AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 8

The Laplace Transform

CHAPTER 8

THE LAPLACE TRANSFORM

8.1 Introduction to integral transforms

In this Chapter we are going to study an integral transform known as the Laplace transform. It is defined by the relation

$$(1) \qquad \mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \quad .$$

Here $F(t)$ is the given function of the real variable t, and the integral (1) converts it into its Laplace transform $f(s)$, a function of the complex variable s. As an example we have

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{s-a} \quad .$$

Throughout this Chapter we will use upper case letters to denote the given function (such as F, G, H) and the corresponding lower case letters to denote the transform (f, g, h).

Every integral transform of use has a corresponding inversion integral. The inversion integral permits us to find the original function $F(t)$ from its transform $f(s)$. The inversion integral for the Laplace transform (1) is

$$(2) \qquad F(t) = \mathcal{L}^{-1}\{f(s)\} = \frac{1}{2\pi i}\int_\Gamma e^{ts} f(s)\, ds$$

where Γ is a straight line parallel to the imaginary axis in the complex s-plane having all the singularities of $f(s)$ to its left.

Other important integral transforms and their corresponding inversion integrals are:

<u>Exponential Fourier Transform</u>

$$f(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \; F(t) \; dt \; , \qquad F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega} \; f(\omega) \; dt$$

<u>Fourier Sine Transform</u>

$$f(\omega) = \int_{0}^{\infty} F(t) \; \sin \omega t \; dt \; , \qquad F(t) = \frac{2}{\pi}\int_{0}^{\infty} f(\omega) \; \sin \omega t \; dt$$

<u>Fourier Cosine Transform</u>

$$f(\omega) = \int_{0}^{\infty} F(t) \; \cos \omega t \; dt \; , \qquad F(t) = \frac{2}{\pi}\int_{0}^{\infty} f(\omega) \; \cos \omega t \; dt$$

<u>Mellin Transform</u>

$$f(s) = \int_{0}^{\infty} F(t) \; t^{s-1} \; dt \quad , \qquad F(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s) \; t^{-s} \; dt$$

Integral transforms are important in applied mathematics. Problems from the real world are often reduced to the solution of ordinary, partial, or integral equations. These equations in the variable t are then reduced to simpler equations in the variable s by means of the appropriate integral transform. The new equation in s is then solved, and the desired solution in the variable t is obtained through the inversion integral.

As an example of this process, consider the differential equation

$$Y''(t) + 4\,Y(t) = 2\,e^{2t}$$

subject to the initial conditions

$$Y(0) = 0, \quad Y'(0) = 1 .$$

We will see later that taking the Laplace transform (1) of both sides of this differential equation yields the simple algebraic equation

$$s^2\,y(s) - 1 + 4\,y(s) = s^{-2}$$

in which $y(s)$ is the Laplace transform of $Y(t)$. Solving this equation for $y(s)$ gives

$$y(s) = \frac{s}{(s-2)(s^2+4)} . \qquad \frac{s^2+1}{s^2(s^2+4)}$$

The inversion integral (2) converts this last expression into the desired solution $Y(t) = [\,e^{2t} + \sin 2t - \cos 2t\,]/4$.

We have only outlined the method by which integral transforms assist us in solving difficult equations. It the next sections, we will examine the process in greater detail. Our main interest will center on the inversion integral and its solution, because it is an important application of the methods of contour integration from Chapter 6.

8.2 The Laplace Transform

The Laplace transform of the function $F(t)$ is defined to be the function $f(s)$ given by

$$(1) \quad \mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t)\, dt \,.$$

We note in particular the following:

1. The given function $F(t)$, is a function of the real variable t, where $0 < t < \infty$.
2. The Laplace transform is a function of the complex variable s .
3. The given function is denoted by an upper case letter, while its transform is denoted by the corresponding lower case letter.
4. The integral (1) defining the Laplace transform is an improper integral. When we investigate this integral for convergence later, we will see that it is defined for all complex numbers s in some right-half plane $\gamma < \mathrm{Re}(s)$, where γ is a constant.

Table 1 shows a few Laplace transforms of elementary functions. We can always use the defining integral (1) to find $f(s)$ from a given $F(t)$, and sometimes we can use a Table of Laplace transforms.

Example 1

Find the Laplace transform of the function $F(t) = 2e^{2t} + 3$.

First Solution

Using the defining relation (1) we have

$$\mathcal{L}\{2\,e^{2t} + 3\} = \int_0^\infty e^{-st}\,[\,2\,e^{2t} + 3\,]\,dt$$

$$= 2\int_0^\infty e^{(2-s)t}\,dt + 3\int_0^\infty e^{-st}\,dt$$

$$= \left.\frac{2\,e^{(2-s)t}}{2-s} + \frac{3\,e^{-st}}{-s}\right|_{t=0}^{\infty}$$

$$= \frac{2}{s-2} + \frac{3}{s}\,.$$

<u>Second Solution</u>

We will now use Table 1. Since the operator $\mathcal{L}$ is defined by the integral (1), it is a linear operator. Therefore

$$\mathcal{L}\{2\,e^{2t} + 3\} = 2\mathcal{L}\{e^{2t}\} + 3\mathcal{L}\{1\}$$

Now $\mathcal{L}\{e^{2t}\}$ and $\mathcal{L}\{1\}$ are given as items 3 and 1 in Table 1. Therefore

$$\mathcal{L}\{2\,e^{2t} + 3\} = 2\left[\frac{1}{s-2}\right] + 3\left[\frac{1}{s}\right]$$

$$= \frac{2}{s-2} + \frac{3}{s}$$

<u>Example 2</u>

Using the defining integral (1), find the Laplace transform of $F(t) = t^p$, where $\mathrm{Re}(p) > -1$.

<u>Solution</u>

Euler's integral from Chapter 7 is

$$\Gamma(p+1) = \int_0^{\infty} e^{-u} u^p \, du ,$$

where we require that $-1 < \mathrm{Re}(p)$ so that this integral will converge. Set $u = ts$ and get entry 2 of Table 1 at once.

Problems

1. Use Table 1 to find the Laplace transforms of the following functions

(a) $2 t^3 + 4 t + 3$, (b) $2 \sin 3t + 4 \cos 3t$, (c) $3 \sin(t + \frac{\pi}{4})$,

(d) $4 e^{2t} \cos t + e^{-t} \sinh 3t$, (e) $e^{4t} \; t^{1/2}$.

2. Verify items 3, 4 and 5 in Table 1. (Hint: Write sine and cosine in exponential form.)

The defining integral (1) is an improper integral because of the upper limit infinity. When does it converge ? We must restrict the growth of $F(t)$ as t nears infinity. In particular, $F(t)$ must not grow so fast as to overcome the rapidly diminishing term e^{-st} in the integrand of (1). Suppose $F(t) = M e^{\gamma t}$ (with γ real), then (1) becomes

$$\mathcal{L}\left\{M e^{\gamma t}\right\} = M \int_0^{\infty} e^{-(s-\gamma)t} \, dt - \left. \frac{M e^{-(s-\gamma)t}}{s-\gamma} \right|_{t=0}^{\infty}$$

As t nears infinity, $e^{-(s-\gamma)t}$ will tend to zero only if $\mathrm{Re}(s-\gamma)$ is positive. In other words we require that the complex variable s be restricted to the half-plane $\mathrm{Re}(s) > \gamma$. In this case we have

TABLE 1

A SHORT LIST OF LAPLACE TRANSFORMS

No.	Function $F(t)$	Laplace transform $f(s)$
1.	1	s^{-1}
2.	$t^p \qquad p > -1$	$p!\ s^{-p-1}$ when $p = 1, 2, 3, \ldots$ $\Gamma(p+1)\ s^{-p-1}$ otherwise
3.	e^{at}	$(s - a)^{-1}$
4.	$\sin at$	$\dfrac{a}{s^2 + a^2}$
5.	$\cos at$	$\dfrac{s}{s^2 + a^2}$
6.	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
7.	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
8.	$e^{bt}\sin at$	$\dfrac{a}{(s-b)^2 + a^2}$
9.	$e^{bt}\cos at$	$\dfrac{s-b}{(s-b)^2 + a^2}$
10.	$e^{bt}\sinh at$	$\dfrac{a}{(s-b)^2 - a^2}$
11.	$e^{bt}\cosh at$	$\dfrac{s-b}{(s-b)^2 - a^2}$
12.	$e^{bt}\ t^p,\ p > -1$	$p!\ (s-b)^{-p-1} \qquad p = 0, 1, 2, \ldots$ $\Gamma(p+1)\ (s-b)^{-p-1}$ otherwise

(2) $\mathcal{L}\{Me^{\gamma t}\} = \frac{M}{s-\gamma}$ for $Re(s) > \gamma$.

This last relation leads us to suspect that if $F(t)$ grows no faster than $M\,e^{\gamma t}$, for some constants M and γ, (and is otherwise well behaved), then the defining integral (1) should converge. We thus define

Functions of Exponential Order

The function $F(t)$ is said to be of exponential order if there exist real constants M and γ such that

$$|F(t)| \leq M\,e^{\gamma t}$$

for all t greater than some value t_o .

Example 3

Which of the following functions are of exponential order, and determine an appropriate constant γ .

(a) $3t + 2$, (b) $t^3 - \sin t$, (c) $e^{2t+1} + 5t$, (d) e^{t^2} , (e) t^t .

Solution

Since polynomials tend to infinity slower than $e^{\gamma t}$ for any positive γ, both (a) and (b) are of exponential order, with γ any positive real constant.

The function (c) is of exponential order since the term e^{2t} dominates all others. The constant γ is 2 or any number greater.

Since both e^{t^2} and t^t grow much more rapidly than $e^{\gamma t}$ for any γ , (d) and (e) are not of exponential order.

Problem

3. Which of the following are of exponential order,? Determine the exponential constant γ.

(a) $4\,e^{-4t} + 3t$, (b) $t^{-t} + t^{2}$, (c) $\sinh t$, (d) $3\,e^{5t}$, (e) $e^{\log t}$, (f) e^{t^4}.

Relation (2) suggests that when $F(t)$ is of exponential order, the integral (1) converges for all s in the right half-plane $Re(s) > \gamma$. It also suggests that $f(s)$ tends to zero like M/s or faster for large s in that half-plane. The following theorem gives precise results:

Theorem 1

Let $F(t)$ and $F'(t)$ be continuous and of exponential order

$$|F(t)| \leq M\,e^{\gamma t}$$

$$|F'(t)| \leq M\,e^{\gamma t}$$

for all $t > 0$. Then the Laplace transform

$$f(s) = \int_0^{\infty} e^{-st}\,F(t)\,dt$$

exists for $Re(s) > \gamma$ and $|f(s)| < \frac{M'}{|s|}$ in this half-plane, for some constant M'.

Problem

4. Observe that each $F(t)$ in Table 1 is of exponential order and that the corresponding Laplace transforms $f(s)$ tend to zero at least as fast as M'/s for s in some right half-plane.

8.3 The inversion integral

We now consider the problem of finding the function $F(t)$ when its Laplace transform $f(s)$ is given. We call $F(t)$ the inverse Laplace transform of $f(s)$ and write

$$F(t) = \mathcal{L}^{-1}\{f(s)\} .$$

In certain simple cases, we can use Table 1 to find inverse Laplace transforms.

Example 1

Find $\mathcal{L}^{-1}\left\{4\ s^{-6} + \dfrac{6s}{s^2+4}\right\}$.

Solution

The operator $\mathcal{L}^{-1}$ is linear and we have

$$F(t) = \mathcal{L}^{-1}\left\{4\ s^{-6} + \frac{6s}{s^2+4}\right\}$$

$$= 4\mathcal{L}^{-1}\{s^{-6}\} + 6\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} .$$

From Table 1 we see that $\mathcal{L}^{-1}\{5!\cdot s^{-6}\} = t^5$. Therefore

$$\mathcal{L}^{-1}\{s^{-6}\} = \frac{t^5}{5!} = \frac{t^5}{120} .$$

We also have from Table 1

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t .$$

Therefore

$$F(t) = 4\left[\frac{t^5}{120}\right] + 6\left[\cos 2t\right]$$

$$= \frac{t^5}{30} + 6 \cos 2t \ .$$

Example 2

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s-2)}\right\}$.

Solution

We do not immediately see this $f(s)$ in our table. However if we use partial fractions techniques which we learned in previous courses we can write

$$(1) \qquad \frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \ .$$

To find the constant A we multiply (1) by s

$$\frac{1}{s-2} = A + \frac{Bs}{s-2}$$

and set $s = 0$ to get $A = -1/2$. To find B we multiply (1) by $(s-2)$

$$\frac{1}{s} = \frac{A(s-2)}{s} + B$$

and set $s = 2$ to get $B = 1/2$. Now (1) becomes

$$\frac{1}{s(s-2)} = \frac{-1/2}{s} + \frac{1/2}{s-2} \ .$$

From Table 1 we have $\mathcal{L}^{-1}\{1/s\} = 1$ and $\mathcal{L}^{-1}\{1/(s-2)\} = e^{2t}$.

