AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 7

Infinite Products, Partial Fractions and the Gamma Function

CHAPTER 7

INFINITE PRODUCTS, PARTIAL FRACTIONS, AND THE GAMMA FUNCTION

7.1 Formal infinite products

In Chapter 4 we saw that polynomials

$$(1) \quad p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_N z^N$$

suggested the use of power series

$$(2) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

in which there are infinitely many terms. We saw that the familiar algebraic manipulations, as well as differentiation and integration, could be applied to the infinite series in almost the same way that they applied to the polynomial. However, there was one difficulty, the power series (1) might not make sense for certain values of z whereas the polynomial can be evaluated for any value of z. The series (2) required an examination for convergence.

Now polynomials (1) can also be written as products. If a polynomial has roots at

$z = r_1$ of multiplicity m_1
$z = r_2$ " " m_2
. . .
$z = r_N$ " " m_N

we can write

$$(3) \quad p(z) = C\,(z - r_1)^{m_1} (z - r_2)^{m_2} \dots (z - r_N)^{m_N}$$

where C is some constant. The constant C can be computed if we know the value of $p(z)$ for some specific value of z. In particular, if we know $p(z)$ at the origin, $p(0)$, then we can write (3) in the slightly different form

$$(4) \quad p(z) = p(0)\left(1 - \frac{z}{r_1}\right)^{m_1}\left(1 - \frac{z}{r_2}\right)^{m_2} \cdots \left(1 - \frac{z}{r_N}\right)^{m_N} .$$

Set $z=0$ in (4) and observe that the formula is correct.

Now consider a function $f(z)$ with infinitely many zeros at the points $r_1, r_2, r_3, \ldots$, with corresponding multiplicities $m_1, m_2, m_3, \ldots$. It now seems reasonable to generalize (4) to the infinitely many factors as

$$(5) \quad f(z) = g(z)\left(1 - \frac{z}{r_1}\right)^{m_1}\left(1 - \frac{z}{r_2}\right)^{m_2}\left(1 - \frac{z}{r_3}\right)^{m_3} \cdots$$

$$= g(z)\prod_{n=1}^{\infty}\left(1 - \frac{z}{r_n}\right)^{m_n} ,$$

where $g(z)$ is some function without any zeros. Here the symbol $\prod$ is used for products just as $\sum$ is used for sums. We call (5) an infinite product. We have generalized the form (4) in preference to (3) because at least when $z = 0$, the product in (5) has one times one times one, etc., which clearly makes sense; whereas the product in the generalization of (3) might not converge. Notice also that the constant $p(0)$ in (4) is now replaced by an arbitrary function having no zeros, $g(z)$, in (5).

Example 1

Find an infinite product representation for $\sin z$.

Solution

We know that $\sin z$ has simple zeros (multiplicity 1) at

$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$. Thus, following (5), it seems reasonable to write

$$(6) \qquad \sin z = g(z) \left[\ldots \left(1 + \frac{z}{2\pi}\right)\left(1 + \frac{z}{\pi}\right)(\ z\ )\left(1 - \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\ldots \right]$$

$$= g(z)\, z \left[\prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) \right] \left[\prod_{n=1}^{\infty} \left(1 + \frac{z}{n\pi}\right) \right]$$

This product is infinite in both the right and the left directions. We will see in the next section that (6) is unsatisfactory because it fails to converge.

However, if we combine the factors in pairs

$$\ldots \underbrace{\left(1 + \frac{z}{2\pi}\right) \underbrace{\left(1 + \frac{z}{\pi}\right) z \left(1 - \frac{z}{\pi}\right)} \left(1 - \frac{z}{2\pi}\right)} \ldots$$

we get, (since $(1+a)(1-a) = 1 - a^2$),

$$(7) \qquad \sin z = g(z)\, z \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right)\left(1 - \frac{z^2}{3^2\pi^2}\right) \ldots$$

$$= g(z) \quad z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2} \right)$$

Oddly, the product (7) does converge, while (6) does not.

