frac{s^2+1}{s^2(s-1)}e^{st}\right|_{s=-1} = -e^{-t}$$

Sum of these three residues is

$$F(t) = \boxed{-t + e^{t} - e^{-t}}$$

8/

$$\operatorname{Res}\left(\frac{e^{st}}{s^2(s^2-1)}, 0\right) = \left.\frac{d\left\{\frac{e^{st}}{s^2-1}\right\}}{ds}\right|_{s=0} = \left.\frac{(s^2-1)te^{st}-2se^{st}}{(s^2-1)^2}\right|_{s=0}$$

$$= -t$$

$$\operatorname{Res}\left(\frac{e^{st}}{s^2(s^2-1)}, 1\right) = \left.\frac{e^{st}}{s^2(s+1)}\right|_{s=1} = \frac{e^{t}}{2}$$

$$\operatorname{Res}\left(\frac{e^{st}}{s^2(s^2-1)}, -1\right) = \left.\frac{e^{st}}{s^2(s-1)}\right|_{s=-1} = -\frac{e^{-t}}{2}$$

Sum of these residues is

$$F(t) = -t + \frac{e^{t}}{2} - \frac{e^{-t}}{2} = \boxed{-t + \sinh t}$$

9/

$$\operatorname{Res}\left(\frac{se^{st}}{(s-2)(s^2+4)}, 2\right) = \left.\frac{se^{st}}{(s^2+4)}\right|_{s=2} = \frac{2e^{2t}}{8} = \frac{e^{2t}}{4}$$

$$\operatorname{Res}\left(\frac{se^{st}}{(s-2)(s^2+4)}, 2i\right) = \left.\frac{se^{st}}{(s-2)(s+2i)}\right|_{s=2i} = \frac{2ie^{2it}}{(2i-2)(4i)}$$

$$= -\frac{1}{4}\,\frac{e^{2ti}}{(1-i)}$$

9/ (continued)

$$\text{Res}\left(\frac{s e^{st}}{(s-2)(s^2+4)}, -i\right) = \left.\frac{s e^{st}}{(s-2)(s-2i)}\right|_{s=-2i}$$

$$= \frac{-2i\, e^{-2it}}{(-2i-2)(-4i)} = -\frac{1}{4}\ \frac{e^{-2it}}{(1+i)}$$

Adding these three residues we have

$$F(t) = \frac{1}{4}\left[e^{2t} - \frac{e^{2ti}}{1-i} - \frac{e^{-2ti}}{1+i}\right]$$

$$= \frac{1}{4}\left[e^{2t} - \frac{(1+i)e^{2ti} + (1-i)e^{-2ti}}{(1-i)(1+i)}\right]$$

$$= \frac{1}{4}\left[e^{2t} - \frac{e^{2ti} + e^{-2ti} + i(e^{2ti} - e^{-2ti})}{2}\right]$$

$$= \boxed{\frac{1}{4}\left[e^{2t} - \cos 2t + \sin 2t\right]}$$

10/ $$\text{Res}\left(\frac{(s^3+4s^2+2s+4)e^{st}}{(s^2+1)(s+2)^2}, -2\right) =$$

$$\left.\frac{d}{ds}\left[\frac{(s^3+4s^2+2s+4)e^{st}}{s^2+1}\right]\right|_{s=-2} =$$

$$= \frac{(s^2+1)\left[e^{st}(3s^2+8s+2) + (s^3+4s^2+2s+4)t e^{st}\right]}{(s^2+1)^2}$$

$$\left. - \frac{2s(s^3+4s^2+2s+4)e^{st}}{(s^2+1)^2}\right|_{s=-2}$$

$$= \left[\frac{22+40t}{25}\right]e^{-2t}$$

10/ (Continued)

$$\text{Res}(i) = \left.\frac{(s^3+4s^2+2s+4)e^{s\tau}}{(s+2)^2(s+i)}\right|_{s=i} = \frac{e^{i\tau}}{(2+i)^2\,2}$$