Therefore

$$F(t) = -1/2 + e^{2t}/2 .$$

Problem

5. Use Table 1 to find the inverse Laplace transforms of

(a) $\dfrac{5}{s+3}$, (b) $4\,(s-2)^{-3}$, (c) $6\,s^{-3} + s^{-5}$,

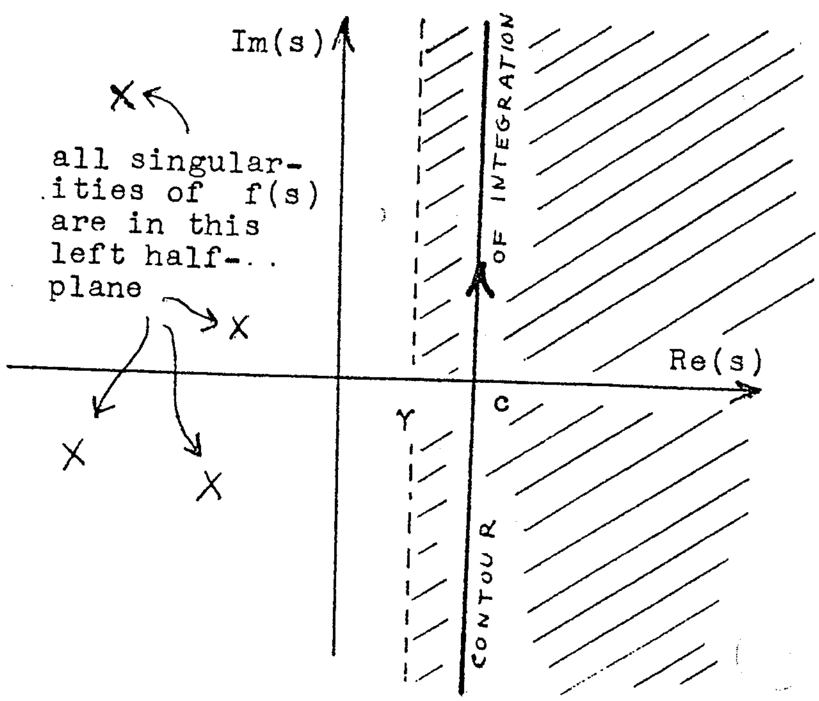
(d) $\dfrac{2}{(s+1)(s+2)}$, (e) $\dfrac{1}{s^2 - 4}$, (f) $\dfrac{1}{s^2(s^2+1)}$.

How can we find inverse Laplace transforms when $f(s)$ cannot be found in a table ? For this purpose, we have the

Inversion Integral

(2) $$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts}\, f(s)\, ds$$

The contour of integration is shown in the figure as a vertical straight line with all singularities of $f(s)$ on its left. We will first use (2) to find several inverse Laplace transforms. At the close of this section, we will suggest why (2) is true.



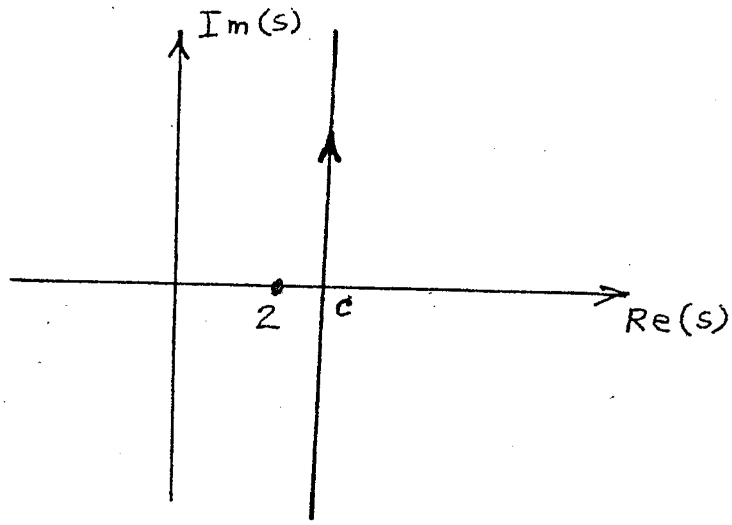
Example 1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$ using the inversion integral (2).

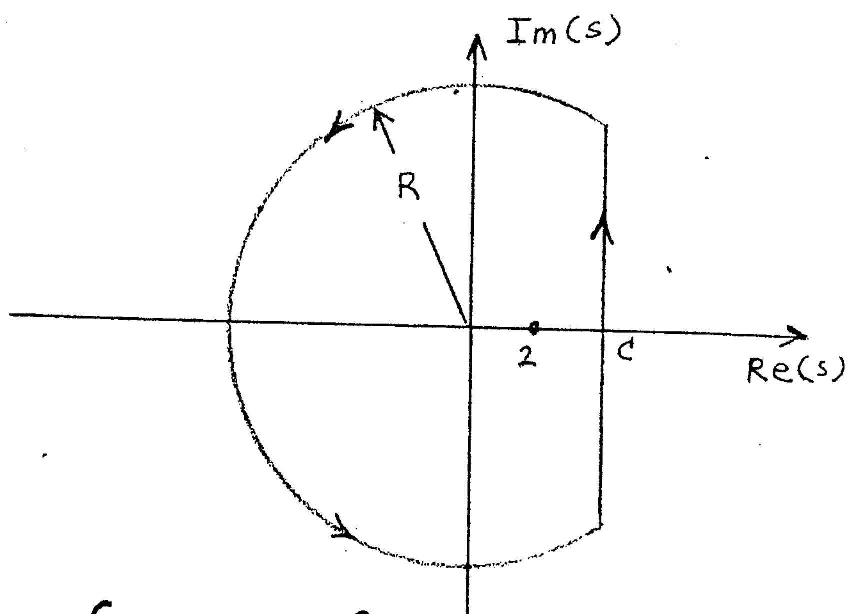
Solution

We have

$$F(t) = \frac{1}{2\pi i}\int_{\uparrow} \frac{e^{ts}\,ds}{(s-2)^2}$$



where the contour of integration is shown as the infinite vertical straight line to the right of the singularity at $s = 2$. Consider now the same integral over the closed contour consisting of two parts: (i) A finite portion of the previous straight line contour; and (ii) the circular arc in the left half-plane. We see that



(3) $$\frac{1}{2\pi i}\oint_{C\uparrow} \frac{e^{ts}\,ds}{(s-2)^2} = \int_{\uparrow} + \int_{C}$$

If the integral over the circular arc tends to zero as the radius R grows, we will be able to replace the contour of (2) by this closed contour and then evaluate the integral by the Residue theorem of Chapter 6. The exponential factor in the integrand

$$\left| e^{ts} \right| = \left| e^{t\,\mathrm{Re}(s)} \quad e^{t\,\mathrm{Im}(s)} \right|,$$

$$= e^{t\,\mathrm{Re}(s)}$$

gives no trouble since $t > 0$ and $\mathrm{Re}(s) < c$. In fact, over most of the circular arc, $\mathrm{Re}(s)$ is negative and this factor is then very small. Thus the integrand $e^{ts}(s-2)^{-2}$ tends to zero as fast as R^{-2} on this circular arc. The length of the contour is less than $2\pi R$. Thus we have

$$\left| \int_{C} \right| < \left\{ \begin{matrix} \text{Estimated size of} \\ \text{the integrand} \end{matrix} \right\} \times \left\{ \begin{matrix} \text{Length of} \\ \text{the contour} \end{matrix} \right\}$$

$$< \frac{\text{constant}}{R^2} \times 2\pi R$$

$$< \frac{\text{constant}}{R}$$

Thus as R grows, the integral over the circular arc tends to zero. Formula (3) now gives us

$$F(t) = \frac{1}{2\pi i} \int_{\uparrow} \frac{e^{st}\, ds}{(s-2)^2} = \oint$$

The only singularity inside this last integral is a pole of order two at $s = 2$. The Residue theorem gives us

$$F(t) = \operatorname{Res}(e^{ts}(s-2)^{-2} , 2)$$

$$= \left.\frac{d\, e^{ts}}{ds}\right|_{s=2} = \left. t\, e^{ts}\right|_{s=2} = t\, e^{2t} .$$

This example suggests that when $f(s)$ has only poles, and no other types of singularities, then

$$(4) \quad F(t) = \frac{1}{2\pi i}\int_{\uparrow} e^{st} f(s)\, ds = \left\{ \begin{array}{l} \text{sum of residues of } e^{st}f(s) \\ \text{at all poles of } f(s) \end{array} \right\}$$

This is certainly true if the integral over the circular part of the contour in the previous example tends to zero. It can be shown that a sufficient condition for the vanishing of this integral is that $f(s)$ tend to zero as rapidly as R^{-k} ($s = R\, e^{i\theta}$) where $k > 0$.

<u>Example 2</u>

Find $\mathcal{L}^{-1}\left\{\frac{2s + 1}{s(s^2+1)}\right\}$ using the inversion integral.

<u>Solution</u>

We see that

$$e^{st}f(s) = \frac{e^{st}(2s+1)}{s(s^2+1)} = \frac{e^{st}(2s+1)}{s(s+i)(s-i)}$$

has simple poles at $s = 0$, i and $-i$. Computing residues we have

$$\text{Res}(0) = \left.\frac{e^{st}(2s+1)}{(s+i)(s-i)}\right|_{s=0} = 1$$

$$\text{Res}(i) = \left.\frac{e^{st}(2s+1)}{s(s+i)}\right|_{s=i} = -\frac{(1+2i)e^{it}}{2}$$

$$\text{Res}(-i) = \left.\frac{e^{st}(2s+1)}{s(s-i)}\right|_{s=-i} = -\frac{(1-2i)e^{-it}}{2}$$

Using (4) we add these residues to get

$$F(t) = 1 - \frac{(1+2i)e^{it} + (1-2i)e^{-it}}{2}$$

$$= 1 - \frac{e^{it} + e^{-it}}{2} - i(e^{it} - e^{-it})$$

$$= 1 - \cos t + 2 \sin t \quad .$$

Problems

Use the inversion integral to find the inverse Laplace transforms of each of the following functions.

6. (a) $1/s$, (b) $(s^2 + 1)^{-1}$.

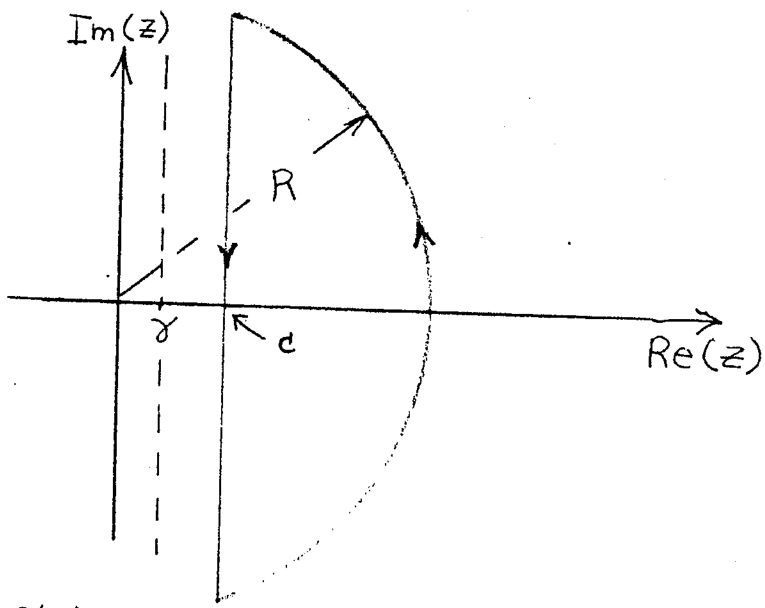
7. $\dfrac{s^2 + 1}{s^2(s^2-1)}$ 8. $\dfrac{1}{s^2(s^2 - 1)}$ 9. $\dfrac{s}{(s-2)(s^2 + 4)}$

10. $\dfrac{s^3 + 4s^2 + 2s + 4}{(s^2 + 1)(s + 2)^2}$ 11. $\dfrac{s^2 - 2s + 3}{(s^2 - 2s + 2)(s^2 + 5s + 6)}$

Now that we have become familiar with the use of the inversion integral (2), we will present an argument that suggests why it is true. Recall from Theorem 1 of the previous section that $f(s)$ is analytic in some right half plane $\gamma < \mathrm{Re}(s)$. We can therefore write, using Cauchy's integral formula

$$f(s) = \frac{1}{2\pi i} \oint_{D} \frac{f(z)\,dz}{z - s}$$

The contour of integration used is shown in the figure where we assume that R is very large, and s is inside this contour. Reversing the direction of this contour we get



$$f(s) = \frac{1}{2\pi i} \oint_{D} \frac{f(z)\,dz}{s - z}$$

$$= \int_{\uparrow} + \int_{)}$$

In Theorem 1 of the previous section we saw that $|f(z)|$ behaves like $\frac{M}{|z|}$ for large z. Thus on the circular arc the integrand tends to zero as fast as R^{-2} while the contour length tends to πR. We have

$$\left| \int_{\gamma} \right| < M R^{-2} (\pi R) = M\pi/R$$

and the integral over the arc vanishes as R grows to infinity. Now we have

$$(5) \qquad f(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(z)\, dz}{s - z}$$

Apply the operator $\mathcal{L}^{-1}$ to both sides of (5) and get

$$F(t) = \mathcal{L}^{-1}\{f(s)\} = \frac{1}{2\pi i} \mathcal{L}^{-1}\left\{ \int_{\uparrow} \frac{f(z)\, dz}{s - z} \right\}$$

$$= \frac{1}{2\pi i} \int_{\uparrow} f(z) \mathcal{L}^{-1}\left\{ \frac{1}{s - z} \right\} dz$$

$$= \frac{1}{2\pi i} \int_{\uparrow} f(z)\, e^{zt}\, dz \, .$$

We have assumed that we can interchange the operators $\mathcal{L}^{-1}$ and $\int_{\uparrow}$. Replace z by s and we have the inversion integral (2) at once.

8.4 Properties of the Laplace transform

The Laplace transform is a useful tool for the solution of certain ordinary, partial and integral equations. In these applications, various properties of the Laplace transform are used. In the following sections we will use the Laplace transform to solve special ordinary and partial differential equations. These applications will require that we be able to express the Laplace transform of the n^{th} derivative of $F(t)$ in terms of the transform of $F(t)$ itself.

