AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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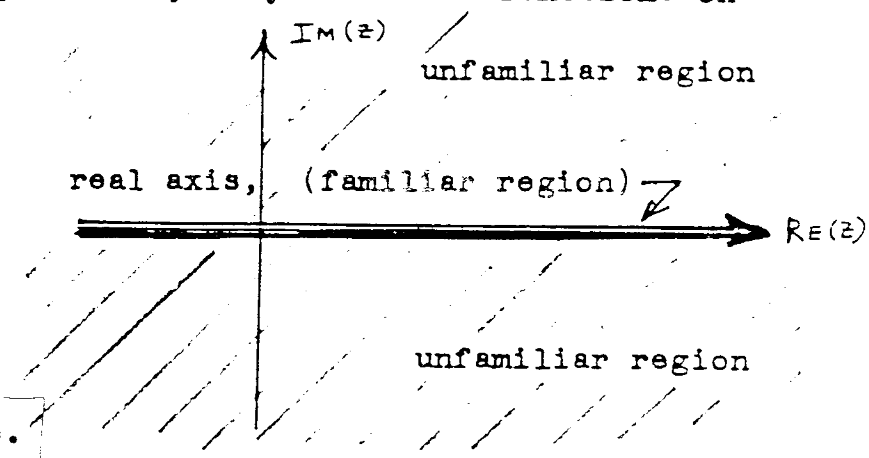
CHAPTER 2

ELEMENTARY FUNCTIONS

The reader has become familiar with many functions. They include x^2, $1/x$, e^x, $\sin x$, $\log x$, $\sinh x$, x^p, and many many more. In previous studies, the independent variable x was always real. Now we replace x by $z = x + iy$ and ask ourselves how the old familiar functions behave now that iy has been added to x. In other words, we previously only saw these functions on that narrow subset of the complex plane called the real axis, We now wish to enlarge our view of these functions by examining their behavior for all points in the complex plane.



There are, of course, many ways in which we can arbitrarily define a given function on the unfamiliar region of the complex plane. For example, we could define $\sin z = \sin(x + iy)$ to be $\sin(x)$, or $\sin x + i \sin y$, or $\sin x + iy$, etc., for all of these would become the familiar sine function when we return to the real axis by setting $y = 0$. However, we are not interested in letting our imaginations roam wildly. Definitions chosen arbitrarily are not likely to preserve many of the relations such as

$$\sin 2z = 2 \sin z \cos z,$$
$$\sin^2 z + \cos^2 z = 1,$$

etc., We seek NATURAL definitions. Natural definitions will evolve by examining formulas and expressions which reason tells us are probably correct. Only in this way will the new mathematics generated be likely to preserve features of the functions already familiar to us. Perhapse, the new mathematics will even lead to new and undxpected insights into the calculus. In the following sections we will explore ways in which natural definitions can be selected.

2.1 Functions and their graphs

In the study of the real calculus, we usually used the notation $y = f(x)$, with the understanding that x is the independent variable and y is the dependent variable. In our complex calculus, we shall most often write $w = f(z)$, where $z = x+iy$ is the independent variable, and $w = u+iv$ is the dependent variable.(x, y, u and v are all real variables.) Note that graphing $w = f(z)$ involves four real variables, and we shall discuss this difficulty momentarily.

As an example, let us examine the function $w = z^2$. It is most natural to simply replace z by $x+iy$ and square to get

$$w = (x+iy)^2 = x^2 - y^2 + 2xyi \ .$$

Since $w = u + iv$ we have

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

It is sometimes useful to use polar coordinates. Writing $z = r\,e^{i\theta}$, and $w = \rho e^{i\phi}$ we have for the function $w = z^2$

$$\rho e^{i\phi} = r^2\,e^{i2\theta}$$

and thus $\rho = r^2$ and $\phi = 2\theta$.

Problems:

1. Let $z = x+iy$ and $w = u+iv$. Find u and v as functions of x and y. (a) $w = 1/z$, (b) $w = z^2 + 2z$, (c) $w = z^{-2}$, (d) $w = z^3$.