$$\text{Res}(-i) = \left.\frac{(s^3+4s^2+2s+4)e^{s\tau}}{(s+2)^2(s-i)}\right|_{s=-i} = \frac{e^{-i\tau}}{(2-i)^2\,2}$$

adding these three residues we have

$$F(\tau) = \left[\frac{22+40\tau}{25}\right]e^{-2\tau} + \frac{1}{2}\left[\frac{e^{i\tau}}{(2+i)^2} + \frac{e^{-i\tau}}{(2-i)^2}\right]$$

$$= \quad " \quad + \frac{1}{2}\left[\frac{(2-i)^2 e^{i\tau} + (2+i)^2 e^{-i\tau}}{(2+i)^2(2-i)^2}\right]$$

$$= \quad " \quad + \frac{1}{2}\cdot\frac{3e^{i\tau} - 4ie^{i\tau} + 3e^{-i\tau} + 4ie^{-i\tau}}{25}$$

$$= \quad " \quad \frac{3}{25}\cos\tau + \frac{4}{25}\sin\tau$$

$$\boxed{F(\tau) = \left[\frac{22+40\tau}{25}\right]e^{-2\tau} + \frac{3}{25}\cos\tau + \frac{4}{25}\sin\tau}$$

11/ $$\text{Res}\left(\frac{(s^2-2s+3)\qquad e^{s\tau}}{(s-1+i)(s-1-i)(s+3)(s+2)}, -2\right) = \left.\frac{(s^2-2s+3)e^{s\tau}}{(s-1+i)(s-1-i)(s+3)}\right|_{s=-2}$$

$$= \frac{11\,e^{-2\tau}}{10}$$

$$\text{Res}(-3) = \left.\frac{(s^2-2s+3)e^{s\tau}}{(s-1+i)(s-1-i)(s+2)}\right|_{s=-3} = -\frac{18}{17}e^{-3\tau}$$

$$\text{Res}(1+i) = \left.\frac{(s^2-2s+3)e^{s\tau}}{(s-1+i)(s+3)(s+2)}\right|_{s=1+i} = \frac{e^{(1+i)\tau}}{2i(11+7i)}$$

11/ (Continued)

$$\text{Res}(1-i) = \left.\frac{(s^2-2s+3)e^{st}}{(s-1-i)(s+3)(s+2)}\right|_{s=1-i} = \frac{e^{(1-i)t}}{-2i\,(11-7i)}$$

$$F(t) = \frac{11}{10}e^{-2t} - \frac{18}{17}e^{-3t} + \frac{1}{2i}\left[\frac{e^{(1+i)t}}{11+7i} - \frac{e^{(1-i)t}}{11-7i}\right]$$

$$= \quad '' \qquad '' \quad + \frac{1}{2i}\left[\frac{(11-7i)e^{(1+i)t} - (11+7i)e^{(1-i)t}}{(11+7i)(11-7i)}\right]$$

$$= \quad '' \qquad '' \quad + \frac{1}{2i}\,\frac{2i\,\text{Im}\{(11-7i)e^{(1+i)t}\}}{170}$$

$$= \quad '' \qquad '' \quad + \frac{e^{t}}{170}\,\text{Im}\{(11-7i)(\cos t + i\sin t)\}$$

$$= \quad '' \qquad '' \quad + \frac{e^{t}}{170}\{11\sin t - 7\cos t\}$$

$$\boxed{F(t) = \frac{11}{10}e^{-2t} - \frac{18}{17}e^{-3t} + \frac{e^{t}}{170}\{11\sin t - 7\cos t\}}$$