Consider now

$$\mathcal{L}\left\{F'(t)\right\} = \int_0^{\infty} e^{-st} F'(t)\, dt \quad .$$

Integrating by parts we get

$$\mathcal{L}\left\{F'(t)\right\} = \int_0^{\infty} e^{-st}\, d\, F(t)$$

$$= e^{-st} F(t)\Big|_0^{\infty} + s\int_0^{\infty} e^{-st} F(t)\, dt$$

(1) $$= - F(0) + s\, f(s).$$

Relation (1) is important and is found in Table 2 as the first entry. Next we find the Laplace transform for the second derivative $F''(t)$. We start with

$$\mathcal{L}\left\{F''(t)\right\} = \mathcal{L}\left\{\frac{d\, F'(t)}{dt}\right\} .$$

TABLE 2

SOME PROPERTIES OF LAPLACE TRANSFORMS

No.	Function $F(t)$	Laplace transform $f(s)$
1.	$F'(t)$	$s\,f(s) - F(0)$
2.	$F''(t)$	$s^2 f(s) - s\,F(0) - F'(0)$
3.	$F'''(t)$	$s^3 f(s) - s^2 F(0) - s\,F'(0) - F''(0)$
4.	$F^{(n)}(t)$	$s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0)$
5.	$t\,F(t)$	$-f'(s)$
6.	$t^2F(t)$	$f''(s)$
7.	$t^n F(t)$	$(-1)^n f^{(n)}(s)$
8.	$e^{bt} F(t)$	$f(s-b)$
9.	$\int_0^t F(u)\,G(t-u)\,du$	$f(s)\,g(s)$
10.	$\int_0^t F(u)\,du$	$\frac{f(s)}{s}$
11.	$\int_0^t \dots \int_0^t F(u)(du)^n = \int_0^t \frac{(t-u)^{n-1} F(u)\,du}{(n-1)!}$	$\frac{f(s)}{s^n}$
12.	$\begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$	$e^{-as} f(s)$
13.	$\frac{F(t)}{t}$	$\int_s^\infty f(u)\,du$

Using (1) we get (replacing F in (1) by F′)

$$\mathcal{L}\{F''(t)\} = -F'(0) + s\mathcal{L}\{F'(t)\} .$$

Finally we apply (1) to the last term of the above expression and get

$$\mathcal{L}\{F''(t)\} = -F'(0) + s[-F(0) + s f(s)]$$

$$(2) \qquad = s^2 f(s) - s F(0) - F'(0) .$$

This last relation is item 2 in Table 2. Continuing in this way we can find the Laplace transform of the derivative of any order (see item 4 of Table 2).

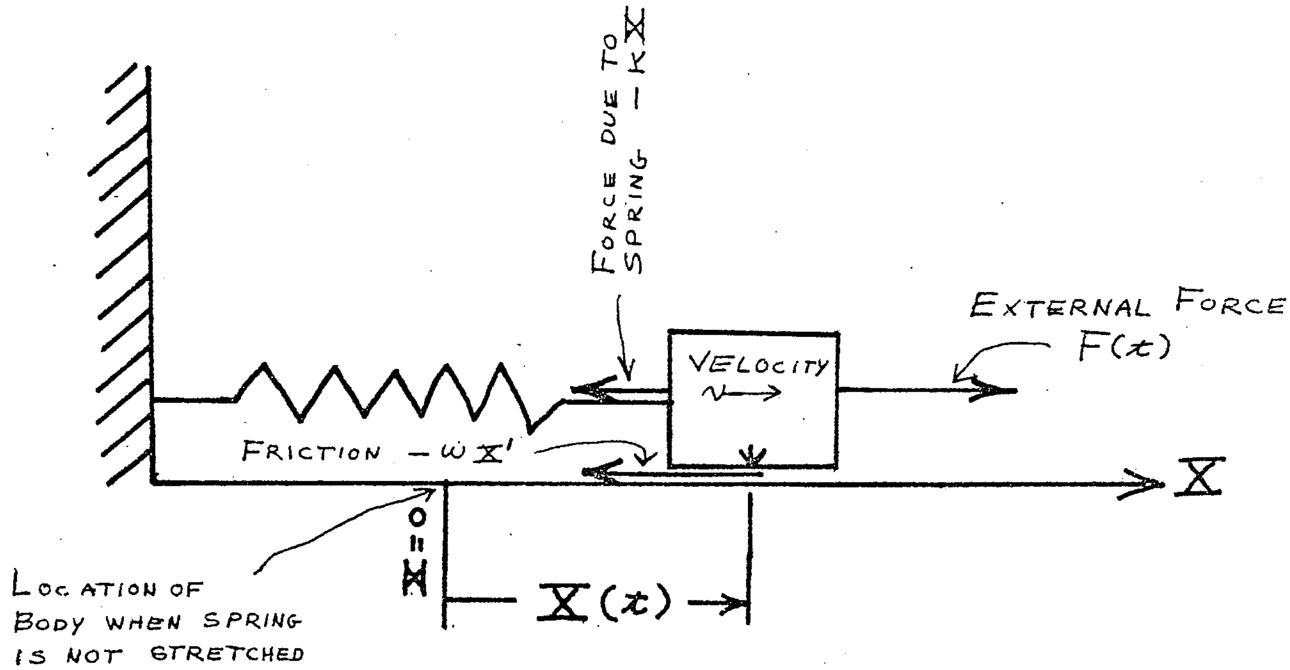
Various other properties of the transform are listed in Table 2. Since we will not be using these other properties in this chapter, we will not derive them here.

8.5 Ordinary Differential Equations

The mathematical formulation of problems from the real world often leads to differential equations. One source of these equations is Newton's law of motion:

(1) FORCE = MASS X ACCELERATION .

As an example, consider a body of mass m attached to a spring as shown. Let $X(t)$ describe the displacement of the body from its rest position at $X = 0$, and let t denote time . We have



$X'(t)$ = VELOCITY OF THE BODY

$X''(t)$ = ACCELERATION OF THE BODY

We see that the right side of (1) is $m\,X''(t)$.

There are three forces acting on the body:

1. Force due to the spring

This force is given by Hooke's law as kX, where k is a constant which measures the strength of the spring. Should we use $+kX$ or $-kX$ on the left side of (1) when accounting for this force ? Note that this force is pointing to the left in the figure when both X and X' are positive. Thus we use the minus sign.

2. Force due to friction

Friction is a complex phenomenon. We make the simplifying assumption that this force is given by $\omega X'$, where ω is a constant. Since this force is to the left in the figure, when X' is positive, we need to attach a minus sign to this term also.

3. Externally applied force

We postulate an external force which might vary with time and denote it by $F(t)$.

Substituting these forces into Newton's law (1) we get

$$- kX - \omega X' + F(t) = mX''$$

(2) $$m X''(t) + \omega X'(t) + k X(t) = F(t) .$$

We also give the intial position and velocity of the body as

(3) $$X(0) = A \quad \text{and} \quad X'(0) = B ,$$

where A and B are constants. The problem of describing the motion of the body is now reduced to solving the differential equation (2) subject to the initial conditions (3). Here m, ω and k are given constants, and $F(t)$ is a given function.

Differential equations, such as (2), involving only one independent variable (in this case t) are called <u>ordinary</u> differential equations. When they involve more than one independent variable they are called <u>partial</u> differential equations.

We will now use our knowledge of the Laplace transform to solve certain ordinary differential equations.

Example 1

Solve $X''(t) + X(t) = 1$ subject to the initial conditions $X(0) = 0$ and $X'(0) = 2$.

Solution

Operate with $\mathcal{L}$ on both sides of

$$X'' + X = 1$$

to get

(4) $$\mathcal{L}\{X''(t)\} + \mathcal{L}\{X(t)\} = \mathcal{L}\{1\}$$

Using Table 2, item 2, we see that

$$\mathcal{L}\{X''(t)\} = s^2 x(s) - s X(0) - X'(0) ,$$

where $x(s)$ is the Laplace transform of $X(t)$. Using the given initial conditions this becomes

$$\mathcal{L}\{X''(t)\} = s^2 x(s) - 0 - 2 .$$

From Table 1, item 1, we have $\mathcal{L}\{1\} = \frac{1}{s}$. Substituting into (4) we have

$$s^2 x(s) - 2 + x(s) = \frac{1}{s}$$

$$(s^2 + 1) x(s) = \frac{1}{s} + 2 = \frac{2s + 1}{s}$$

$$x(s) = \frac{2s + 1}{s(s^2+1)}$$

We now use the inversion integral to obtain $X(t)$ from $x(s)$. This particular inversion was performed in Example 2 of section 8.3 . We have

$$X(t) = 1 + 2 \sin t - \cos t .$$

Example 2

A body having mass $m = 1$ is attached to a spring with constant $k = 4$. The coefficient of friction ω is given as 4, and an external force $F(t) = 2$ is applied. Describe the motion if the body is displaced 2 units initially and is at rest.

Solution

Equations (2) and (3) become

$$(5) \qquad X'' + 4X' + 4X = 2$$

$$(6) \qquad X(0) = 2, \quad X'(0) = 0 .$$

Taking the Laplace transform of both sides of (5) gives

$$\mathcal{L}\{X''\} + 4\mathcal{L}\{X'\} + 4\mathcal{L}\{X\} = 2\mathcal{L}\{1\} .$$

Using Tables 1 and 2 we now have

$$[s^2 x(s) - s X(0) - X'(0)] + 4[sx(s) - X(0)] + 4x(s) = \frac{2}{s} .$$

Substituting the intitial conditions (6) we get

$$[s^2 x(s) - 2s - 0] + 4[sx(s) - 2] + 4x(s) = \frac{2}{s}$$

$$(s^2 + 4s + 4)\,x(s) = \frac{2}{s} + 2s + 8 = \frac{2s^2 + 8s + 2}{s}$$

$$x(s) = \frac{2s^2 + 8s + 2}{s\,(s+2)^2} .$$

The inversion integral now gives

$$X(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}(2s^2 + 8s + 2)}{s(s+2)^2}\, ds$$

$$= \text{Res}(0) + \text{Res}(-2) .$$

Now

$$\text{Res}(0) = \left.\frac{e^{st}(2s^2 + 8s + 2)}{(s+2)^2}\right|_{s=0} = \frac{1}{2} .$$

At $s = -2$ we have a pole of order two. Therefore

$$\text{Res}(-2) = \frac{d}{ds}\left.\left\{\frac{e^{st}(2s^2 + 8s + 2)}{s}\right\}\right|_{s=-2}$$

$$= \left.\frac{s[\,t\,e^{st}(2s^2 + 8s + 2) + e^{st}(4s + 8)] - e^{st}(2s^2 + 8s + 2)}{s^2}\right|_{s=-2}$$

$$= \left(3t + \frac{3}{2}\right) e^{-2t}.$$

Adding these two residues we get the solution

$$X(t) = \frac{1}{2} + \left(3t + \frac{3}{2}\right) e^{-2t} .$$

<u>Problems</u>

Solve the following ordinary differential equations.

12. $X'' - X = t$, $X(0) = 0$ and $X'(0) = 1$.

13. $X^{(4)} - X = t$, $X(0) = X'(0) = X''(0) = 0$, $X'''(0) = 1$.

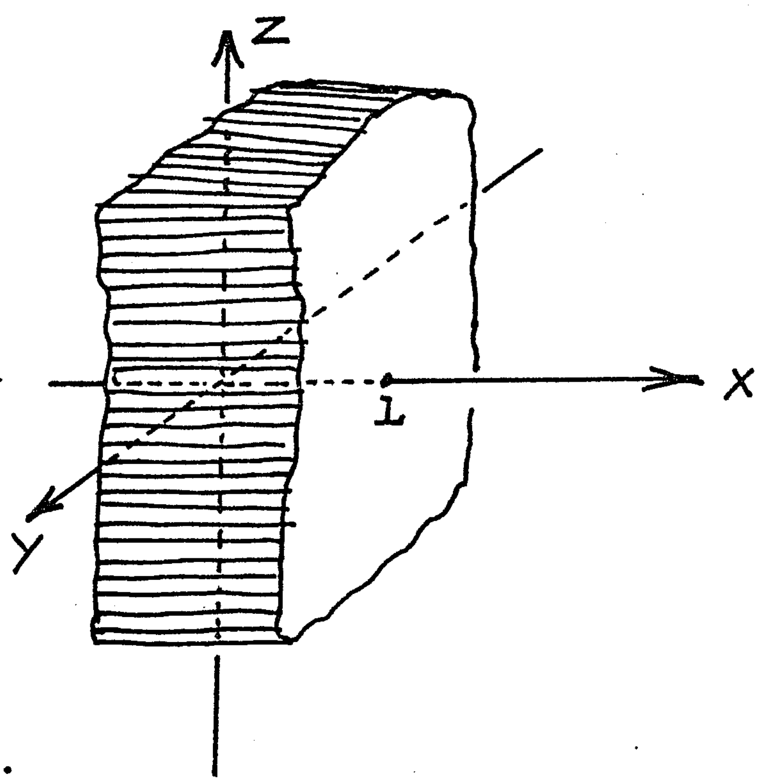
14. $X'' + 4X = 2e^{2t}$, $X(0) = 0$, $X'(0) = 1$.

15. A body having mass 1 is attached to a spring. The spring constant is 4, and the coefficient of friction is 4 . An external force cos t is applied. The body is initially at rest and displaced 1 unit. Describe the motion.