Now we must try to find $g(z)$. When z is very small, $\sin z \approx z$ and $\left(1 - \frac{z^2}{n^2\pi^2}\right) \approx 1$. Thus for small z (7) becomes

$$z \approx g(0)\, z$$

Thus $g(0) = 1$. The function $\sin z$ is fundamental in analysis. Our experience shows that formulas for natural functions like $\sin z$ are often quite simple and beautiful in appearance. Perhaps, then

$$(8) \qquad g(z) \equiv g(0) = 1 .$$

This is, in fact, correct. We will not prove (8) however. Thus

$$\boxed{(9) \qquad \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)}$$

Example 2

Evaluate the product $\prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)$.

Solution

Set $z = \pi/2$ in (9) and see at once that the product equals $2/\pi$.

Example 3

Find an infinite product for the function $e^{2zi} - 1$.

Solution

We can manipulate $e^{2zi} - 1$ into the exponential form for the sine function $\sin z = (e^{iz} - e^{-iz})/2i$.

$$f(z) = e^{i2z} - 1 = e^{iz}(e^{iz} - e^{-iz})$$

$$= 2i\, e^{iz}(e^{iz} - e^{-iz})/2i$$

$$= 2i\, e^{iz} \sin z .$$

Substituting (9) for $\sin z$ we have

$$f(z) = 2i\, e^{iz}\, z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) .$$

Example 4

Find an infinite product representation for $\sin \pi z$.

First Solution

The function $\sin \pi z$ has simple zeros at $z = 0, \pm 1, \pm 2, \ldots,$ Therefore we expect

$$\sin \pi z = g(z)\, z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) .$$

When z is very small, $\sin \pi z \approx \pi z$ and $1 - \frac{z^2}{n^2} \approx 1$. Thus

we have $\pi z \approx g(0)\, z$. Thus $g(0) = \pi$, and as before we strongly suspect that the function $g(z)$ is identically $g(0) = \pi$.

We then expect

$$(10) \qquad \sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) .$$

Second Solution

Replace z in (9) by πz and get (10) at once .

Problems

1. Find infinite product representations for the following functions:

(a) $\sin 2z$, (b) $\sin 3z$, (c) $\sinh z$, (d) $\sinh 2z$

(e) $\sinh \pi z$, (f) $e^{2z} - 1$, (g) $(e^{z} - 1)/z$, (h) $e^{az} - e^{-bz}$

2. Find infinite product representations for

(a) $\cos z$, (b) $\cosh z$, (c) $e^{i2\pi z} + 1$, (d) $z \cos \pi z$.

(e) $e^{az} + e^{-bz}$.

3. Evaluate the following infinite products.

(a) $(1 + 1^{-2})(1 + 2^{-2})(1 + 3^{-2})(1 + 4^{-2}) \ldots$

(b) $(1 + 1^{-2})(1 + 3^{-2})(1 + 5^{-2})(1 + 7^{-2}) \ldots$

(c) $(1 - 4/1^{2})(1 - 4/3^{2})(1 - 4/5^{2})(1 - 4/7^{2}) \ldots$

Conjecture 1

Looking back over the previous examples and problems from this section, we see that all the functions $f(z)$ expanded in infinite products of the form (5) were entire functions . (Recall that an entire function is an analytic function having no singularities in the finite plane.) What type of function is $g(z)$ in (5) when $f(z)$ is entire ?

Conjecture 2

The function e^{z} is entire and has no zeros. Conjecture the general form of the function $g(z)$ in (5) when $f(z)$ is entire.

7.2 The convergence of infinite products

We will now, intuitively, investigate the convergence of infinite products. The rigorous theory of convergence of infinite products, like that of infinite series, requires extensive and careful thought. However, as before, we will see that certain useful aspects of the theory can be conjectured without great effort.

In the theory of infinite series, it is quite evident that for the series $\sum_{n=0}^{\infty} u_n$, the term u_n should approach zero as n tends to infinity. If this were not so, the series would not converge to a finite value. This is called the "n^{th} term test," and it gives only a necessary condition for convergence.

Conjecture 3

Conjecture a necessary condition similar to the above "n^{th} term test" for the product

$$\prod_{n=1}^{\infty}(1 + u_n)$$

to converge.