12/ $\mathcal{L}\{X''\} - \mathcal{L}\{X\} = \mathcal{L}\{t\}$

$s^2 x(s) - X(0)s - X'(0) - x(s) = \frac{1}{s^2}$

$s^2 x(s) - 1 - x(s) = \frac{1}{s^2}$

$(s^2-1)x(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$

$x(s) = \frac{x^2+1}{s^2(s^2-1)}$

Problem 7 now yields $\boxed{X(t) = -t + 2\sinh t}$

13/ $\mathcal{L}\{X^{(4)}\} - \mathcal{L}\{X\} = \mathcal{L}\{t\}$

$s^4 x(s) - 1 - x(s) = \frac{1}{s^2}$

$(s^4-1)\,x(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$

$x(s) = \frac{s^2+1}{s^2(s^4-1)} = \frac{1}{s^2(s^2-1)}$, Problem 8 now yields

$\boxed{X(t) = -t + \sinh t}$

14/ $\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = 2\mathcal{L}\{e^{2t}\}$

$$s^2 X(s) - 1 + 4X(s) = \frac{2}{s-2}$$

$$(s^2+4)X(s) = \frac{2}{s-2} + 1 = \frac{s}{s-2}$$

$$X(s) = \frac{s}{(s^2+4)(s-2)}$$

Problem 9 now yields

$$\boxed{x(t) = \frac{1}{4}\left[e^{2t} + \sin 2t - \cos 2t\right]}$$

15/ $x''(t) + 4x'(t) + 4x(t) = \cos t$

$x(0) = 1, \quad x'(0) = 0,$

$\mathcal{L}\{x''(t)\} + 4\mathcal{L}\{x'(t)\} + 4\mathcal{L}\{x(t)\} = \mathcal{L}\{\cos t\}$

$$s^2 X(s) - s + 4(sX(s) - 1) + 4X(s) = \frac{s}{s^2+1}$$

$$(s^2+4s+4)X(s) = \frac{s}{s^2+1} + s + 4 = \frac{s^3+4s^2+2s+4}{s^2+1}$$

$$X(s) = \frac{s^3+4s^2+2s+4}{(s+2)^2(s^2+1)}$$

?

Problem 10 now yields

$$\boxed{x(t) = \left[\frac{22+40t}{25}\right]e^{-2t}} + \frac{3}{25}\cos t + \frac{4}{25}\sin t$$

16/ $\mathcal{L}\{x''\} + 5\mathcal{L}\{x'\} + 6\mathcal{L}\{x\} = \mathcal{L}\{e^{t}\sin t\}$

$$s^2 X(s) - 1 + 5(sX(s)) + 6X(s) = \frac{1}{(s-1)^2+1}$$

$$(s^2+5s+6)X(s) = \frac{1}{s^2-2s+2} + 1 = \frac{s^2-2s+3}{s^2-2s+2}$$

$$X(s) = \frac{s^2-2s+3}{(s+3)(s+2)(s^2-2s+2)}$$

16/ (Continued)

Problem 11 now yields

$$\boxed{X(t) = \frac{11}{10} e^{-2t} - \frac{18}{17} e^{-3t} + \frac{e^{t}}{170}\left[11 \sin t - 7 \cos t\right]}$$

17/ $\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

$$s u(x,t) - u(x,0) = u_{xx}(x,s)$$

$$s u(x,t) - 2 \sin 2x = u_{xx}(x,s)$$

$$u_{xx} - s u = -2 \sin 2x$$

Solve for u using (11) and (12) of section 8.6 to get

$$u = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{2 \sin 2x}{s+4}$$

As in Example 1, $c_1 = c_2 = 0$ and thus

$u = \dfrac{2 \sin 2x}{s+4}$, Inverting we get

$$\boxed{u(x,t) = 2 e^{-4t} \sin 2x}$$

18/ $\mathcal{L}\{u_t\} = 4\mathcal{L}\{u_{xx}\}$

$$s u(x,s) - u(x,0) = 4 u_{xx}(x,s)$$

$$s u - 2 \sin \pi x - 3 \sin 2\pi x = 4 u_{xx}$$

$$u_{xx} - \frac{s}{4} u = -\frac{1}{2} \sin \pi x - \frac{3}{2} \sin 2\pi x$$

maybe $-\frac{3}{4}$

Solving this equation we have

$$u(x,s) = c_1 e^{\frac{\sqrt{s}}{2} x} + c_2 e^{-\frac{\sqrt{s}}{2} x} + \frac{2 \sin \pi x}{s + 4\pi^2} + \frac{6 \sin 2\pi x}{s + 16\pi^2}$$