16. Solve $X'' + 5X' + 6X = e^t \sin t$, $X(0) = 0$ and $X'(0) = 1$. Give a physical interpretation to the solution.

8.6 The heat equation and its solution

We will now formulate and solve problems in the conduction of heat that involve partial differential equations. Suppose we seek the temperature at each point inside the slab of material shown between $x = 0$ and $x = L$. The slab extends to infinity in the y and the z directions. In general the temperature, which we denote by U, is a function of the three space variables x, y, z and the time t . Thus $U = U(x,y,z,t)$. To simplify matters, we shall assume that the temperature is independent of y and z. This means that on any plane surface $x = x_0$, the temperature is the same regardless of y and z. In this case, the temperature involves only two variables, x and t, and $U = U(x,t)$.



We will now give motivation for the basic equation which the temperature $U(x,t)$ satisfies. The quantity

$$\frac{\partial U(x,t)}{\partial t} = U_t(x,t)$$

is the rate at which the temperature is rising at the point x at time t. When is U_t positive, when is it zero, and when is it negative ?

Consider the graph of temperature versus x shown in Figure 1 at a certain instant of time. Will the temperature at the point P be rising or falling during the next second (in other words, will U_t be positive or negative) . From the bend in the curve, it appears that the temperatures near P are, on the average, higher than it is at P, and thus the temperature at P should ride.

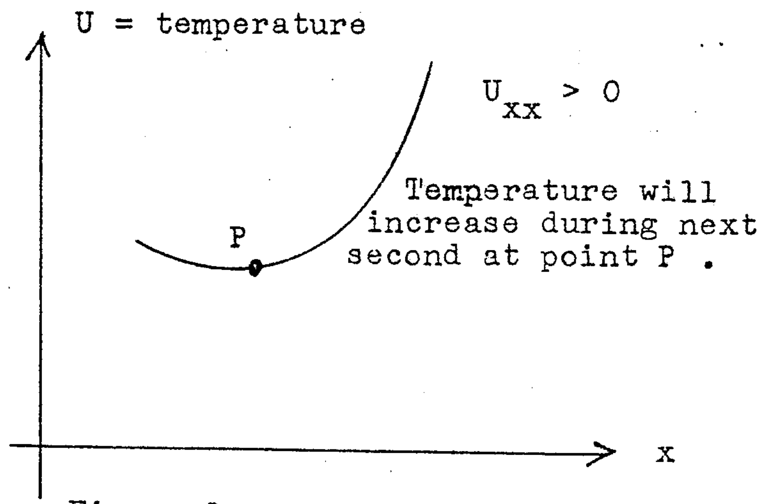


Figure 1

Now examine Figure 2. The temperatures to the right of P are higher and those to the left of P are lower than the temperature at P itself. These should balance each other, and the temperature at P should be

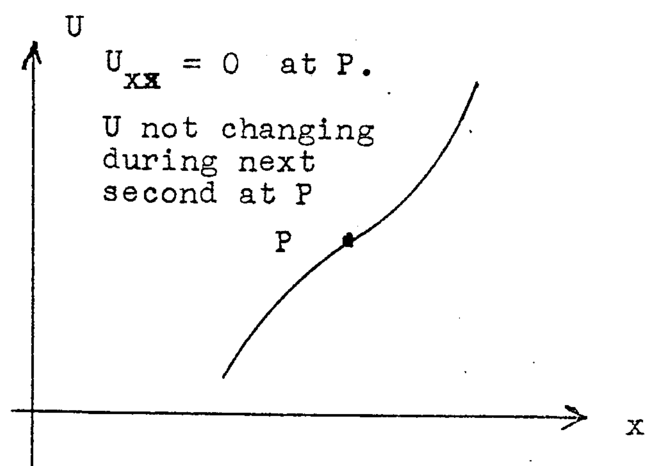


Figure 2

momentarily steady (in other words, $U_t = 0$ at P).

In Figure 3, the temperature at P appears to be on the decline, and thus U_t should be negative. What mathematical quantity distinguishes these three graphs ? In the elementary calculus we learned that

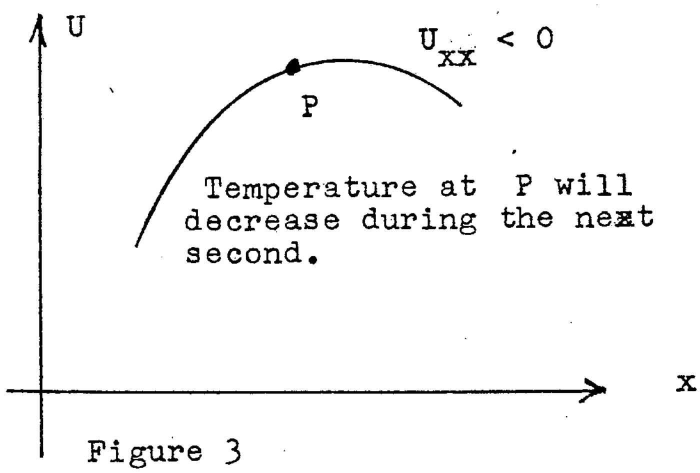


Figure 3

in Fig. 1 the graph is <u>concave upward</u>, and U_{xx} is positive,

in Fig. 2, point P is an <u>inflection point</u>, and $U_{xx} = 0$,

in Fig. 3, the graph is <u>concave downward</u>, and U_{xx} is negative.

In summary we see that

in Fig. 1 both U_t and U_{xx} are positive,

in Fig. 2 both U_t and U_{xx} are zero, and

in Fig. 3 both U_t and U_{xx} are negative.

It now seems reasonable to assume that U_t is proportional to U_{xx}

$$(1) \qquad U_t(x,t) = k \; U_{xx}(x,t) \quad ,$$

where k is some constant which reflects the thermal properties of the material out of which our slab is constructed. Relation (1) is called the <u>heat equation</u> .

Along with the heat equation (1), the behavior of the temperature on the left and right faces of our slab must be specified for all time. Also, the temperature distribution in the slab at the initial time $t = 0$ must be known. These are called <u>boundary</u>

conditions.

Example 1

The temperature $U(x,t)$ in a slab of material between $x = 0$ and $x = 2$ is governed by the heat equation

(2) $$U_t = 3\, U_{xx} \quad .$$

The temperature on both the left and the right faces of the slab is maintained at zero for all time. Thus

(3) $$U(0,t) = U(2,t) = 0 \; .$$

The initial temperature distribution in the slab is found to be

(4) $$U(x,0) = 5 \sin \pi x \; .$$

Determine $U(x,t)$ for $t \geq 0$ and $0 \leq x \leq 2$.

Solution

The partial differential equation (2) involves two independent variables, x and t. For the moment, hold x constant and take the Laplace transform of both sides of (2) with respect to the variable t . We have

(5) $$\mathcal{L}\{U_t(x,t)\} = 3\,\mathcal{L}\{U_{xx}(x,t)\} \; .$$

Denote $\mathcal{L}\{U(x,t)\}$ by $u(x,s)$, where

(6) $$u(x,s) = \int_0^{\infty} e^{-st}\, U(x,t)\, dt \quad .$$

Using item 1 of Table 2 we have

(7) $$\begin{aligned} \mathcal{L}\{U_t(x,t)\} &= s\, u(x,s) - U(x,0) \\ &= s\, u(x,s) - 5 \sin \pi x \end{aligned}$$

where the last term comes from (4). Operating with $\frac{\partial^2}{\partial x^2}$

on both sides of (6) we get

$$\frac{\partial^2 u(x,s)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}\, U(x,t)\, dt$$

$$= \int_0^\infty e^{-st}\, \frac{\partial^2 U(x,t)}{\partial x^2}\, dt$$

$$= \mathcal{L}\{U_{xx}\} \tag{8}$$

Substituting (7) and (8) into (5) we have

$$s\, u(x,s) - 5 \sin \pi x = 3\, u_{xx}(x,s)$$

$$u_{xx}(x,s) - \frac{s}{3}\, u(x,s) = -\frac{5}{3} \sin \pi x\ . \tag{9}$$

For the moment, treat s as a constant in (9), and observe that it is then an ordinary differential equation in x. We will assume that the the solution of such ordinary differential equations is familiar. They can be solved by elementary methods learned in separate courses in differential equations, or by the Laplace transform technique of the previous section. To simplify matters, we outline the solution of these ordinary differential equations bellow:

The solution of the ordinary differential equation

$$(10) \qquad u_{xx} - m^2 u = A \sin ax + B \cos bx$$

is

$$(11) \qquad u = c_1 e^{mx} + c_2 e^{-mx} - \frac{A \sin ax}{a^2 + m^2} - \frac{B \cos bx}{b^2 + m^2} ,$$

where c_1 and c_2 are arbitrary constant to be determined from given conditions on the solution $u(x)$.

Using (11) we see at once that the solution of (9) is

$$(12) \qquad u(x,s) = c_1 \exp(\sqrt{\tfrac{s}{3}}\, x) + c_2 \exp(-\sqrt{\tfrac{s}{3}}\, x) + \frac{5 \sin \pi x}{s + 3\pi^2}$$

We must now find c_1 and c_2 . Set $x = 0$ in (6) and get with the help of (3)

$$(13) \qquad u(0,s) = \int_0^{\infty} e^{-st}\, U(0,t)\, dt = 0 .$$

In the same way we see that

$$(14) \qquad u(2,s) = 0 .$$

Now set $x = 0$ in (12) and use (13) to get

$$(15) \qquad c_1 + c_2 = 0 .$$

Next set $x = 2$ in (12) and get using (14)

(16) $\quad \exp(2\sqrt{\frac{s}{3}})\, c_1 + \exp(-2\sqrt{\frac{s}{3}})\, c_2 = 0 \,.$

The simultaneous solution of equations (15) and (16) is easily seen to be $c_1 = c_2 = 0$. Thus (12) becomes

(17) $\quad u(x,s) = \dfrac{5 \sin \pi x}{s + 3\pi^2} \,.$

Treat x as fixed in this last expression and find the inverse Laplace transform. Since we recognize $\frac{1}{s + a} = \mathcal{L}\{e^{-at}\}$ we have

$$U(x,t) = 5 \exp(-3\pi^2 t) \sin \pi x$$

which is the solution to our problem.

Example 2

Solve the heat equation $U_t = 4\, U_{xx}$ subject to the boundary conditions $U_x(0,t) = 0$, $U(\pi,t) = 0$ and $U(x,0) = 40 \cos \frac{x}{2}$.

Solution

The boundary condition $U_x(0,t) = 0$ tells us that no heat can enter or leave through the left face of the slab (insulated) .

Operating with $\mathcal{L}$ on both sides of

$$U_t = 4\, U_{xx}$$

and holding x constant we get

$$\mathcal{L}\{U_t\} = 4\mathcal{L}\{U_{xx}\} \,.$$

which yields as before

$$s\ u(x,s) - U(x,0) = 4\ u_{xx}(x,s)$$

$$s\ u(x,s) - 40 \cos \frac{x}{2} = 4\ u_{xx}(x,s)$$

(18) $$u_{xx}(x,s) - \frac{s}{4} u(x,s) = -10 \cos \frac{x}{2}\ .$$

As before, we are using the notation

(19) $$u(x,s) = \int_0^{\infty} e^{-st}\ U(x,t)\ dt\ .$$

We see that (18) is an ordinary differential equation in x when s is held fixed. Using (10) and (11) we can solve for $u(x,s)$ as

(20) $$u(x,s) = c_1 \exp\left(\frac{\sqrt{s}}{2} x\right) + c_2 \exp\left(-\frac{\sqrt{s}}{2} x\right) + \frac{40 \cos(x/2)}{s+1}$$

We must now solve for c_1 and c_2. From (19) and the given boundary conditions $U_x(0,t) = 0$ and $U(\pi,t) = 0$ we have

(21) $$u_x(0,s) = 0$$

(22) $$u(\pi,s) = 0$$

To use (21), we must first differentiate (20) with respect to x,

$$u_x(x,s) = \frac{\sqrt{s}}{2} c_1 \exp\left(\frac{\sqrt{s}}{2} x\right) - \frac{\sqrt{s}}{2} c_2 \exp\left(-\frac{\sqrt{s}}{2} x\right) - \frac{20 \sin(x/2)}{s+1}\ .$$

Set $x = 0$ and use (21) to get

(23) $$c_1 - c_2 = 0$$

Set $x = \pi$ in (20) and use (22) to get

$$\exp\left(\frac{\sqrt{s}}{2} \pi \right) c_1 + \exp\left(\frac{\sqrt{s}}{2} \pi \right) c_2 = 0 . \tag{24}$$

The simultaneous solution of (23 and (24) is clearly $c_1 = c_2 = 0$, and thus (20) becomes

$$u(x,s) = \frac{40 \cos(x/2)}{s + 1} .$$

Hold x fixed and find the inverse Laplace transform from Table 1 item 3 as

$$U(x,t) = 40 e^{-t} \cos(x/2) .$$

Problems

Solve the following partial differential equations subject to the given boundary conditions.

17. $U_t = U_{xx}$, $U(0,t) = U(\pi,t) = 0$ and $U(x,0) = 2 \sin 2x$.

18. $U_t = 4 U_{xx}$, $U(0,t) = U(1,t) = 0$,
$U(x,0) = 2 \sin \pi x + 3 \sin 2\pi x$

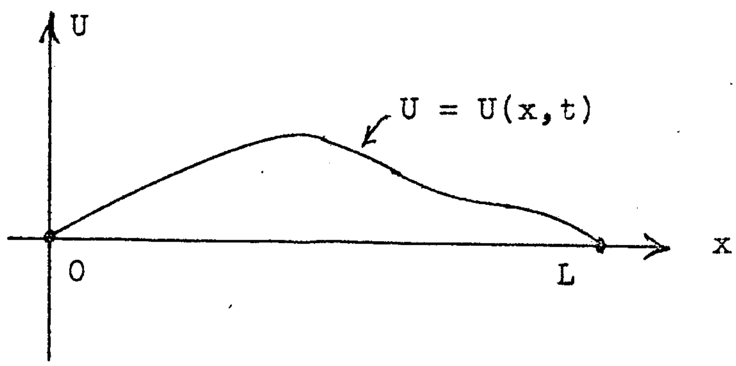
19. $U_t = 9 U_{xx}$, $U_x(0,t) = U(\pi/2, t) = 0$, and $U(x,0) = 4 \cos x$

20. $U_t = U_{xx}$, $U_x(0,t) = U(2,0) = 0$,
$U(x,0) = 2 \cos(\pi x/4) + 4 \cos(3\pi x/4)$.