2. Let $z = re^{i\theta}$, and $w = \rho e^{i\phi}$. Find ρ and ϕ as functions of r and θ. (a) $w = 1/z$, (b) $w = z^3$, (c) $w = \bar{z}$, (d) $w = |z|$.

How can we visualize complex valued functions of a complex variable ? The four dimensional nature of the expression $u+iv = f(x+iy)$ makes this difficult, but nevertheless, good techniques are available.

One method is to plot lines of constant u and constant v directly over the complex z- plane. Then for each value of z, we can estimate u and v and thereby approximate $w = u+iv$.

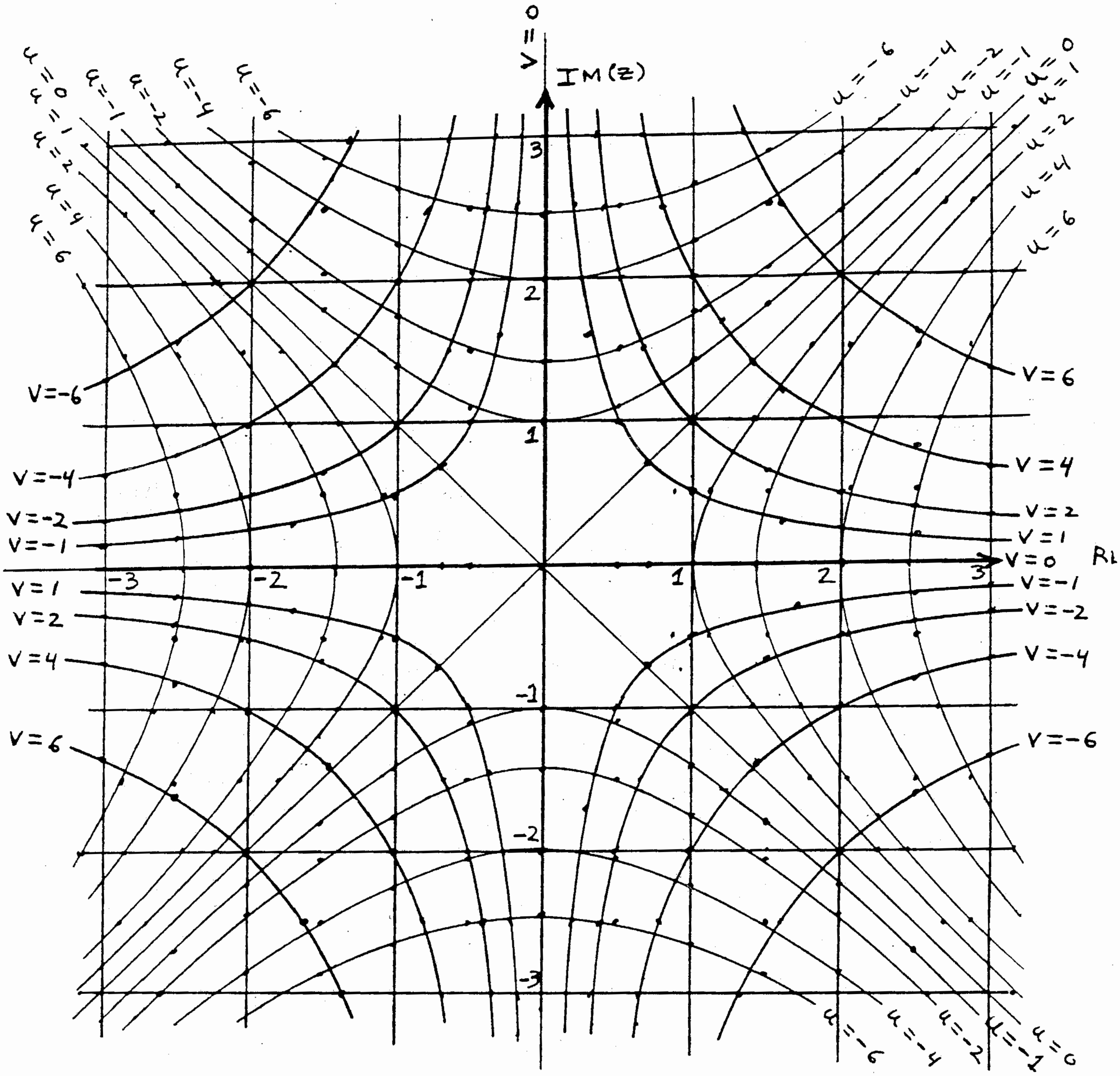
As an example, consider the function $w = z^2$. Previously we showed that $u = x^2-y^2$ and $v = 2xy$. Our previous experience in analytic geometry reveals that the curves u = constant and v = constant are hyperbolas. These "level lines" are shown in Figure 2.1 . This "contour map" of the function $w = z^2$ shows that at $z = 1+i$, $u = 0$, and $v = 2$; thus $w = 2i$.