18/ (continued)

As in Example 1, $c_1 = c_2 = 0$. Thus we have

$$u(x,s) = \frac{2\sin \pi x}{s+4\pi^2} + \frac{6\sin 2\pi x}{s+16\pi^2},$$

Inverting we have

$$\boxed{u(x,t) = 2e^{-4\pi^2 t}\sin \pi x + 6e^{-16\pi^2 t}\sin 2\pi x.}$$

maybe 3?

19/ $\mathcal{L}\{u_t\} = 9\,\mathcal{L}\{u_{xx}\}$

$$s\,u(x,s) - u(x,0) = 9\,u_{xx}(x,s)$$

$$s\,u(x,s) - 4\cos x = 9\,u_{xx}(x,s)$$

$$u_{xx} - \frac{s}{9}u = -\frac{4}{9}\cos x$$

Solving for u we have

$$u(x,s) = c_1 e^{\frac{\sqrt{s}}{3}x} + c_2 e^{-\frac{\sqrt{s}}{3}x} + \frac{4\cos x}{s+9}$$

As in Example 2, $c_1 = c_2 = 0$ and thus

$u(x,s) = \dfrac{4\cos x}{s+9}$, Inverting we have

$$\boxed{u(x,s) = 4e^{-9t}\cos x}$$

20/ $\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

$$s\,u(x,s) - u(x,0) = u_{xx}$$

$$s\,u(x,s) - 2\cos\frac{\pi x}{4} - 4\cos\frac{3\pi x}{4} = u_{xx}(x,s)$$

20/ (Continued)

$$u_{xx} - su = -2\cos\frac{\pi x}{4} - 4\cos\frac{3\pi x}{4}$$

Solving for u we have

$$u = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{2\cos\frac{\pi x}{4}}{s+\frac{\pi^2}{16}} + \frac{4\cos\frac{3\pi x}{4}}{s+\frac{9\pi^2}{16}}$$

As in Example 2, $c_1 = c_2 = 0$, Thus

$$u(x,s) = \frac{2\cos\frac{\pi}{4}x}{s+\frac{\pi^2}{16}} + \frac{4\cos\frac{3\pi x}{4}}{s+\frac{9\pi^2}{16}}$$

Inverting we have

$$\boxed{u(x,t) = 2e^{-\frac{\pi^2}{16}t}\cos\frac{\pi x}{4} + 4e^{-\frac{9\pi^2 t}{16}}\cos\frac{3\pi x}{4}}$$

21/ $\mathcal{L}\{u_{tt}\} = 9\mathcal{L}\{u_{xx}\}$

$$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = 9u_{xx}(x,s)$$

$$s^2 u(x,s) - 2s\sin x = 9u_{xx}(x,s)$$

$$u_{xx} - \frac{s^2}{9}u = \frac{-2s}{9}\sin x$$

This is an ordinary differential equation in x if we hold s fixed, The solution is found from (11) of section 8.6 as

$$u(x,s) = c_1 e^{\frac{s}{3}t} + c_2 e^{-\frac{s}{3}t} + \frac{2s\sin x}{s^2+9}$$

21/ (continued)