8.7 The wave equation and its solution

We will now introduce a second partial differential equation known as the wave equation. We will solve certain boundary value problems involving this equation by means of the Laplace transform.

Consider a uniform string (such as a violin string) stretched along the x-axis from $x = 0$ to $x = L$. If this string is vibrating, can we describe its motion ? Let $U(x,t)$ denote the vertical displacement of the string from the axis at point x and time t as shown in the figure. We will now obtain the basic equation which $U(x,t)$ must satisfy.



Imagine that a very small segment of length Δx is cut out from the string. To cause this small segment of the string to continue vibrating as before, we attach the forces T_L and T_R (left tension and right tension) that the remainder of the string would have been exerting. This is shown in Figure 1. We see that the horizontal components of T_L and T_R cancel each other since they point in opposite directions. However, the vertical components add and cause the segment to experience an upward acceleration. The vertical components add because of the bend in the segment. What mathematical quantity measures the

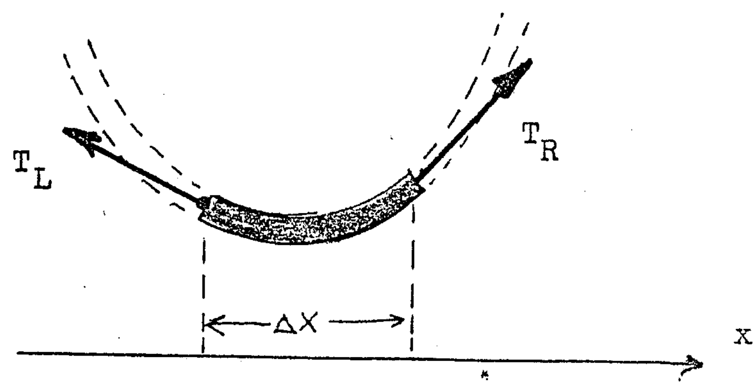


Figure 1

bend in the segment seen in Figure 1 ? Recall from the previous section that it is U_{xx} that measures this tendency to bend. Thus we can postulate that :

> "The resultant vertical force on a small segment of the string is proportional to U_{xx} " .

We know that Newton's basic law of motion reads:

$$\text{acceleration} = \frac{\text{force}}{\text{mass}} \quad .$$

The acceleration of our small segment is given by U_{tt} and the force on it is proportional to U_{xx} . Thus Newton's law of motion assumes the mathematical form

$$(1) \qquad U_{tt} = k^2 U_{xx} \quad .$$

where k^2 is some constant which reflects the density and tension of the string. Relation (1) is called the <u>wave equation</u> . It is little more than Newton's basic law of motion applied to a continuous body (string) .

Boundary conditions are always associated with (1) before a solution is obtained. Usually the nature of U at the left and right ends of the string is specified. Also, the initial shape and velocity of the string are usually given for $0 \le x \le L$.

Example 1

A string is stretched along the x-axis from $x = 0$ to $x = 2$, and its motion is governed by the equation

(2) $$U_{tt} = 4\, U_{xx} \quad .$$

The ends of the string are held fixed on the axis.Therefore

(3) $$U(0,t) = 0$$

(4) $$U(2,t) = 0 \quad .$$

The initial displcement and velocity of the string are

(5) $$U(x,0) = 3 \sin \pi x \quad \text{and}$$

(6) $$U_t(x,0) = 0 \ .$$

Describe the motion of the string.

Solution

Taking the Laplace transform (with respect to t) of both sides of (2) we get

$$\mathcal{L}\{U_{tt}\} = 4\mathcal{L}\{U_{xx}\}$$

where x is held fixed. Using item 2 of Table 2 we have

$$s^2\, u(x,s) - s\, U(x,0) - U_t(x,0) = 4\, u_{xx}(x,s) \quad .$$

Substituting the boundary conditions (5) and (6) we get

$$s^2\, u(x,s) - 3\, s \sin \pi x = 4\, u_{xx}(x,s)$$

(7) $$u_{xx} - \frac{s^2}{4} u = - \frac{3s}{4} \sin \pi x \quad .$$

Notice that (7) is an ordinary differential equation in x if we hold s fixed. Using (10) and (11) of the previous section we can write the solution of (7) at once as

$$(8) \qquad u(x,s) = c_1\, e^{sx/2} + c_2\, e^{-sx/2} + \frac{3s \sin \pi x}{s^2 + 4\pi^2} .$$

We must now solve for c_1 and c_2. Because

$$u(x,s) = \int_0^{\infty} e^{-st}\, U(x,t)\, dt$$

we see at once from (3) and (4) that

$$(9) \qquad u(0,s) = 0$$

$$(10) \qquad u(2,s) = 0 .$$

When we set $x = 0$ in (8) and also use (9) we get

$$(11) \qquad c_1 + c_2 = 0 .$$

When we set $x = 2$ in (8) and use (10) we get

$$(12) \qquad e^{s}\, c_1 + e^{-s}\, c_2 = 0 .$$

The simultaneous solution of (11) and (12) is seen at once to be $c_1 = c_2 = 0$. Thus (8) becomes

$$(13) \qquad u(x,s) = \frac{(\,3 \sin \pi x\,)\, s}{s^2 + 4\pi^2} .$$

In Table 1 we see that $\mathcal{L}\left\{\cos at\right\} = \dfrac{s}{s^2 + a^2}$,. Therefore the

inverse Laplace transform of (13) is

$$U(x,t) = 3 \sin \pi x \cos 2\pi t \quad .$$

Problems

Solve the following boundary value problems.

21. $U_{tt} = 9\, U_{xx}$, $U(0,t) = U(\pi,t) = 0$, $U(x,0) = 2 \sin x$, $U_t(x,0) = 0$.

22. $U_{tt} = 4\, U_{xx}$, $U(0,t) = U(1,t) = 0$, $U(x,0) = 0$ $U_t(x,0) = 3 \sin \pi x$.

23. $U_{tt} = U_{xx}$, $U(0,t) = U(4,t) = 0$, $U(x,0) = \sin \pi x$, $U_t(x,0) = \sin 2\pi x$.

24. $U_{tt} = 16\, U_{xx}$, $U_x(0,t) = U(\pi/2, t) = 0$, $U(x,0) = 2 \cos x$, $U_t(x,0) = 0$. What is the physical meaning of the boundary condition $U_x(0,t) = 0$.?

8.8 Inverting transforms with branch cuts

In this section we will solve another problem in heat conduction. In this new problem, the solution is complicated by the appearance of a branch cut in the Laplace transform $f(s)$. The residue theorem cannot be used to find the inverse Laplace transform as before, but the methods of contour integration about branch cuts seen in Chapter 6 will be employed.

Consider now an infinitely thick slab of material extending from $x = 0$ to plus infinity in the direction of the x-axis. We wish to find the temperature $U(x,t)$ inside this slab which is governed by the heat equation

(1) $$U_t = U_{xx} \quad \text{where } 0 < x < \infty \text{ , and } 0 < t < \infty \text{ .}$$

Initially, the temperature of the entire slab is zero,

(2) $U(x,0) = 0$.

The left face of the slab is kept at unit temperature

(3) $U(0,t) = 1$.

As in the previous two sections, we begin by operating with $\mathcal{L}$ on (1) to get

$$\mathcal{L}\{U_t\} = \mathcal{L}\{U_{xx}\}$$

$$s\,u(x,s) - U(x,0) = u_{xx}(x,s) .$$

Using (2) we have

$$s\,u(x,s) = u_{xx}(x,s)$$

$$u_{xx} - s\,u = 0 .$$

This is a simple ordinary differential equation whose solution is (see (11) of section 8.6)

(4) $$u(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} .$$

We must now determine the constants c_1 and c_2 . Since

(5) $$u(x,s) = \int_0^{\infty} e^{-st}\, U(x,t)\, dt ,$$

and since $U(x,t)$ is clearly less than or equal to 1 for all x and t from the boundary values, we see that $u(x,s)$ must tend to zero as s tends to plus infinity through positive real values. Thus c_1 in (4) must be zero since $e^{\sqrt{s}\,x}$ goes to infinity as s tends to plus infinity.

Now we determine c_2. Set $x = 0$ in (5) and get

$$u(0,s) = \int_0^\infty e^{-st}\, U(0,t)\, dt \quad .$$

Now use (3) and obtain

$$(6) \qquad u(0,s) = \int_0^\infty e^{-st}\, dt = \frac{1}{s} \quad .$$

Set $x = 0$ in (4) and compare the result with (6) to get

$$u(0,s) = c_2 = \frac{1}{s} \quad .$$

Thus we have

$$(7) \qquad u(x,s) = \frac{e^{-x\sqrt{s}}}{s} \quad .$$

We must now invert (7). The inversion integral gives us

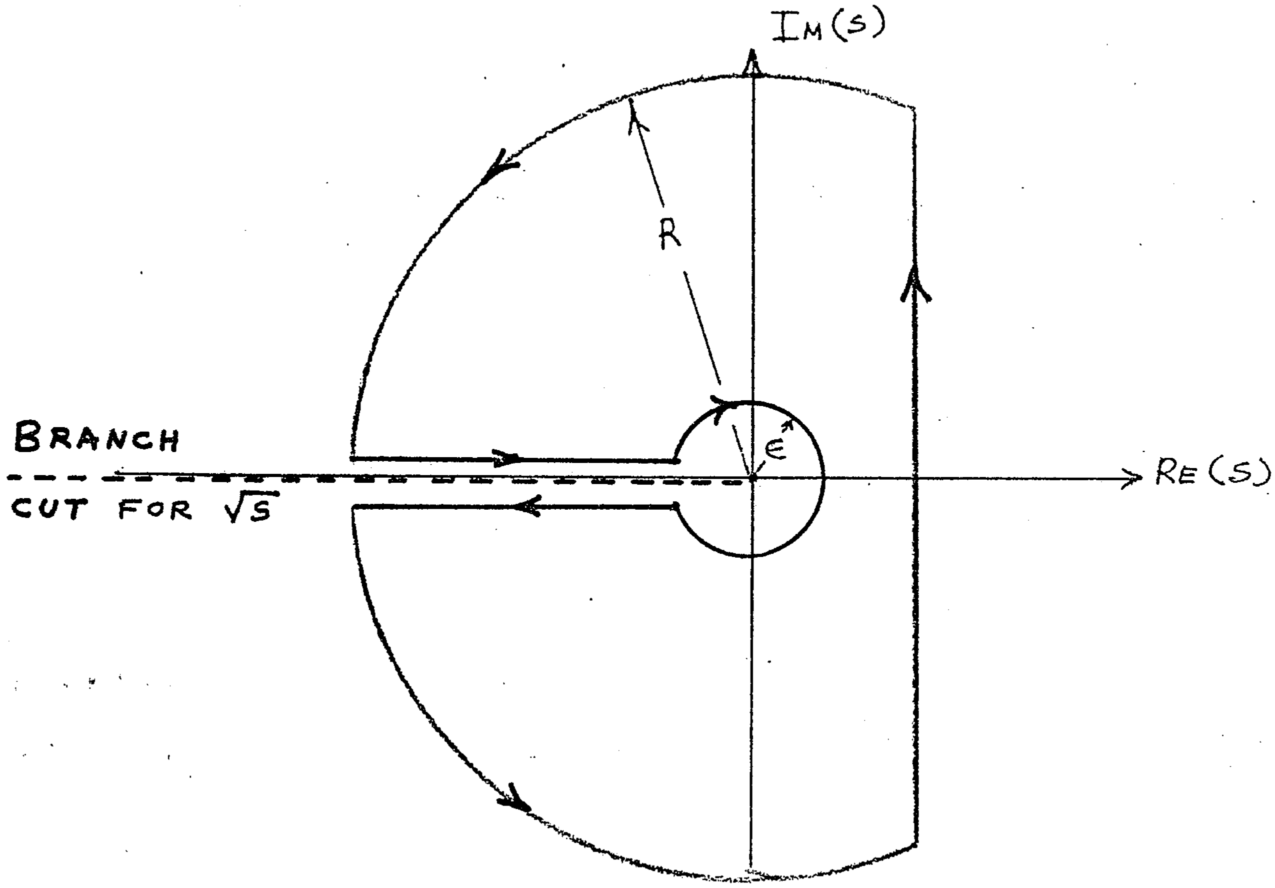
$$(8) \qquad U(x,t) = \frac{1}{2\pi i} \int_{\uparrow} e^{st}\, \frac{e^{-x\sqrt{s}}}{s}\, ds$$

The function $\sqrt{s}$ is double valued. By selecting

$$(9) \qquad s = r\, e^{i\theta} \quad , \text{ with } -\pi < \theta < \pi \;, \text{ we get}$$

$$\sqrt{s} = \sqrt{r}\, e^{i\theta/2}$$

which makes the negative x-axis our branch cut and defines $\sqrt{s}$ as a single valued branch.



Consider now the integral

$$(10) \qquad \frac{1}{2\pi i}\int e^{st}\,\frac{e^{-x\sqrt{s}}}{s}\,ds = \int + \int + \int + \int + \int + \int$$

The right side of (10) contains six integrals. We will now examine each of these individually.

<u>Integral over the vertical straight line</u>

Relation (8) shows us at once that this integral is $U(x,t)$.