Example 1

What does the "n^{th} term test" for products tell us about

(a) $\prod_{n=1}^{\infty}(1+n)$, (b) $\prod_{n=1}^{\infty}(1 + \log n)$, (c) $\prod_{n=1}^{\infty}(1 + n^{-1})$.

Solution

Both (a) and (b) diverge because the terms n and $\log n$ do not tend to zero as n grows large. The test tells us nothing about the convergence of (c) since n^{-1} does approach zero. From this test, (c) might converge or diverge.

Now consider the infinite product

$$(1) \qquad P = \prod_{n=0}^{\infty} (1 + u_n) \quad .$$

Since we have some familiarity with the convergence properties of infinite series, it would be very useful to convert (1) into a series. This is achieved by taking logarithms.

$$\log P = \log \prod_{n=0}^{\infty} (1 + u_n)$$

$$(2) \qquad = \sum_{n=0}^{\infty} \log(1 + u_n) \quad .$$

This series suggests that the infinite product (1) converges if and only if the infinite series (2) converges. This is in fact true, but the series (2) is awkward an often difficult to test. Can we replace (2) by a more convenient series ? Recall that the power series for the logarithm is

$$\log(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^5}{5} + \dots$$

and since the u_n are very small for large n in a convergent product, it seems reasonable to make the approximation

$$\log(1 + u) \approx u$$

in (2) to get

$$(3) \qquad \sum_{n=0}^{\infty} u_n \quad .$$

We now suspect that the infinite product (1) converges if and only if the infinite series (3) converges. This is almost true, but not quite. The following Theorem gives the true facts.

Theorem 1

The infinite product $\prod_{n=0}^{\infty} (1 + u_n)$ converges if

$$\sum_{n=0}^{\infty} |u_n| \tag{4}$$

converges. If

$$\sum_{n=0}^{\infty} u_n \tag{5}$$

diverges, the infinite product diverges.

We will need the following result from the elementary theory of infinite series when using the above theorem.

Theorem 2

The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Remark

If one of the terms $u_n = -1$, then one factor of the infinite product is zero and thus the entire product should converge to zero regardless of the size of the remaining factors. This case, where one factor of the product is zero, is therefore not applicable for the above theory.

Example 2

Test the following products for convergence.

(a) $\prod_{n=1}^{\infty}(1+n^{-1})$, (b) $\prod_{n=1}^{\infty}(1+n^{-2})$, (c) $\prod_{n=3}^{\infty}(1+n^{-1/2})$

(d) $\prod_{n=2}^{\infty}(1+(-1)^n/n)$.

Solution

The products (a) and (c) diverge by (5) since $\sum n^{-1}$ and $\sum n^{-1/2}$ are divergent series (see Theorem 2).

The product (b) converges because $\sum |n^{-2}| = \sum n^{-2}$ converges.

Theorem 1 fails to give us any information concerning the product (d) since $\sum |(-1)^n/n| = \sum 1/n$ diverges, yet $\sum (-1)^n/n$ converges.

Problem

4. Test the following products for convergence.

(a) $\prod_{n=1}^{\infty}(1+n^{2})$, (b) $\prod_{n=1}^{\infty}(1+n^{-3})$, (c) $\prod_{n=1}^{\infty}(1+e^{-n})$,

(d) $\prod_{n=1}^{\infty}(1+e^{n})$, (e) $\prod_{n=2}^{\infty}(1+1/\log n)$, (f) $\prod_{n=1}^{\infty}(1+n^{-n})$.

In the previous section, we obtained two infinite products for the function sin z . We now test them for convergence.

Example 3

Test the two products

$$(6) \quad \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right), \quad \text{and}$$

$$(7) \quad = z \left[\prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) \right] \left[\prod_{n=1}^{\infty} \left(1 + \frac{z}{\pi n}\right) \right]$$

for convergence and determine the region in the z plane in which they are valid.

Solution

The product in (6) converges for all z since

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| \frac{-z^2}{\pi^2 n^2} \right| = \frac{|z^2|}{\pi^2} \sum_{n=1}^{\infty} n^{-2}$$

converges for all z.