<u>Problems:</u>

3. Estimate the value of $w = z^2$ from Figure 2.1 at the following points: (a) $z = 2-i$, (b) $z = -1+2i$, (c) $z = 1.5 -2i$, (d) $z = -i$, (e) $z = 2.25 - 1.75i$.

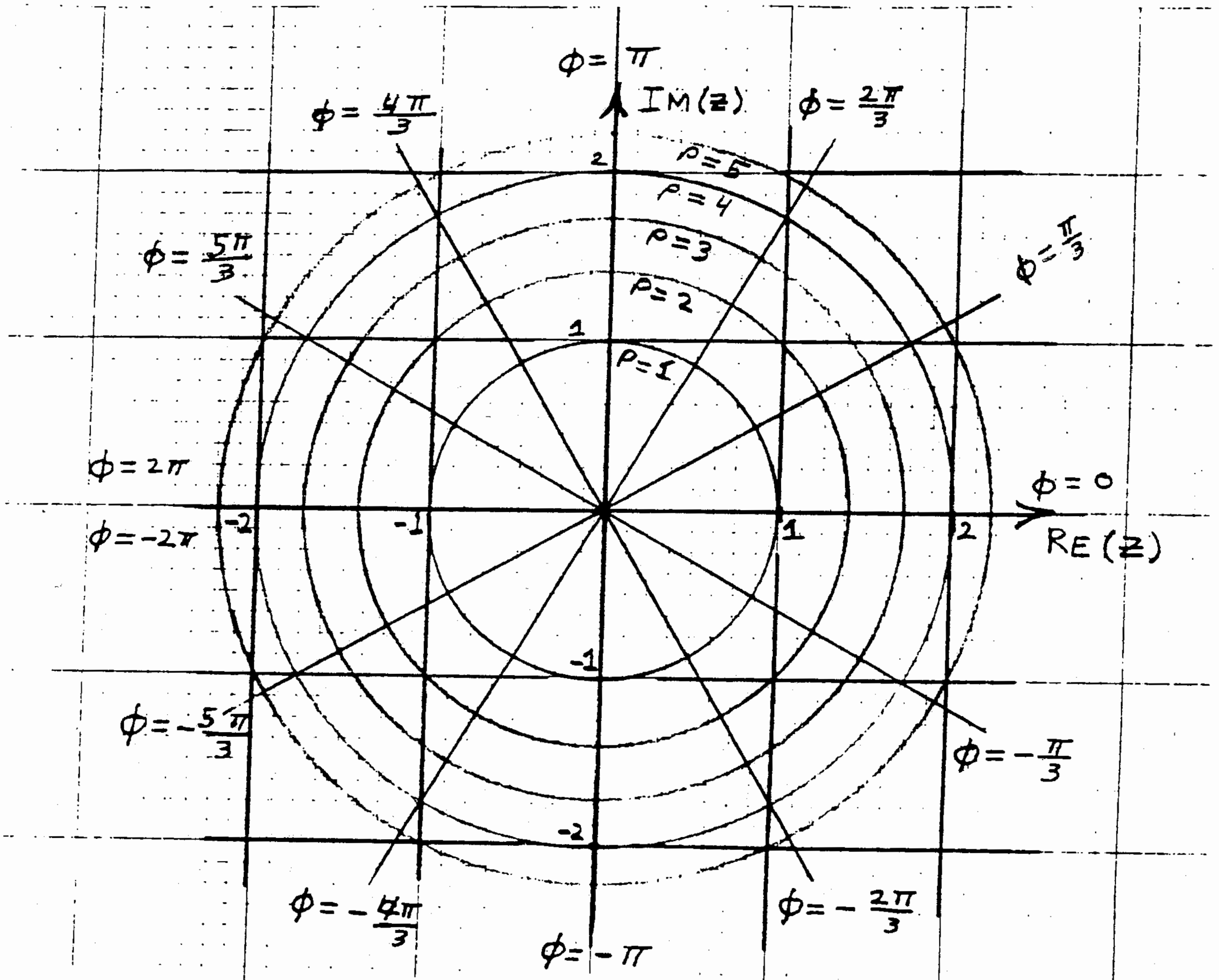
4. Estimate the values of z associated with the following values of w given by the function $w = z^2$. Use Figure 2.1 .
(a) $w = 2i$, (b) $w = -6-6i$, (c) $w = -i$, (d) $w = 2+2i$.