<u>The two integrals over the large quarter circles</u>

On page 8.15 we learned that if $f(s)$ tends to zero as fast as R^{-k}, where k is any positive constant, then the integral over these circular arcs tends to zero. (Here we set $s = R\,e^{i\theta}$ on these arcs.) Our Laplace transform tends to zero as fast as R^{-1} and thus these two integrals over the quarter circles tend to zero as R tends to infinity.

<u>The integral over the small circle of radius ε</u>

Set $s = \varepsilon e^{i\theta}$, where ε is very small and get

$$\int_{\circlearrowright} = \frac{1}{2\pi i}\int_{\pi}^{-\pi} \frac{e^{t\varepsilon e^{i\theta}} \; e^{-x\sqrt{\varepsilon}\, e^{i\theta/2}} \; i\,\varepsilon\, e^{i\theta}\, d\theta}{\varepsilon\, e^{i\theta}}$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} e^{\varepsilon t e^{i\theta} - x\sqrt{\varepsilon}\, e^{i\theta/2}} \, d\theta \, .$$

Now let ε approach zero and get

$$(11) \qquad \lim_{\varepsilon \to 0} \int_{\circlearrowright} = \frac{1}{2\pi}\int_{\pi}^{-\pi} 1 \; d\theta = -1 \, .$$

<u>The integral along the upper part of the branch cut</u>

Return to relations (9) which define the branch we are using for the square root. We see that on the upper part of the branch cut we must take $\theta = \pi$. Thus we have

$$s = r\, e^{i\pi} = -r \quad \text{and}$$

$$\sqrt{s} = \sqrt{r}\; e^{i\pi/2} = i\sqrt{r} \, .$$

Substituting these relations into the integral gives

$$(12) \qquad \int_{\rightarrow} = \frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt}\, \frac{e^{-ix\sqrt{r}}}{-r}\,(-dr) = -\frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\, \frac{e^{-ix\sqrt{r}}}{r}\, dr$$

The integral along the lower part of the branch cut

From relations (9) which define the appropriate values for the square root, we see that we must select $\theta = -\pi$. Thus we now have

$$s = r\,e^{-i\pi} = -r \quad \text{and}$$

$$\sqrt{s} = \sqrt{r}\,e^{-i\pi/2} = -i\sqrt{r}\,.$$

Substituting into the integral we get

$$(13) \int_{\leftarrow} = \frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\,\frac{e^{ix\sqrt{r}}}{-r}\,(-dr) = \frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\,\frac{e^{ix\sqrt{r}}}{r}\,dr$$

The integral over the entire closed contour

This integral, which is the left side of (7) is zero by Cauchy's integral theorem since there are no singularities of the integrand inside the contour.

Now we collect relations (8) through (13) and substitute them back into (7) to get

$$0 = U(x,t) + 0 - \frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\,\frac{e^{-ix\sqrt{r}}}{r}\,dr \;-1\; + \frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\,\frac{e^{ix\sqrt{r}}}{r}\,dr$$

$$+\,0$$

Solving for $U(x,t)$ we have

$$U(x,t) = 1 - \frac{1}{\pi}\int_0^{\infty} e^{-rt}\left[\frac{e^{ix\sqrt{r}} - e^{-ix\sqrt{r}}}{2i}\right]\frac{dr}{r}$$

$$(14)\quad U(x,t) = 1 - \frac{1}{\pi}\int_0^{\infty} \frac{e^{-rt}\ \sin x\sqrt{r}}{r}\, dr\ .$$

At first glance, our solution (14) looks too complicated. The variables x and t appear as parameters in the integrand of an improper integral. However, we can convert (14) into a form which is suitable for computation. There exists a higher transcendental function known as the <u>error function</u> defined by

$$(15)\qquad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^{z} e^{-u^2}\, du$$

which is closely related to (14). In fact, one can show that

$$U(x,t) = 1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\ .$$

Tables of the error function are readily available.

<u>Problems</u>

Use the inversion integral to find the inverse Laplace transforms of the following functions.

25. $f(s) = s^{-1/2}$

26. $f(s) = \exp(-s^{+1/2})$.

Review Problems for Chapter 8

1. Show that the Laplace transform of $e^{at}F(t)$ is $f(s-a)$.

2. Use the inversion integral to find the inverse Laplace transform of

$$\frac{s^3 + 3s^2 + s + 4}{(s^2 + 3s + 2)(s^2 + 1)} .$$

3. A body of mass 2 is attached to a spring with spring constant 4. Friction is present and is assumed proportional to the velocity with proportionality constant 6. An external force $F(t) = 2 \sin t$ is also applied. Initially, the spring is stretched one unit, and the body is not moving. Describe the motion.

4. An infinite wall has its left face desribed by $x = 0$ and its right face described by $x = 1$. The temperature $U(x,t)$ at point x and time t inside the wall is governed by the heat equation $U_t = U_{xx}$. Both faces of the wall are maintained at temperature zero, but the temperature distribution within the wall at time $t = 0$ is given by $U(x,0) = \sin \pi x$. Find $U(x,t)$.

5. A uniform string is held fixed at the points $x = 0$ and $x = 2\pi$ on the x-axis. Let $U(x,t)$ describe the vertical displacement of the string, let the initial displacement be described by $U(x,0) = \sin(x/2)$, and let the string be at rest initially. The motion of the string is governed by the wave equation $U_{tt} = U_{xx}$. Find $U(x,t)$.

6. Use the inversion integral to find $\mathcal{L}^{-1}\left\{\frac{1}{s^{1/3}}\right\}$.

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 8

1/ (a) $2\left[\frac{3!}{s^4}\right] + 4\left[\frac{1!}{s^2}\right] + 3\left[\frac{1}{s}\right] = \boxed{\frac{12}{s^4} + \frac{4}{s^2} + \frac{3}{s}}$

(b) $2\left[\frac{3}{s^2+9}\right] + 4\left[\frac{s}{s^2+9}\right] = \boxed{\frac{4s+6}{s^2+9}}$

(c) $3\sin\left(t+\frac{\pi}{4}\right) = 3\sin t\cos\frac{\pi}{4} + 3\cos t\sin\frac{\pi}{4}$

$= \frac{3}{\sqrt{2}}\sin t + \frac{3}{\sqrt{2}}\cos t$, Therefore we have

$f(s) = \frac{3}{\sqrt{2}}\left[\frac{1}{s^2+1}\right] + \frac{3}{\sqrt{2}}\left[\frac{s}{s^2+1}\right] = \boxed{\frac{3}{\sqrt{2}}\ \frac{s+1}{s^2+1}}$

(d) $4\left[\frac{s-2}{(s-2)^2+1}\right] + \frac{3}{(s+1)^2-9} = \boxed{\frac{4(s-2)}{s^2-4s+5} + \frac{3}{s^2+2s-8}}$

(e) $\frac{\Gamma(\frac{1}{2}+1)}{(s-4)^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{(s-4)^{3/2}} = \boxed{\frac{\sqrt{\pi}}{2(s-4)^{3/2}}}$

2/ (Item 3) $f(s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \boxed{\frac{1}{s-a}}$

(Item 4) $f(s) = \int_0^\infty e^{-st}\,\frac{e^{iat}-e^{-iat}}{2i}\,dt =$

$= \frac{1}{2i}\int_0^\infty e^{-(s-ia)t}\,dt - \frac{1}{2i}\int_0^\infty e^{-(s+ia)t}\,dt$

$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{s+ia-s+ia}{(s-ia)(s+ia)}\right]$

$= \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right] = \boxed{\frac{a}{s^2+a^2}}$

2/ (continued)

(Item 5) $f(s)=\int_0^\infty e^{-st}\cos at\,dt=\int_0^\infty e^{-st}\frac{e^{iat}+e^{-iat}}{2}\,dt$

$=\frac{1}{2}\int_0^\infty e^{-(s-ia)t}\,dt+\frac{1}{2}\int_0^\infty e^{-(s+ia)t}\,dt$

$=\frac{1}{2}\left[\frac{1}{s-ia}+\frac{1}{s+ia}\right]=\boxed{\frac{s}{s^2+a^2}}$

3/ (a) is of exponential order with γ any positive constant since both e^{-4t} and t are dominated by $e^{\gamma t}$.

(b) is of exponential order with γ any positive constant since x^{-x} tends to zero and t^2 is dominated by $e^{\gamma t}$.

(c) $\sinh t=\frac{e^{t}-e^{-t}}{2}\approx\frac{1}{2}e^{t}$ for large t. Thus (c) is of exponential order with $\gamma=1$ or greater.

(d) is of exponential order with $\gamma=5$ or greater.

(e) Since $\log t<\gamma t$ for any positive γ and sufficiently large t, (e) is of exponential order with γ any positive constant.

(f) is not of exponential order since t^4 grows much faster than γt.

5/(a) $\boxed{5e^{-3t}}$, (b) $\frac{4}{2}e^{2t}t^2 = \boxed{2e^{2t}t^2}$

(c) $\frac{6}{2}t^2 + \frac{t^4}{4!} = \boxed{3t^2 + \frac{t^4}{24}}$

(d) $\frac{2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{s+2}$

$A = \left.\frac{2}{s+2}\right|_{s=-1} = 2$; $B = \left.\frac{2}{s+1}\right|_{s=-2} = -2$

$F(t) = \boxed{2e^{-t} - 2e^{-2t}}$

(e) $\boxed{\frac{1}{2}\sinh 2t}$

(f) $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$

$F(t) = \boxed{t - \sin t}$

6/(a) $F(t) = \frac{1}{2\pi i}\int_{\uparrow} \frac{e^{st}}{s}\,ds = \text{Res}\left(\frac{e^{st}}{s}, 0\right) = e^{st}\Big|_{s=0} = \boxed{1}$

(b) $\text{Res}\left(\frac{e^{st}}{s^2+1}, i\right) + \text{Res}\left(\frac{e^{st}}{s^2+1}, -i\right) =$

$\frac{e^{it}}{2i} - \frac{e^{-it}}{2i} = \boxed{\sin t}$

7/ $\text{Res}\left(\frac{s^2+1}{s^2(s^2-1)}e^{st}, 0\right) = \left.\frac{d\left\{\frac{s^2+1}{s^2-1}e^{st}\right\}}{ds}\right|_{s=0} =$

$= \left.\frac{(s^2-1)\left[2se^{st} + t(s^2+1)e^{st}\right] - (s^2+1)e^{st}2s}{(s^2-1)^2}\right|_{s=0}$

$= -t$

7/ (continued)

$$\operatorname{Res}\left(\frac{s^2+1}{s^2(s^2-1)}e^{s\tau}, 1\right) = \left.\frac{s^2+1}{s^2(s+1)}e^{s\tau}\right|_{s=1} = e^{\tau}$$

$$\operatorname{Res}\left(\frac{s^2+1}{s^2(s^2-1)}e^{s\tau}, -1\right) = \left.\frac{s^2+1}{s^2(s-1)}e^{s\tau}\right|_{s=-1} = -e^{-\tau}$$

Sum of these three residues is

$$F(\tau) = \boxed{-\tau + e^{\tau} - e^{-\tau}}$$

8/ $$\operatorname{Res}\left(\frac{e^{s\tau}}{s^2(s^2-1)}, 0\right) = \left.\frac{d\left\{\frac{e^{s\tau}}{s^2-1}\right\}}{ds}\right|_{s=0} = \left.\frac{(s^2-1)\tau e^{s\tau} - 2se^{s\tau}}{(s^2-1)^2}\right|_{s=0}$$

$$= -\tau$$

$$\operatorname{Res}\left(\frac{e^{s\tau}}{s^2(s^2-1)}, 1\right) = \left.\frac{e^{s\tau}}{s^2(s+1)}\right|_{s=1} = \frac{e^{\tau}}{2}$$

$$\operatorname{Res}\left(\frac{e^{s\tau}}{s^2(s^2-1)}, -1\right) = \left.\frac{e^{s\tau}}{s^2(s-1)}\right|_{s=-1} = -\frac{e^{-\tau}}{2}$$

Sum of these residues is

$$F(\tau) = -\tau + \frac{e^{\tau}}{2} - \frac{e^{-\tau}}{2} = \boxed{-\tau + \sinh\tau}$$

9/ $$\operatorname{Res}\left(\frac{se^{s\tau}}{(s-2)(s^2+4)}, 2\right) = \left.\frac{se^{s\tau}}{(s^2+4)}\right|_{s=2} = \frac{2e^{2\tau}}{8} = \frac{e^{2\tau}}{4}$$

$$\operatorname{Res}\left(\frac{se^{s\tau}}{(s-2)(s^2+4)}, 2i\right) = \left.\frac{se^{s\tau}}{(s-2)(s+2i)}\right|_{s=2i} = \frac{2ie^{2i\tau}}{(2i-2)(4i)}$$

$$= -\frac{1}{4}\frac{e^{2\tau i}}{(1-i)}$$

9/ (continued)

$$\text{Res}\left(\frac{s e^{st}}{(s-2)(s^2+4)}, -i\right) = \left.\frac{s e^{st}}{(s-2)(s-2i)}\right|_{s=-2i}$$

$$= \frac{-2i\, e^{-2it}}{(-2i-2)(-4i)} = -\frac{1}{4}\,\frac{e^{-2it}}{(1+i)}$$

Adding these three residues we have

$$F(t) = \frac{1}{4}\left[e^{2t} - \frac{e^{2ti}}{1-i} - \frac{e^{-2ti}}{1+i}\right]$$

$$= \frac{1}{4}\left[e^{2t} - \frac{(1+i)e^{2ti} + (1-i)e^{-2ti}}{(1-i)(1+i)}\right]$$

$$= \frac{1}{4}\left[e^{2t} - \frac{e^{2ti} + e^{-2ti} + i(e^{2ti} - e^{-2ti})}{2}\right]$$

$$= \boxed{\frac{1}{4}\left[e^{2t} - \cos 2t + \sin 2t\right]}$$

10/ $$\text{Res}\left(\frac{(s^3+4s^2+2s+4)e^{st}}{(s^2+1)(s+2)^2}, -2\right) =$$

$$\left.\frac{d}{ds}\left[\frac{(s^3+4s^2+2s+4)e^{st}}{s^2+1}\right]\right|_{s=-2} =$$

$$= \frac{(s^2+1)\left[e^{st}(3s^2+8s+2) + (s^3+4s^2+2s+4)t e^{st}\right]}{(s^2+1)^2}$$

$$\left. - \frac{2s(s^3+4s^2+2s+4)e^{st}}{(s^2+1)^2}\right|_{s=-2}$$

$$= \left[\frac{22+40t}{25}\right]e^{-2t}$$

10/ (Continued)

$$\text{Res}(i) = \left.\frac{(s^3+4s^2+2s+4)\,e^{s\tau}}{(s+2)^2(s+i)}\right|_{s=i} = \frac{e^{i\tau}}{(2+i)^2\,2}$$

$$\text{Res}(-i) = \left.\frac{(s^3+4s^2+2s+4)\,e^{s\tau}}{(s+2)^2(s-i)}\right|_{s=-i} = \frac{e^{-i\tau}}{(2-i)^2\,2}$$