Neither of the products in (7) converge because

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{z}{\pi n} = \frac{z}{\pi} \sum_{n=1} n^{-1}$$

diverges for all z.

Problem

5. Test all the products obtained in problems 1 and 2 and determine the region in the z-plane in which they converge.

7.3 Making divergent products converge

Divergent products like

(1) $$\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)$$

can be altered so that they will converge. This is done by the insertion of appropriately selected exponential convergence factors. The altered product

(2) $$\prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{n}\right) e^{z/n} \right]$$

does converge, and clearly has the same zeros as (1).

We now show how the convergence factor $e^{z/n}$ is selected.

Example 1

Find appropriate exponential convergence factors to make the product (1) converge.

Solution

Generally, the exponential convergence factor will have the form

$$e^{az + bz^2 + cz^3 + \cdots} = e^{az}\, e^{bz^2}\, e^{cz^3} \cdots .$$

We begin by trying the simple factor e^{az}. For very small a we have

$$e^{az} = 1 + az + \frac{(az)^2}{2!} + \frac{(az)^3}{3!} + \cdots$$

$$\approx 1 + az + \frac{a^2 z^2}{2}$$

Thus for small a, the product $\left(1 - \frac{z}{n}\right) e^{az}$ behaves like

$$\begin{array}{l}
1 + az + \frac{a^2z^2}{2} \\
1 - \frac{z}{n} \\
\hline
1 + az + \frac{a^2z^2}{2} \\
\quad - \frac{z}{n} - \frac{a\,z^2}{n} + \dots \\
\hline
1 + (a - \frac{1}{n})\, z + (\frac{a^2}{2} - \frac{a}{n})\, z^2 + \dots
\end{array}$$

If we select $a = \frac{1}{n}$, the coefficient of z becomes zero and we get

$$(1 - \frac{z}{n})\, e^{z/n} = 1 - \frac{z^2}{2n^2} + \dots$$

This is fine, because the product

$$\prod_{n=1}^{\infty} (1 - \frac{z^2}{2n^2})$$

is convergent and behaves like

$$(3) \qquad \prod_{n=1}^{\infty} [\, (1 - \frac{z}{n})\, e^{z/n} \,]$$

for large n. Thus we have found the desired convergence factors.

Looking back over the previous example, we see that it is based on the following ideas:

1. The product $\prod (1 - z/n)$ diverges because the term z/n does not tend to zero with sufficient speed as n tends to infinity. We need terms like n^{-2} or n^{-3} or $n^{-3/2}$ etc..

2. We think always of the simple algebraic identity

$$(1+u)(1-u) = 1 - u^2 .$$

We can multiply the slow factor $(1 - z/n)$ by $(1 + z/n)$ to get a fast factor $1 - z^2/n^2$. However we cannot simply use $1 + z/n$ as a convergence factor since it introduces new unwanted zeros.

3. Rather than use $1 + z/n$ as our convergence factor, we use $e^{z/n} = 1 + z/n + \ldots$ which approximates the factor $1 + z/n$ for large n and has no zeros.

Example 2

Introduce convergence factors for the product

$$\prod_{n=1}^{\infty} (1 + \frac{z}{n}) .$$

Solution

We see at once that we should try $e^{-z/n} = 1 - \frac{z}{n} + \ldots$ since this factor times $1 + z/n$ will kill the unwanted $1/n$ type terms. We have

$$\begin{array}{lllll} 1 & - z/n & + z^2/2n^2 & + \ldots \\ 1 & + z/n & & \\ \hline 1 & - z/n & + z^2/2n^2 & + \ldots \\ & + z/n & - z^2/n^2 & + \ldots \\ \hline 1 & & - z^2/2n^2 & + \ldots \end{array}$$

Thus we see that for large n,

$$(1 + z/n)\, e^{-z/n} \approx 1 - z^2/2n^2$$

and therefore

$$\prod_{n=1}^{\infty} (1 + z/n)\, e^{-z/n} \tag{4}$$

is a convergent product.