5. Using $u = x^2-y^2$ and $v = 2xy$, sketch the graphs of $u = -4$ and $v = 2$. Check your results by comparing them with Figure 2.1.



<u>Figure 2.1</u> The level lines u = constant and v = constant for the function $w = z^2$ plotted over the complex z plane. ($w = u + iv$, and $z = x + iy$.)

Another "contour map" for $w = z^2$ can be given in which we draw lines of constant modulus of w ($|w| = \rho$) and lines of constant argument of w ($\arg(w) = \phi$) directly over the complex z-plane. We demonstrated previously that $\rho = r^2$ and $\phi = 2\theta$. This contour map is shown in Figure 2.2.



~~**Figure 2.2** Level lines for the function $w = z^2$. $w = \rho e^{i\phi}$~~
Lines of constant modulus ρ, and lines of constant argument ϕ are plotted above the complex z-plane.

From Figure 2.2 we can also estimate the values of the function $w = z^2$. As an example, examine the point $z = -1 + 1.75\, i$. Here we see that $\rho = 4$ and $\phi = 4\pi/3$. Thus $w = 4\, e^{i4\pi/3} = -2-2\sqrt{3}\, i$. Look closely at the values of ϕ along the negative real axis. Notice that ϕ approaches the value -2π from below the axis, and 2π from above the axis. This abrupt change in ϕ does not, however cause an abrupt change in the value of $w = \rho e^{i\phi}$ since $e^{-2\pi i} = e^{2\pi i} = 1$.

Problems:

6. From Figure 2.2, estimate the values of the function $w = z^2$ at the following points: (a) $z = 2$, (b) $z = -1.1 + 2i$, (c) $z = 1.5 - 0.9i$.

7. From Figure 2.2, determine the values of z associated with each of the following values of w governed by the equation $w=z^2$. Notice that for each value of w , there are two values of ϕ, one in the range $-2\pi < \phi \leq 0$, and the other in $0 < \phi \leq 2\pi$.
(a) $w = \sqrt{3} + i$, (b) $w = 4\, e^{i2\pi/3}$, (c) $w = -1$.

8. Construct similar "contour maps" for the functions (a) $w = z^3$, and (b) $w = 1/z$, which show level lines for the modulus and the argument of w.