Adding these three residues we have

$$F(\tau) = \left[\frac{22+40\tau}{25}\right]e^{-2\tau} + \frac{1}{2}\left[\frac{e^{i\tau}}{(2+i)^2} + \frac{e^{-i\tau}}{(2-i)^2}\right]$$

$$= \quad '' \quad + \frac{1}{2}\left[\frac{(2-i)^2 e^{i\tau} + (2+i)^2 e^{-i\tau}}{(2+i)^2(2-i)^2}\right]$$

$$= \quad '' \quad + \frac{1}{2}\,\frac{3e^{i\tau} - 4ie^{i\tau} + 3e^{-i\tau} + 4ie^{-i\tau}}{25}$$

$$= \quad '' \quad \frac{3}{25}\cos\tau + \frac{4}{25}\sin\tau$$

$$\boxed{F(\tau) = \left[\frac{22+40\tau}{25}\right]e^{-2\tau} + \frac{3}{25}\cos\tau + \frac{4}{25}\sin\tau}$$

11/ $$\text{Res}\left(\frac{(s^2-2s+3)\quad e^{s\tau}}{(s-1+i)(s-1-i)(s+3)(s+2)},\ -2\right) = \left.\frac{(s^2-2s+3)e^{s\tau}}{(s-1+i)(s-1-i)(s+3)}\right|_{s=-2}$$

$$= \frac{11\,e^{-2\tau}}{10}$$

$$\text{Res}(-3) = \left.\frac{(s^2-2s+3)\,e^{s\tau}}{(s-1+i)(s-1-i)(s+2)}\right|_{s=-3} = -\frac{18}{17}e^{-3\tau}$$

$$\text{Res}(1+i) = \left.\frac{(s^2-2s+3)\,e^{s\tau}}{(s-1+i)(s+3)(s+2)}\right|_{s=1+i} = \frac{e^{(1+i)\tau}}{2i\,(11+7i)}$$

11/ (continued)

$$\operatorname{Res}(1-i) = \left.\frac{(s^2-2s+3)e^{st}}{(s-1-i)(s+3)(s+2)}\right|_{s=1-i} = \frac{e^{(1-i)t}}{-2i\,(11-7i)}$$

$$F(t) = \tfrac{11}{10}e^{-2t} - \tfrac{18}{17}e^{-3t} + \tfrac{1}{2i}\left[\frac{e^{(1+i)t}}{11+7i} - \frac{e^{(1-i)t}}{11-7i}\right]$$

$$= \quad " \qquad " \quad + \tfrac{1}{2i}\left[\frac{(11-7i)e^{(1+i)t} - (11+7i)e^{(1-i)t}}{(11+7i)(11-7i)}\right]$$

$$= \quad " \qquad " \quad + \tfrac{1}{2i}\,\frac{2i\,\operatorname{Im}\{(11-7i)e^{(1+i)t}\}}{170}$$

$$= \quad " \qquad " \quad + \frac{e^{t}}{170}\operatorname{Im}\{(11-7i)(\cos t + i\sin t)\}$$

$$= \quad " \qquad " \quad + \frac{e^{t}}{170}\{11\sin t - 7\cos t\}$$

$$\boxed{F(t) = \tfrac{11}{10}e^{-2t} - \tfrac{18}{17}e^{-3t} + \frac{e^{t}}{170}\{11\sin t - 7\cos t\}}$$

12/ $\mathcal{L}\{X''\} - \mathcal{L}\{X\} = \mathcal{L}\{t\}$

$$s^2x(s) - X(0)s - X'(0) - x(s) = \frac{1}{s^2}$$

$$s^2x(s) - 1 - x(s) = \frac{1}{s^2}$$

$$(s^2-1)x(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$$

$$x(s) = \frac{x^2+1}{s^2(s^2-1)}$$

Problem 7 now yields $\boxed{X(t) = -t + 2\sinh t}$

13/ $\mathcal{L}\{X^{(4)}\} - \mathcal{L}\{X\} = \mathcal{L}\{t\}$

$$s^4x(s) - 1 - x(s) = \frac{1}{s^2}$$

$$(s^4-1)x(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2}$$

$$x(s) = \frac{s^2+1}{s^2(s^4-1)} = \frac{1}{s^2(s^2-1)}$$, Problem 8 now yields

$$\boxed{X(t) = -t + \sinh t}$$

14/ $\mathcal{L}\{X''\} + 4\mathcal{L}\{X\} = 2\mathcal{L}\{e^{2t}\}$

$$s^2X(s) - 1 + 4X(s) = \frac{2}{s-2}$$

$$(s^2+4)X(s) = \frac{2}{s-2} + 1 = \frac{s}{s-2}$$

$$X(s) = \frac{s}{(s^2+4)(s-2)}$$

Problem 9 now yields

$$\boxed{X(t) = \frac{1}{4}\left[e^{2t} + \sin 2t - \cos 2t\right]}$$

15/ $X''(t) + 4X'(t) + 4X(t) = \cos t$

$X(0) = 1, \quad X'(0) = 0,$

$$\mathcal{L}\{X''(t)\} + 4\mathcal{L}\{X'(t)\} + 4\mathcal{L}\{X(t)\} = \mathcal{L}\{\cos t\}$$

$$s^2X(s) - s + 4(sX(s) - 1) + 4X(s) = \frac{s}{s^2+1}$$

$$(s^2+4s+4)X(s) = \frac{s}{s^2+1} + s + 4 = \frac{s^3+4s^2+2s+4}{s^2+1}$$

$$X(s) = \frac{s^3+4s^2+2s+4}{(s+2)^2(s^2+1)}$$

?

Problem 10 now yields

$$\boxed{X(t) = \left[\frac{22+40t}{25}\right]e^{-2t}} + \frac{3}{25}\cos t + \frac{4}{25}\sin t$$

16/ $\mathcal{L}\{X''\} + 5\mathcal{L}\{X'\} + 6\mathcal{L}\{X\} = \mathcal{L}\{e^{t}\sin t\}$

$$s^2X(s) - 1 + 5(sX(s)) + 6X(s) = \frac{1}{(s-1)^2+1}$$

$$(s^2+5s+6)X(s) = \frac{1}{s^2-2s+2} + 1 = \frac{s^2-2s+3}{s^2-2s+2}$$

$$X(s) = \frac{s^2-2s+3}{(s+3)(s+2)(s^2-2s+2)}$$

16/ (Continued)

Problem 11 now yields

$$\boxed{X(t) = \frac{11}{10}e^{-2t} - \frac{18}{17}e^{-3t} + \frac{e^{t}}{170}\left[11\sin t - 7\cos t\right]}$$

17/ $\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

$$s\,u(x,t) - u(x,0) = u_{xx}(x,s)$$

$$s\,u(x,t) - 2\sin 2x = u_{xx}(x,s)$$

$$u_{xx} - s\,u = -2\sin 2x$$

Solve for u using (11) and (12) of section 8.6 to get

$$u = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{2\sin 2x}{s+4}$$

As in Example 1, $c_1 = c_2 = 0$ and thus

$u = \dfrac{2\sin 2x}{s+4}$, Inverting we get

$$\boxed{u(x,t) = 2e^{-4t}\sin 2x}$$

18/ $\mathcal{L}\{u_t\} = 4\mathcal{L}\{u_{xx}\}$

$$s\,u(x,s) - u(x,0) = 4u_{xx}(x,s)$$

$$s\,u - 2\sin\pi x - 3\sin 2\pi x = 4u_{xx}$$

$$u_{xx} - \frac{s}{4}u = -\frac{1}{2}\sin\pi x - \frac{3}{2}\sin 2\pi x$$

maybe $-\frac{3}{4}$

Solving this equation we have

$$u(x,s) = c_1 e^{\frac{\sqrt{s}}{2}x} + c_2 e^{-\frac{\sqrt{s}}{2}x} + \frac{2\sin\pi x}{s+4\pi^2} + \frac{6\sin 2\pi x}{s+16\pi^2}$$

18/ (Continued)

As in Example 1, $c_1 = c_2 = 0$. Thus we have

$$u(x,s) = \frac{2\sin \pi x}{s+4\pi^2} + \frac{6 \sin 2\pi x}{s+16\pi^2},$$

Inverting we have

$$\boxed{\mathcal{U}(x,t) = 2e^{-4\pi^2 t}\sin \pi x + 6 e^{-16\pi^2 t}\sin 2\pi x.}$$

maybe 3 ?

19/ $\mathcal{L}\{\mathcal{U}_t\} = 9\,\mathcal{L}\{\mathcal{U}_{xx}\}$

$$s\,u(x,s) - \mathcal{U}(x,0) = 9\,u_{xx}(x,s)$$

$$s\,u(x,s) - 4\cos x = 9\,u_{xx}(x,s)$$

$$u_{xx} - \frac{s}{9}u = -\frac{4}{9}\cos x$$

Solving for u we have

$$u(x,s) = c_1 e^{\frac{\sqrt{s}}{3}x} + c_2 e^{-\frac{\sqrt{s}}{3}x} + \frac{4\cos x}{s+9}$$

As in Example 2, $c_1 = c_2 = 0$ and thus

$u(x,s) = \dfrac{4\cos x}{s+9}$, Inverting we have

$$\boxed{\mathcal{U}(x,s) = 4e^{-9t}\cos x}$$

20/ $\mathcal{L}\{\mathcal{U}_t\} = \mathcal{L}\{\mathcal{U}_{xx}\}$

$$s\,u(x,s) - \mathcal{U}(x,0) = u_{xx}$$

$$s\,u(x,s) - 2\cos\frac{\pi x}{4} - 4\cos\frac{3\pi x}{4} = u_{xx}(x,s)$$

20/ (Continued)

$$u_{xx} - s u = -2 \cos \frac{\pi x}{4} - 4 \cos \frac{3\pi x}{4}$$

Solving for u we have

$$u = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{2 \cos \frac{\pi x}{4}}{s + \frac{\pi^2}{18}} + \frac{4 \cos \frac{3\pi x}{4}}{s + \frac{9\pi^2}{16}}$$

As in Example 2, $c_1 = c_2 = 0$, Thus

$$u(x,s) = \frac{2 \cos \frac{\pi}{4} x}{s + \frac{\pi^2}{16}} + \frac{4 \cos \frac{3\pi x}{4}}{s + \frac{9\pi^2}{16}}$$

Inverting we have

$$\boxed{u(x,t) = 2 e^{-\frac{\pi^2}{16} t} \cos \frac{\pi x}{4} + 4 e^{-\frac{9\pi^2 t}{16}} \cos \frac{3\pi x}{4}}$$

21/ $\mathcal{L}\{u_{tt}\} = 9\,\mathcal{L}\{u_{xx}\}$

$$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = 9\,u_{xx}(x,s)$$

$$s^2 u(x,s) - 2 s \sin x = 9\,u_{xx}(x,s)$$

$$u_{xx} - \frac{s^2}{9} u = \frac{-2s}{9} \sin x$$

This is an ordinary differential equation in x if we hold s fixed. The solution is found from (11) of section 8.6 as

$$u(x,s) = c_1 e^{\frac{s}{3} t} + c_2 e^{-\frac{s}{3} t} + \frac{2 s \sin x}{s^2 + 9}$$

21/ (continued)

We show that $c_1 = c_2 = 0$ as in Example 1. Thus

$u(x,s) = \dfrac{2s \sin x}{s^2+9}$. Since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$

we have

$$\boxed{u(x,t) = 2 \sin x \cos 3t}$$

22/ $\mathcal{L}\{u_{tt}\} = 4\mathcal{L}\{u_{xx}\}$

$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = 4\,u_{xx}(x,s)$

$s^2 u(x,s) - 3 \sin \pi x = 4\,u_{xx}(x,s)$

$u_{xx} - \dfrac{s^2}{4} u = -\dfrac{3}{4} \sin \pi x$

The solution of this ordinary differential equation is

$u(x,s) = c_1 e^{\frac{s}{2}x} + c_2 e^{-\frac{s}{2}x} + \dfrac{3 \sin \pi x}{s^2+4\pi^2}$