We show that $c_1 = c_2 = 0$ as in Example 1, Thus

$$u(x,s) = \frac{2s\sin x}{s^2+9}, \quad \text{Since } \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos a t$$

we have

$$\boxed{u(x,t) = 2\sin x \cos 3t}$$

22/ $\mathcal{L}\{u_{tt}\} = 4\mathcal{L}\{u_{xx}\}$

$$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = 4\,u_{xx}(x,s)$$

$$s^2 u(x,s) - 3\sin \pi x = 4\,u_{xx}(x,s)$$

$$u_{xx} - \frac{s^2}{4}u = -\frac{3}{4}\sin \pi x$$

The solution of this ordinary differential equation is

$$u(x,s) = c_1 e^{\frac{s}{2}x} + c_2 e^{-\frac{s}{2}x} + \frac{3\sin \pi x}{s^2+4\pi^2}$$

Again, $c_1 = c_2 = 0$ as in Example 1, Thus

$$u(x,s) = \frac{3\sin \pi x}{s^2+4\pi^2}, \quad \text{Since } \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin a t,$$

we have

$$\boxed{u(x,t) = \frac{3\sin \pi x \ \sin 2\pi t}{2\pi}}$$

23/ $\mathcal{L}\{u_{tt}\} = \mathcal{L}\{u_{xx}\}$

$$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = u_{xx}(x,s)$$

$$s^2 u(x,s) - s\sin \pi x - \sin 2\pi x = u_{xx}(x,s)$$

23/ (Continued)

$u_{xx} - s^2 u = -s \sin \pi x - \sin 2\pi x$.

The solution of this ordinary differential equation is

$$u(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{s \sin \pi x}{s^2+\pi^2} + \frac{\sin 2\pi x}{s^2 + 4\pi^2},$$

As in Example 1, $c_1 = c_2 = 0$ and we get

$$u(x,s) = \frac{s \sin \pi x}{s^2+\pi^2} + \frac{\sin 2\pi x}{s^2+4\pi^2}.$$

Inverting we get

$$\boxed{u(x,t) = \sin \pi x \cos \pi t + \frac{\sin 2\pi x \sin 2\pi t}{2\pi}}$$

24/ $\mathcal{L}\{u_{tt}\} = 16\, \mathcal{L}\{u_{xx}\}$

$$s^2 u(x,s) - s\, u(x,0) - u_t(x,0) = 16\, u_{xx}(x,s)$$

$$s^2 u(x,s) - 2s \cos x = 16\, u_{xx}(x,s)$$

$$u_{xx} - \frac{s^2}{16} u = -\frac{s}{8} \cos x$$

The solution of this ordinary differential equation is

$$u(x,s) = c_1 e^{\frac{s}{4}x} + c_2 e^{-\frac{s}{4}x} + \frac{2s \cos x}{s^2+16}$$

We must now solve for c_1 and c_2. Set $x=0$ and get

24/ (Continued)

$0 = c_1 + c_2$. Next, differentiate $u(x,s)$ with respect to x and get

$$u_x(x,s) = c_1 \tfrac{s}{4} e^{\frac{s}{4}x} - c_2 \tfrac{s}{4} e^{-\frac{s}{4}x} - \frac{2s \sin x}{s^2+16}$$

Set $x=0$ and note that

$$u_x(0,s) = \int_0^\infty e^{-st}\, \mathcal{U}_x(0,t)\,dt = 0 ,$$

Therefore $0 = \frac{s}{4} c_1 - \frac{s}{4} c_2$, Thus $c_1 = c_2 = 0$,

and
$$u(x,s) = \frac{2s\cos x}{s^2+16} ,$$
Inverting we have

$$\boxed{\mathcal{U}(x,t) = 2\cos x \cos 4t}$$

25/ Employ the same closed contour as was used in Section 8.8, Consider

(1) $$\frac{1}{2\pi i}\int e^{st}\frac{ds}{\sqrt{s}} = \int_{\uparrow} + \int_{F} + \int_{\rightarrow} + \int_{\circlearrowleft} + \int_{\leftarrow} + \int_{\swarrow}$$

Both $\int_F$ and $\int_{\swarrow}$ tend to zero as R tends to infinity,

Also $\int_{\circlearrowleft}$ tends to zero as $\epsilon \to 0$.