We can now combine (3) and (4) to get an entire function having a simple zero at each integer

$$(5) \qquad f(z) = z\left[\prod_{n=1}^{\infty}(1 - z/n)e^{z/n}\right]\left[\prod_{n=1}^{\infty}(1 + z/n)e^{-z/n}\right].$$

What is this function $f(z)$? Multiplying the corresponding factors in each product together we get

$$(1 - z/n)e^{z/n}(1 + z/n)e^{-z/n} = 1 - z^2/n^2 .$$

But the product

$$z\prod_{n=1}^{\infty}(1 - z^2/n^2)$$

was obtained in equation (10) of section 7.1 . as

$$\frac{\sin \pi z}{\pi} = f(z) .$$

We see that (5) gives us another infinite product for the sine function.

Example 3

Find the most general entire function having zeros at $z = n^{1/2}$, $n = 1, 2, 3, 4, \ldots$, of multiplicity two.

Solution

We immediately try the product

$$(6) \qquad \prod_{n=1}^{\infty}\left(1 - \frac{z}{n^{1/2}}\right)^2 ,$$

but it fails to converge because the series $\sum n^{-1/2}$ diverges. Thus we try as convergence factor

$$(7) \qquad e^{z/n^{1/2}} = 1 + z/n^{1/2} + z^2/2n + \ldots$$

Multiplying $(1 - z/n^{1/2})$ times the series for $e^{z/n^{1/2}}$ we get

$$\begin{array}{cccccccc}
1 & + & z/n^{1/2} & + & z^2/2n & + & \ldots \\
1 & - & z/n^{1/2} & & & & \\
\hline
1 & + & z/n^{1/2} & + & z^2/2n & + & \ldots \\
 & - & z/n^{1/2} & - & z^2/n & - & \ldots \\
\hline
1 & & & - & z^2/2n & + & \ldots
\end{array}$$

Thus we see that the factor $(1 - z/n^{1/2})\ e^{z/n^{1/2}}$ is also unsuitable since it behaves like $1 - z^2/2n$ for large n.

It remains to remove the $1/n$ type term from

$$(8) \quad (1 - z/n^{1/2})\ e^{z/n^{1/2}} \approx (1 - z^2/2n)$$

by means of yet another exponential convergence factor. We see that a factor of the type $e^{z^2/2n} = 1 + z^2/2n + \ldots.$ will remove the undesirable $1/n$ type term from the right side of (8) and replace it by a term which tends to zero faster than $1/n$.

Therefore the proper factors are

$$(1 - z/n^{1/2})\ e^{z/n^{1/2}} \quad e^{z^2/2n}$$

and the most general entire function having the required zeros is

$$e^{h(z)} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\sqrt{n}}\right) \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{2n} \right) \right]^2 \tag{9}$$

where $h(z)$ is an arbitrary entire function.

Problems

6. Find the most general entire function having zeros at $z = n^2$, where $n = 1, 2, 3, 4, \ldots$, of multiplicity two.

7. Find the most general entire function having zeros at $z = in$, where $n = 0, 1, 2, 3, \ldots$, of multiplicity three.

8. Find the most general entire function having zeros at $z = n^{2/3}$, where $n = 1, 2, 3, \ldots$, of multiplicity two.

9. Find the most general entire function having zeros at $z = n^{1/3}$, where $n = 1, 2, 3, \ldots$, of multiplicity one.

10. Obtain an infinite product expansion for $\cos z$ of the type (5).

Conjecture 4. (Weierstrass' factor theorem)

Describe the infinite product representation of the most general entire function having zeros at the points r_n of multiplicity m_n, $n = 1, 2, 3, \ldots$.