The level lines shown in Figure 2.2 can also be described by means of a three dimensional "relief map " shown in Figure 2.3. In this graphic visualization of the function $w=z^2$, the complex z-plane is the base plane, and the vertical altitudes are the moduli ρ of w. The lines of constant argument ϕ are then drawn on this surface. While it is more difficult to use a relief map to actually estimate values of the function, the relief map provides a vivid picture of the behavior of the function.

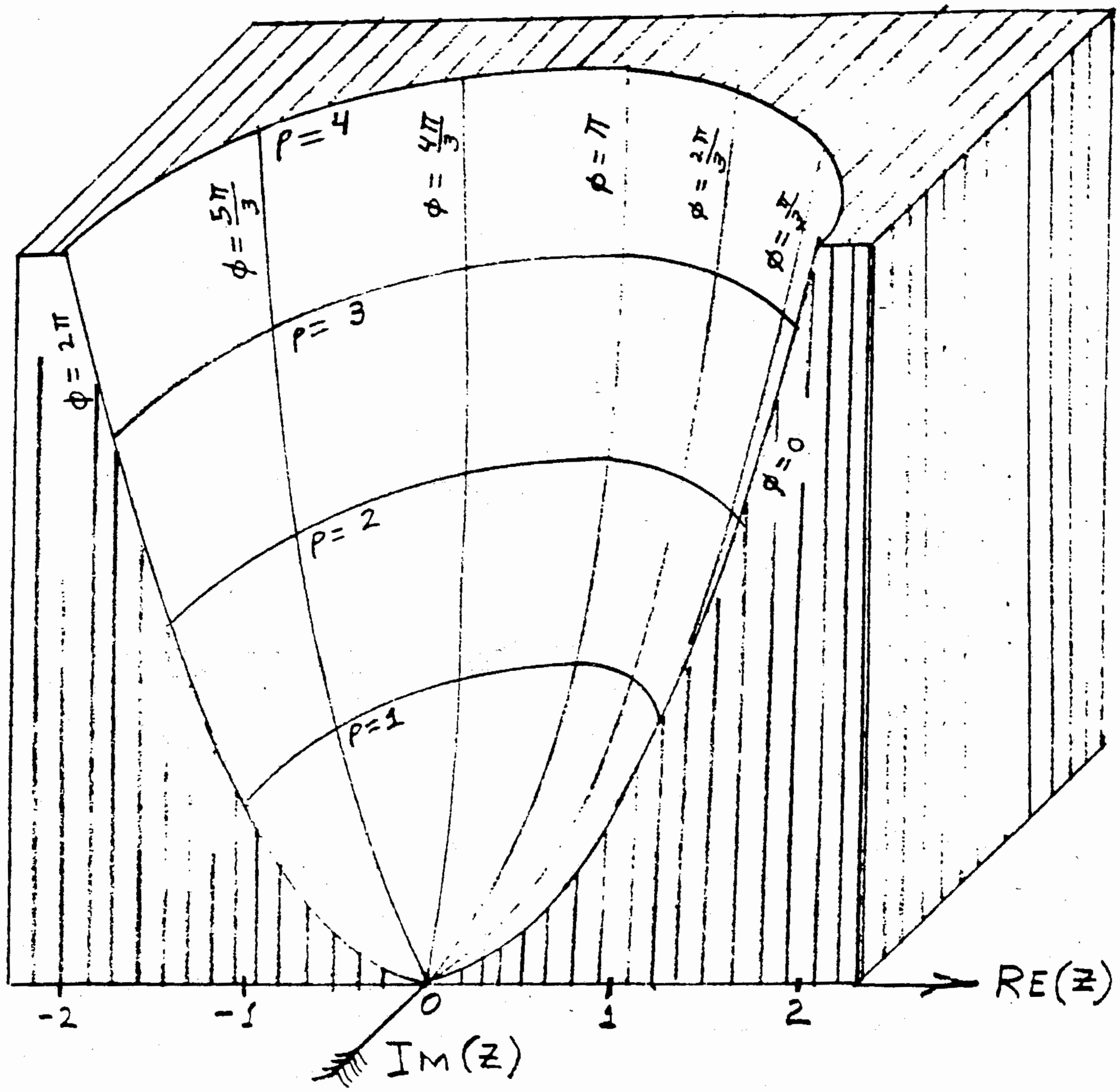


Figure 2.3 Relief map for the function $w = z^2$. The modulus of w is graphed vertically upward over the corresponding points on the complex z-plane. Lines of constant argument ϕ are also shown.