On the upper part of the branch cut, $s = re^{i\pi} = -r$, $\sqrt{s} = \sqrt{r}\, e^{i\pi/2} = i\sqrt{r}$. Thus

$$\int_{\rightarrow} = \frac{1}{2\pi i}\int_\infty^0 e^{-rt}\frac{(-dr)}{i\sqrt{r}} = -\frac{1}{2\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr .$$

25/ (continued)

On the lower part of the branch cut we have
$s = r e^{-i\pi} = -r, \quad \sqrt{s} = \sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r}$.

$$\int_{\leftarrow} = \frac{1}{2\pi i}\int_0^\infty e^{-rt}\frac{(-dr)}{-i\sqrt{r}} = -\frac{1}{2\pi}\int_0^\infty e^{-rt}\frac{dr}{\sqrt{r}}$$

Since there are no singularities inside the closed contour, the left side of (1) is zero. Thus (1) now reads

$$0 = F(t) + 0 - \frac{1}{2\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr + 0 - \frac{1}{2\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr + 0$$

Thus

$$F(t) = \frac{1}{\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr, \qquad \text{Set } rt = u \text{ and get}$$

$$F(t) = \frac{1}{\pi}\int_0^\infty e^{-u}\left(\frac{u}{t}\right)^{-1/2}\frac{du}{t}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-u}\, u^{1/2-1}\,du}{\sqrt{t}} = \frac{\Gamma(\frac{1}{2})}{\pi\sqrt{t}} = \boxed{\frac{1}{\sqrt{\pi t}}}$$

26/ Again we use the same contour as was used in section 8.8. Consider

$$(1) \quad \frac{1}{2\pi i}\int_{\circlearrowleft} e^{st} e^{-\sqrt{s}}\,ds = \int_{\uparrow} + \int_{\swarrow} + \int_{\rightarrow} + \int_{\circlearrowleft} + \int_{\leftarrow} + \int_{\searrow}$$

Both the integrals over the two large quarter circles tend to zero as $R \to \infty$.
Set $s = \epsilon e^{i\theta}$ in the integral over the small circle to get

$$\int_{\circlearrowleft} = \frac{1}{2\pi i}\int_{\pi}^{-\pi} e^{t\epsilon e^{i\theta}} e^{-\sqrt{\epsilon}\, e^{i\theta/2}}\, i\epsilon e^{i\theta}\,d\theta,$$

26/ (Continued)

As $\epsilon \to 0$ we see that the integral over the small circle tends to zero.

On the upper part of the branch cut we have

$s = r e^{i\pi} = -r$, and $\sqrt{s} = \sqrt{r}\, e^{i\pi/2} = i\sqrt{r}$.

Thus

$$\underset{\rightarrow}{\int} = \frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt}\, e^{-i\sqrt{r}}(-dr) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt-i\sqrt{r}}\, dr.$$

On the lower part of the branch cut we have

$s = r e^{-i\pi} = -r$, and $\sqrt{s} = \sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r}$.

Thus

$$\underset{\leftarrow}{\int} = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\, e^{i\sqrt{r}}(-dr)$$

Substituting into (1) we get

$$0 = F(t) + 0 + \frac{1}{2\pi i}\int_0^\infty e^{-rt}\, e^{-i\sqrt{r}}\, dr + 0$$

$$-\frac{1}{2\pi i}\int_0^\infty e^{-rt}\, e^{i\sqrt{r}}\, dr + 0$$

$$F(t) = \frac{1}{\pi}\int_0^\infty e^{-rt}\left[\frac{e^{i\sqrt{r}} - e^{-i\sqrt{r}}}{2i}\right] dr$$

$$= \boxed{\frac{1}{\pi}\int_0^\infty e^{-rt}\sin\sqrt{r}\, dr}$$

Solutions to Review Problems from Chapter 8

1/ $\mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\,dt =$

$\int_0^\infty e^{-(s-a)t} F(t)\,dt = f(s-a)$.