7.4 Partial fraction expansions

We begin by deriving a useful formula involving the derivative of the infinite product of functions

$$(1) \qquad P(z) = \prod_{n=1}^{\infty} f_n(z) = f_1(z)\ f_2(z)\ f_3(z) \dots .$$

If $P(z)$ is only the product of two functions

$$P = f_1 f_2$$

then the formula for the derivative of a product gives

$$P' = f_1' f_2 + f_1 f_2' .$$

Now divide by P and get

$$\frac{P'}{P} = \frac{f_1' f_2}{f_1 f_2} + \frac{f_1 f_2'}{f_1 f_2}$$

$$= \frac{f_1'}{f_1} + \frac{f_2'}{f_2} .$$

Suppose next that $P(z)$ is the product of three functions

$$P = f_1 f_2 f_3 .$$

Differentiating gives

$$P' = f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3' .$$

Dividing by P as before gives

$$\frac{P'}{P} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3} .$$

It now seems reasonable that for the infinite product (1) we should have

$$\boxed{(2) \qquad \frac{P'}{P} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n} \qquad \text{where} \quad P(z) = \prod_{n=1}^{\infty} f_n(z)}$$

Example 1

Apply (2) to the product $P(z) = \dfrac{\sin \pi z}{\pi} = z \displaystyle\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$.

Solution

Substituting directly into (2) we have

$$\frac{\cos \pi z}{\frac{\sin \pi z}{\pi}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}}$$

which simplifies to

$$(3) \qquad \pi \cot \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} .$$

Thus we have obtained a partial fractions type expansion of the cotangent function.

Example 2

Apply (2) to the product

$$P(z) = \frac{\sin \pi z}{\pi} = z \prod_{n=-\infty}^{\infty}{}' \left[\left(1 - \frac{z}{n}\right) e^{z/n} \right] .$$

Note: A prime on a product or summation symbol means that the term involving n=0 is to be deleted.

Solution

Direct substitution into (2) gives

$$\frac{\cos \pi z}{\frac{\sin \pi z}{\pi}} = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{-\frac{1}{n}}{1 - \frac{z}{n}} + \frac{\frac{1}{n}\ e^{z/n}}{e^{z/n}} \right] ,$$

which simplifies to

$$(4) \qquad \pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z - n} + \frac{1}{n} \right] .$$

Notice that both (3) and (4) give partial fractions type expansions of $\pi \cot \pi z$. The series (3) converges because the terms behave like n^{-2} as n nears infinity. If the term $\frac{1}{n}$ were missing from (4), the series would diverge because it would behave like $\sum 1/n$ for large n . However, with the $\frac{1}{n}$ term present we have

$$\frac{1}{z - n} + \frac{1}{n} = \frac{z}{n\,(z - n)}$$

which behaves like n^{-2} for large n and thus (4) converges.

Problem

11. Apply formula (2) to the infinite products obtained previously as solutions to the following problems:

(a) Problem 1(e) ; (b) Problem 2(a) ; (c) Problem 2(b) ;
(d) Problem 10 .

An analytic function whose only singularities are poles (with the exception of the singularity at infinity, which might be essential) is called meromorphic . All the functions expanded in partial fractions in this section were meromorphic. Can any meromorphic function be expanded in a series of partial fractions ? What will be the form of such a series ? These are questions which will be answered after the study of a few more examples and problems.

Example 3

Let $f(z)$ be a meromorphic function having simple poles at the points $z = n$, where $n = 1, 2, 3, \ldots$. Near the point $z = n$, we know that $f(z)$ can be expanded in a convergent Laurent series. Suppose near $z = n$ we have

$$f(z) = \frac{3}{z-n} + a_0 + a_1(z-n) + a_2(z-n)^2 + \ldots .$$

We call the term $\frac{3}{z-n}$ the principal part of $f(z)$ at the singularity $z = n$.

Find the most general partial fractions expansion of $f(z)$.

Solution

We naturally try as a first series

$$(5) \qquad f(z) \overset{?}{=} \sum_{n=1}^{\infty} \frac{3}{z-n} .$$

A moments reflection tells us that (5) diverges because for large n it behaves like the divergent series $\sum n^{-1}$. For large n, the terms of the series (5) behave precisely like

$$\frac{1}{z-n} = -\frac{1}{n}\frac{1}{(1-\frac{z}{n})} = -\frac{1}{n}\left[1+\frac{z}{n}+\frac{z^2}{n^2}+\dots\right]$$

$$= -\frac{1}{n} - \frac{z}{n^2} - \dots$$

Thus we see that

$$\frac{1}{z-n} + \frac{1}{n} \approx \frac{z}{n^2}$$

should replace $\frac{1}{z-n}$ in (5) since it behaves like n^{-2} which is the general term of a convergent series. Thus the series

$$(6) \qquad f(z) \overset{?}{=} 3\sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right]$$

is convergent.