<u>Problem :</u>

9. Sketch a relief map for the function $w = 1/z$ in which the modulus of w is plotted vertically above the complex z-plane, and the lines of constant argument of w are also drawn on this surface. (See Figure 2.4 for the final relief map.)

It is not surprising that the Relief map for the function $w = 1/z$ shown in Figure 2.4 goes off to infinity at the point z=0. Such a point, where the entire relief map is urged to infinity about a particular value of z, is called a "pole" . This is a "singular point" for the function. We will have much to say about singular points later.

We have yet another method for visualizing the behavior of a complex valued function of a complex variable. In this method we reveal certain "mapping properties"of the function $w = f(z)$. That is, we shall determine pairs of regions, one region on the z-plane, and the corresponding region on the w-plane, such that the function $w = f(z)$ maps each point in the z - region onto a point in the w-region. For example, consider the function $w=z^2$. Figure 2.1 reveals that the region in the z-plane bound by the hyperbolas $x^2-y^2 = 2$, $x^2-y^2= 4$, $2xy = 4$ and $2xy = 6$, maps onto the region bound by the lines $u=2, u=4$, $v=4$, and $v=6$ in the w-plane.

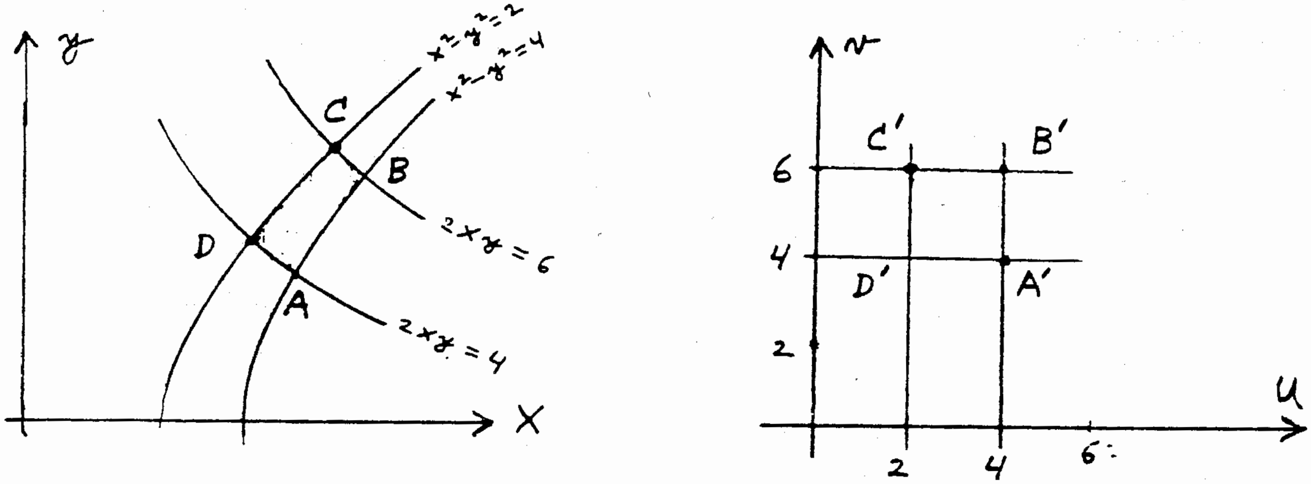


Figure 2.4 Relief map for the function $w = 1/z$ showing the modulus of w plotted vertically over the complex z-plane. Lines of constant argument of w are also plotted on this surface.