2/ $$F(t) = \frac{1}{2\pi i}\int_{\uparrow} e^{st}\left[\frac{s^3+3s^2+s+4}{(s+1)(s+2)(s+i)(s-i)}\right] ds$$

$$= \text{Res}(-1) + \text{Res}(-2) + \text{Res}(i) + \text{Res}(-i)$$

We have a simple pole at each residue.

$$\text{Res}(-1) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s+2)(s+i)(s-i)}\right|_{s=-1} = \frac{5}{2}e^{-t}$$

$$\text{Res}(-2) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s+1)(s^2+1)}\right|_{s=-2} = -\frac{6}{5}e^{-2t}$$

$$\text{Res}(i) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s^2+3s+2)(s+i)}\right|_{s=i} = \frac{e^{it}}{-6+2i}$$

$$\text{Res}(-i) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s^2+3s+2)(s-i)}\right|_{s=-i} = \frac{e^{-it}}{-6-2i}$$

Adding these four residues we get

$$\boxed{F(t) = \frac{5}{2}e^{-t} - \frac{6}{5}e^{-2t} - \frac{3}{10}\cos t + \frac{1}{10}\sin t}$$

3/ $2X''(t) + 6X'(t) + 4X(t) = 2\sin t$

$X(0) = 1, \quad X'(0) = 0,$

$\mathcal{L}\{X''\} + 3\mathcal{L}\{X'\} + 2\mathcal{L}\{X\} = \mathcal{L}\{\sin t\}$

$$\left[s^2 x(s) - sX(0) - X'(0)\right] + 3\left[s\,x(s) - X(0)\right] + 2x(s) = \frac{1}{s^2+1}$$

$$s^2 x(s) - s + 3s\,x(s) - 3 + 2x(s) = \frac{1}{s^2+1}$$

$$(s^2+3s+2)x(s) = \frac{1}{s^2+1} + s + 3 = \frac{s^3+3s^2+s+4}{s^2+1}$$

$$x(s) = \frac{s^3+3s^2+s+4}{(s^2+3s+2)(s^2+1)}$$

We inverted this transform in the previous problem,

4/ $u_t = u_{xx}, \quad u(0,t) = u(1,t) = 0$ and

$u(x,0) = \sin \pi x,$

$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

$s\,u(x,s) - u(x,0) = u_{xx}(x,s)$

$s\,u(x,s) - \sin \pi x = u_{xx}(x,s)$

$u_{xx} - s\,u = -\sin \pi x$

This is an ordinary differential equation in the variable x if we hold s fixed.

$$u(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{\sin \pi x}{s+\pi^2}.$$

This solution was obtained for (11) of section 8.6,

4/ (CONTINUED)

We show that $c_1 = c_2 = 0$ in the same way as in Example 1 of section 8.6. Thus

$$u(x,s) = \frac{\sin \pi x}{s + \pi^2},$$

Since $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$ we have

$$\boxed{\mathcal{U}(x,t) = e^{-\pi^2 t} \sin \pi x}$$

5/ We must solve $\mathcal{U}_{tt} = \mathcal{U}_{xx}$ subject to the boundary conditions $\mathcal{U}(0,t) = \mathcal{U}(2\pi, t) = 0$, and $\mathcal{U}(x,0) = \sin \frac{x}{2}$ while $\mathcal{U}_t(x,0) = 0$,

$$\mathcal{L}\{\mathcal{U}_{tt}\} = \mathcal{L}\{\mathcal{U}_{xx}\}$$

$$s^2 u(x,s) - s\,\mathcal{U}(x,0) - \mathcal{U}_t(x,0) = u_{xx}(x,s)$$

$$s^2 u(x,s) - s \sin \frac{x}{2} = u_{xx}(x,s)$$

$$u_{xx} - s^2 u = -s \sin \frac{x}{2}$$

The solution of this ordinary differential equation is obtained from (11) of section 8,6 as

$$u(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{s \sin\left(\frac{x}{2}\right)}{s^2 + \frac{1}{4}}$$