Again, $c_1 = c_2 = 0$ as in Example 1. Thus

$u(x,s) = \dfrac{3 \sin \pi x}{s^2+4\pi^2}$, Since $\mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$,

we have

$$\boxed{u(x,t) = \frac{3 \sin \pi x \sin 2\pi t}{2\pi}}$$

23/ $\mathcal{L}\{u_{tt}\} = \mathcal{L}\{u_{xx}\}$

$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = u_{xx}(x,s)$

$s^2 u(x,s) - s \sin \pi x - \sin 2\pi x = u_{xx}(x,s)$

23/ (Continued)

$$u_{xx} - s^2 u = -s\sin \pi x - \sin 2\pi x.$$

The solution of this ordinary differential equation is

$$u(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{s\sin \pi x}{s^2+\pi^2} + \frac{\sin 2\pi x}{s^2+4\pi^2},$$

As in Example 1, $c_1 = c_2 = 0$ and we get

$$u(x,s) = \frac{s\sin \pi x}{s^2+\pi^2} + \frac{\sin 2\pi x}{s^2+4\pi^2}.$$

Inverting we get

$$\boxed{u(x,t) = \sin \pi x \cos \pi t + \frac{\sin 2\pi x \sin 2\pi t}{2\pi}}$$

24/ $\mathcal{L}\{u_{tt}\} = 16\,\mathcal{L}\{u_{xx}\}$

$$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = 16\,u_{xx}(x,s)$$

$$s^2 u(x,s) - 2s\cos x = 16\,u_{xx}(x,s)$$

$$u_{xx} - \frac{s^2}{16}u = -\frac{s}{8}\cos x$$

The solution of this ordinary differential equation is

$$u(x,s) = c_1 e^{\frac{s}{4}x} + c_2 e^{-\frac{s}{4}x} + \frac{2s\cos x}{s^2+16}$$

We must now solve for c_1 and c_2. Set $x=0$ and get

24/ (Continued)

$0 = c_1 + c_2$. Next, differentiate $u(x,s)$ with respect to x and get

$$u_x(x,s) = c_1 \tfrac{s}{4} e^{\frac{s}{4}x} - c_2 \tfrac{s}{4} e^{-\frac{s}{4}x} - \frac{2s\sin x}{s^2+16}$$

Set $x=0$ and note that

$$u_x(0,s) = \int_0^\infty e^{-st}\, \mathcal{U}_x(0,t)\,dt = 0 ,$$

Therefore $0 = \frac{s}{4} c_1 - \frac{s}{4} c_2$, Thus $c_1 = c_2 = 0$,

and
$$u(x,s) = \frac{2s\cos x}{s^2+16} ,$$ Inverting

we have

$$\boxed{\mathcal{U}(x,t) = 2\cos x \cos 4t}$$

25/ Employ the same closed contour as was used in Section 8, 8, Consider

(1) $$\frac{1}{2\pi i}\int_{\circlearrowleft} e^{st} \frac{ds}{\sqrt{s}} = \int_{\uparrow} + \int_{F} + \int_{\rightarrow} + \int_{\circlearrowright} + \int_{\leftarrow} + \int_{\searrow}$$

Both $\int_F$ and $\int_{\searrow}$ tend to zero as R tends to infinity,

Also $\int_{\circlearrowright}$ tends to zero as $\epsilon \to 0$.

On the upper part of the branch cut, $s = re^{i\pi} = -r$,
$\sqrt{s} = \sqrt{r}\, e^{i\pi/2} = i\sqrt{r}$. Thus

$$\int_{\rightarrow} = \frac{1}{2\pi i}\int_\infty^0 e^{-rt} \frac{(-dr)}{i\sqrt{r}} = -\frac{1}{2\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr .$$

25/ (continued)

On the lower part of the branch cut we have

$s = re^{-i\pi} = -r, \quad \sqrt{s} = \sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r},$

$$\int_{\leftarrow} = \frac{1}{2\pi i}\int_0^\infty e^{-rt}\frac{(-dr)}{-i\sqrt{r}} = -\frac{1}{2\pi}\int_0^\infty e^{-rt}\frac{dr}{\sqrt{r}}$$

Since there are no singularities inside the closed contour, the left side of (1) is zero. Thus (1) now reads

$$0 = F(t) + 0 - \frac{1}{2\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr + 0 - \frac{1}{2\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr + 0$$

Thus

$$F(t) = \frac{1}{\pi}\int_0^\infty e^{-rt} r^{-1/2}\,dr. \quad \text{Set } rt = u \text{ and get}$$

$$F(t) = \frac{1}{\pi}\int_0^\infty e^{-u}\left(\frac{u}{t}\right)^{-1/2}\frac{du}{t}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-u}\,u^{1/2-1}\,du}{\sqrt{t}} = \frac{\Gamma(\frac{1}{2})}{\pi\sqrt{t}} = \boxed{\frac{1}{\sqrt{\pi t}}}$$

26/ Again we use the same contour as was used in section 8.8. Consider

$$(1)\quad \frac{1}{2\pi i}\int e^{st}e^{-\sqrt{s}}\,ds = \int_{\uparrow} + \int + \int_{\rightarrow} + \int + \int_{\leftarrow} + \int$$

Both the integrals over the two large quarter circles tend to zero as $R \to \infty$.

Set $s = \epsilon e^{i\theta}$ in the integral over the small circle to get

$$\int = \frac{1}{2\pi i}\int_{\pi}^{-\pi} e^{t\epsilon e^{i\theta}} e^{-\sqrt{\epsilon}\,e^{i\theta/2}}\, i\epsilon e^{i\theta}\,d\theta.$$

26/ (Continued)

As $\epsilon \to 0$ we see that the integral over the small circle tends to zero.

On the upper part of the branch cut we have

$s = r e^{i\pi} = -r$, and $\sqrt{s} = \sqrt{r}\, e^{i\pi/2} = i\sqrt{r}$.

Thus

$$\int_{\rightarrow} = \frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt}\, e^{-i\sqrt{r}}(-dr) = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt - i\sqrt{r}}\, dr.$$

On the lower part of the branch cut we have

$s = r e^{-i\pi} = -r$, and $\sqrt{s} = \sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r}$.

Thus

$$\int_{\leftarrow} = \frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\, e^{i\sqrt{r}}(-dr)$$

Substituting into (1) we get

$$0 = F(t) + 0 + \frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\, e^{-i\sqrt{r}}\, dr + 0$$

$$-\frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\, e^{i\sqrt{r}}\, dr + 0$$

$$F(t) = \frac{1}{\pi}\int_{0}^{\infty} e^{-rt}\left[\frac{e^{i\sqrt{r}} - e^{-i\sqrt{r}}}{2i}\right] dr$$

$$= \boxed{\frac{1}{\pi}\int_{0}^{\infty} e^{-rt} \sin\sqrt{r}\, dr}$$

Solutions to Review Problems from Chapter 8

1/ $\mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\,dt =$

$\int_0^\infty e^{-(s-a)t} F(t)\,dt = f(s-a).$

2/ $F(t) = \frac{1}{2\pi i} \int_\uparrow e^{st} \left[\frac{s^3+3s^2+s+4}{(s+1)(s+2)(s+i)(s-i)}\right] ds$

$= \text{Res}(-1) + \text{Res}(-2) + \text{Res}(i) + \text{Res}(-i)$

We have a simple pole at each residue.

$\text{Res}(-1) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s+2)(s+i)(s-i)}\right|_{s=-1} = \frac{5}{2} e^{-t}$

$\text{Res}(-2) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s+1)(s^2+1)}\right|_{s=-2} = -\frac{6}{5} e^{-2t}$

$\text{Res}(i) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s^2+3s+2)(s+i)}\right|_{s=i} = \frac{e^{it}}{-6+2i}$

$\text{Res}(-i) = \left.\frac{(s^3+3s^2+s+4)e^{st}}{(s^2+3s+2)(s-i)}\right|_{s=-i} = \frac{e^{-it}}{-6-2i}$

Adding these four residues we get

$$\boxed{F(t) = \frac{5}{2} e^{-t} - \frac{6}{5} e^{-2t} - \frac{3}{10}\cos t + \frac{1}{10}\sin t}$$

3/ $2\,X''(t) + 6X'(t) + 4X(t) = 2\sin t$

$X(0) = 1, \quad X'(0) = 0,$

$\mathcal{L}\{X''\} + 3\mathcal{L}\{X'\} + 2\mathcal{L}\{X\} = \mathcal{L}\{\sin t\}$

$$\left[s^2 x(s) - sX(0) - X'(0)\right] + 3\left[s\,x(s) - X(0)\right] + 2\,x(s) = \frac{1}{s^2+1}$$

$$s^2 x(s) - s + 3s\,x(s) - 3 + 2\,x(s) = \frac{1}{s^2+1}$$

$$(s^2+3s+2)\,x(s) = \frac{1}{s^2+1} + s + 3 = \frac{s^3+3s^2+s+4}{s^2+1}$$

$$x(s) = \frac{s^3+3s^2+s+4}{(s^2+3s+2)(s^2+1)}$$

We inverted this transform in the previous problem,

4/ $u_t = u_{xx}, \quad u(0,t) = u(1,t) = 0$ and

$u(x,0) = \sin \pi x$,

$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

$s\,u(x,s) - u(x,0) = u_{xx}(x,s)$

$s\,u(x,s) - \sin \pi x = u_{xx}(x,s)$

$u_{xx} - s\,u = -\sin \pi x$

This is an ordinary differential equation in the variable x if we hold s fixed.

$$u(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{\sin \pi x}{s+\pi^2}.$$

This solution was obtained for (11) of section 8.6,

4/ (CONTINUED)

We show that $c_1 = c_2 = 0$ in the same way as in Example 1 of section 8.6. Thus

$$u(x,s) = \frac{\sin \pi x}{s + \pi^2},$$

Since $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$ we have

$$\boxed{u(x,t) = e^{-\pi^2 t} \sin \pi x}$$

5/ We must solve $u_{tt} = u_{xx}$ subject to the boundary conditions $u(0,t) = u(2\pi, t) = 0$, and $u(x,0) = \sin \frac{x}{2}$ while $u_t(x,0) = 0$,

$$\mathcal{L}\{u_{tt}\} = \mathcal{L}\{u_{xx}\}$$

$$s^2 u(x,s) - s\,u(x,0) - u_t(x,0) = u_{xx}(x,s)$$

$$s^2 u(x,s) - s \sin \frac{x}{2} = u_{xx}(x,s)$$

$$u_{xx} - s^2 u = -s \sin \frac{x}{2}$$

The solution of this ordinary differential equation is obtained from (11) of section 8.6 as

$$u(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{s \sin(\frac{x}{2})}{s^2 + \frac{1}{4}}$$

We show that $c_1 = c_2 = 0$ as in Example 1 of section 8.7. Thus

$$u(x,s) = \frac{s \sin \frac{x}{2}}{s^2 + \frac{1}{4}}, \quad \text{Since } \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

we have $u(x,t) = \boxed{\sin \frac{x}{2} \cos \frac{t}{2}}$

6/ We use the contour of integration described in section 8.8. We have

$$(1)\qquad \frac{1}{2\pi i}\int e^{st}\frac{ds}{\sqrt[3]{s}} = \int_{\uparrow} + \int_{\rightarrow} + \int_{\leftarrow} + \underbrace{\int + \int + \int}_{\text{Three integrals over circular arcs}}$$

As $R \to \infty$ and $\epsilon \to 0$, the three integrals over circular arcs tend to zero. The integral $\int_{\uparrow}$ is the desired function $F(t)$. On the integral alove the branch cut we have

$$s = re^{i\theta} \qquad -\pi \le \theta \le \pi$$

where $\theta = \pi$. Thus $s = re^{i\pi} = -r$ and $\sqrt[3]{s} = \sqrt[3]{r}\, e^{i\pi/3}$, Therefore

$$\int_{\rightarrow} = \frac{1}{2\pi i}\int_{\infty}^{0} \frac{e^{-rt}\,(-dr)}{e^{i\pi/3}\,\sqrt[3]{r}} = \frac{e^{-\frac{i\pi}{3}}}{2\pi i}\int_{0}^{\infty} e^{-rt}\frac{dr}{\sqrt[3]{r}},$$

On the integral bellow the branch cut we have

$$s = re^{i\theta}$$

with $\theta = -\pi$, Thus $s = re^{-i\pi} = -r$ and

$\sqrt[3]{s} = \sqrt[3]{r}\, e^{-\frac{i\pi}{3}}$, Therefore

6/ (Continued)

$$\int_{\leftarrow} = \frac{1}{2\pi i}\int_0^\infty \frac{e^{-tr}\,(-dr)}{\sqrt[3]{r}\;e^{-i\frac{\pi}{3}}} = \frac{-e^{i\frac{\pi}{3}}}{2\pi i}\int_0^\infty e^{-tr}\frac{dr}{\sqrt[3]{r}},$$

By Cauchy's integral theorem, the integral on the left side of (1) is zero because there are no singularities inside the contour,

Now (1) reads

$$0 = F(t) + \left[\frac{e^{-i\frac{\pi}{3}}}{2\pi i} - \frac{e^{i\frac{\pi}{3}}}{2\pi i}\right]\int_0^\infty e^{-tr}\,r^{-\frac{1}{3}}\,dr + 0$$

$$F(t) = \frac{\sin\frac{\pi}{3}}{\pi}\int_0^\infty e^{-tr}\,r^{\frac{2}{3}-1}\,dr$$

Set $tr = u$ and get

$$F(t) = \frac{\sqrt{3}}{2\pi}\,t^{-\frac{2}{3}}\int_0^\infty e^{-u}\,u^{\frac{2}{3}-1}\,du$$

This last integral is $\Gamma(\frac{2}{3})$ (See Chapter 7).

Thus

$$\boxed{F(t) = \frac{\sqrt{3}}{2\pi}\,\frac{\Gamma(\frac{2}{3})}{t^{2/3}}}$$