Is (6) the most general form for our meromorphic function ? No . We could add any entire function $g(z)$ to the right side of (6) without effecting the principal parts of $f(z)$ at its singularities. Thus the most general form seems to be

$$(7) \qquad f(z) = g(z) + 3\sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right],$$ where $g(z)$ is some entire function.

Example 4

Find the most general meromorphic function having poles at the points $z = n^{1/2}$, where $n = 1, 2, 3, \ldots$ and having principal parts

$$\frac{1}{(z - n^{1/2})^3} + \frac{5}{z - n^{1/2}}$$

at these singularities.

Solution

We try at first

$$(8) \quad f(z) = g(z) + \sum_{n=1}^{\infty} \left[\frac{1}{(z - n^{1/2})^3} + \frac{5}{z - n^{1/2}} \right]$$

The first term on the right side of (8) needs no alteration because it behaves like $n^{-3/2}$ for large n which is the general term of a convergent series. However, the second term behaves like $n^{-1/2}$ which is the genral term of a divergent series. Therefore this second term must be altered. Expanding this term in a power series we have

$$\frac{5}{z - n^{1/2}} = -\frac{5}{n^{1/2}} \frac{1}{\left(1 - \frac{z}{n^{1/2}}\right)} = -\frac{5}{n^{1/2}} \left[1 + \frac{z}{n^{1/2}} + \frac{z^2}{n} + \ldots \right.$$

$$= -\frac{5}{n^{1/2}} - \frac{5z}{n} - \frac{5z^2}{n^{3/2}} - \ldots .$$

The terms involving $n^{-1/2}$ and n^{-1} create divergence, but the term involving $n^{-3/2}$ and all following terms create convergence. Moving the divergent terms to the left side we get

$$\frac{5}{z - n^{1/2}} + \frac{5}{n^{1/2}} + \frac{5z}{n} \approx - \frac{5z^2}{n^{3/2}}$$

and since this term on the left acts like $n^{-3/2}$, it is suitable as a general term in an infinite series. Thus we have

$$f(z) = g(z) + 5\sum_{n=1}^{\infty}\left[\frac{1}{(z - n^{1/2})^3} + \frac{1}{z - n^{1/2}} + \frac{1}{n^{1/2}} + \frac{z}{n}\right]$$

where $g(z)$ is an arbitrary entire function.

Problems

12. Express the most general meromorphic function with poles of the form

$$\frac{1}{(z-n)^2}$$

at the points $n = 0, \pm 1, \pm 2, \ldots$, in the partial fractions expansion.

13. Express the most general meromorphic function with poles of the form

$$\frac{(-1)^n}{z - n},$$

where z is any integer n, in a partial fractions expansion.

14. Conjecture the partial fractions expansion for $\csc \pi z$. (Hint: Look at problem 13.)

15. A meromorphic function has simple poles at the points $z = n^{1/3}$, where n is a positive integer. Each pole has residue 4. Find the most general partial fractions expansion.

<u>Conjecture 5</u> <u>Mittag - Leffler's partial fractions theorem</u>

Suppose a meromorphic function is given having poles at the points r_n, where n is a positive integer. Suppose that the principal part of the function at each pole r_n is described as

$$\sum_{k=1}^{m_n} \frac{a_{n,k}}{(z - r_n)^k} .$$

Conjecture the form of the partial fractions expansion of this meromorphic function.