We show that $c_1 = c_2 = 0$ as in Example 1 of section 8.7, Thus

$$u(x,s) = \frac{s \sin \frac{x}{2}}{s^2 + \frac{1}{4}},$$ Since $\mathcal{L}\{\cancel{\cos}\ at\} = \frac{s}{s^2+a^2}$

we have $\mathcal{U}(x,t) = \boxed{\sin \frac{x}{2} \cos \frac{t}{2}}$

6/ We use the contour of integration described in section 8.8. We have

$$(1)\quad \frac{1}{2\pi i}\int e^{st}\,\frac{ds}{\sqrt[3]{s}} = \int_{\uparrow} + \int_{\rightarrow} + \int_{\leftarrow} + \underbrace{\int+\int+\int}_{\text{Three integrals over circular arcs}}$$

As $R \to \infty$ and $\epsilon \to 0$, the three integrals over circular arcs tend to zero. The integral $\int_{\uparrow}$ is the desired function $F(t)$. On the integral above the branch cut we have

$$s = r e^{i\theta} \qquad -\pi \le \theta \le \pi$$

where $\theta = \pi$. Thus $s = r e^{i\pi} = -r$ and $\sqrt[3]{s} = \sqrt[3]{r}\, e^{i\pi/3}$. Therefore

$$\int_{\rightarrow} = \frac{1}{2\pi i}\int_{\infty}^{0} \frac{e^{-rt}\,(-dr)}{e^{i\pi/3}\,\sqrt[3]{r}} = \frac{e^{-\frac{i\pi}{3}}}{2\pi i}\int_{0}^{\infty} e^{-rt}\,\frac{dr}{\sqrt[3]{r}},$$

On the integral bellow the branch cut we have

$$s = r e^{i\theta}$$

with $\theta = -\pi$. Thus $s = r e^{-i\pi} = -r$ and

$\sqrt[3]{s} = \sqrt[3]{r}\, e^{-\frac{i\pi}{3}}$, Therefore

6/ (Continued)

$$\int_{\leftarrow} = \frac{1}{2\pi i}\int_0^\infty \frac{e^{-tr}\,(-dr)}{\sqrt[3]{r}\;e^{-i\frac{\pi}{3}}} = \frac{-e^{i\frac{\pi}{3}}}{2\pi i}\int_0^\infty e^{-tr}\frac{dr}{\sqrt[3]{r}},$$

By Cauchy's integral theorem, the integral on the left side of (1) is zero because there are no singularities inside the contour,

Now (1) reads

$$0 = F(t) + \left[\frac{e^{-i\frac{\pi}{3}}}{2\pi i} - \frac{e^{i\frac{\pi}{3}}}{2\pi i}\right]\int_0^\infty e^{-tr}\,r^{-\frac{1}{3}}\,dr + 0$$

$$F(t) = \frac{\sin\frac{\pi}{3}}{\pi}\int_0^\infty e^{-tr}\,r^{\frac{2}{3}-1}\,dr$$

Set $tr = u$ and get

$$F(t) = \frac{\sqrt{3}}{2\pi}\,t^{-\frac{2}{3}}\int_0^\infty e^{-u}\,u^{\frac{2}{3}-1}\,du$$

This last integral is $\Gamma(\frac{2}{3})$ (See Chapter 7).

Thus

$$\boxed{F(t) = \frac{\sqrt{3}}{2\pi}\frac{\Gamma(\frac{2}{3})}{t^{2/3}}}$$