7.5 A quick look at the gamma function

The so called gamma function, denoted by the symbol

$$w = \Gamma(z) \, ,$$

is important in mathematics and its applications because it provides a natural extension of the concept of "factorial". We know that $3! = 1\cdot2\cdot3 = 6$, and that $4! = 1\cdot2\cdot3\cdot4 = 24$, but what is $(1/2)!$? The Swiss mathematician, Leonhard Euler (1707-1783), invented the gamma function in an attempt to provide a natural answer to this question. In fact,

$$\Gamma(z+1) = z! \tag{1}$$

so that for integer values of z we have

$$\begin{aligned} \Gamma(1) &= 0! = 1 \\ \Gamma(2) &= 1! = 1 \\ \Gamma(3) &= 2! = 2 \\ \Gamma(4) &= 3! = 6 \\ \Gamma(5) &= 4! = 24 \\ &\cdots \end{aligned}$$

The simplest way to gain information about something new is to look at it. Therefore, before discussing the mathematical theory of the gamma function, we will take a quick look at three graphic representations of the function.

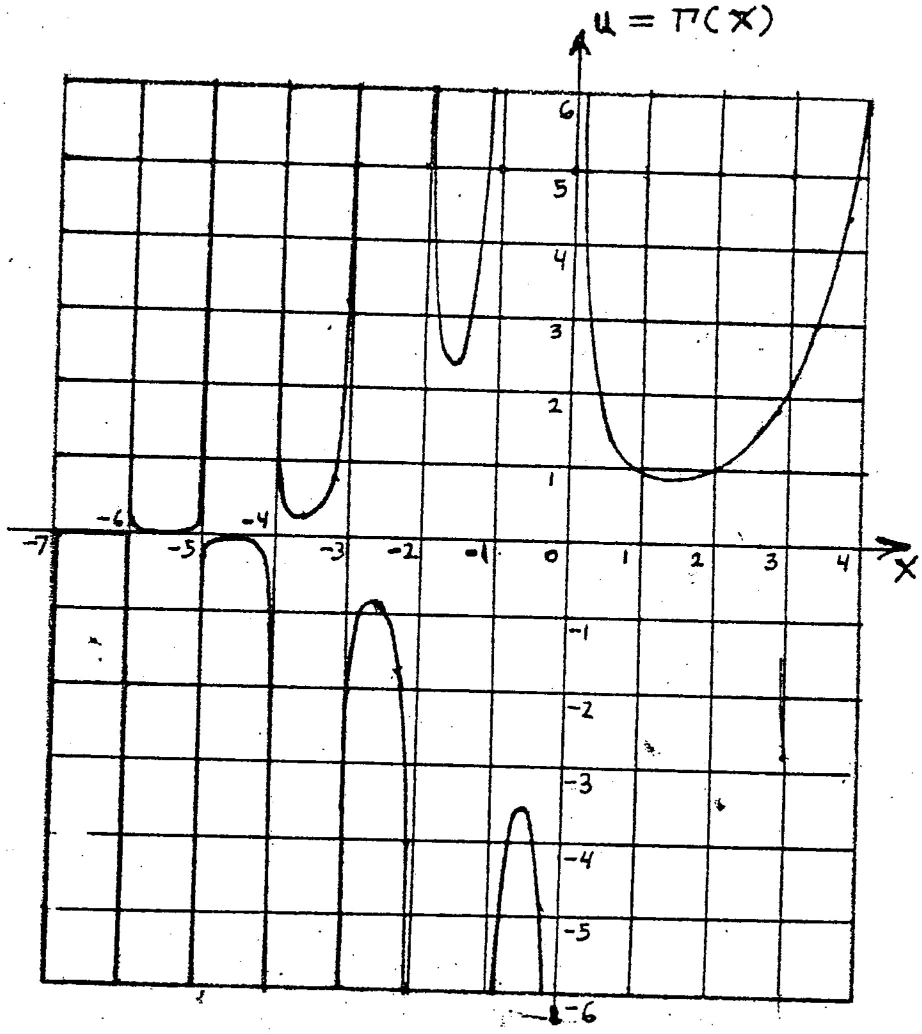
Figure 1 shows the graph of $\Gamma(x)$ for real numbers x between -7 and $+4$. From this graph we can roughly estimate $(1/2)!$ as

$$(1/2)! = \Gamma(3/2) \approx 0.8 \, .$$

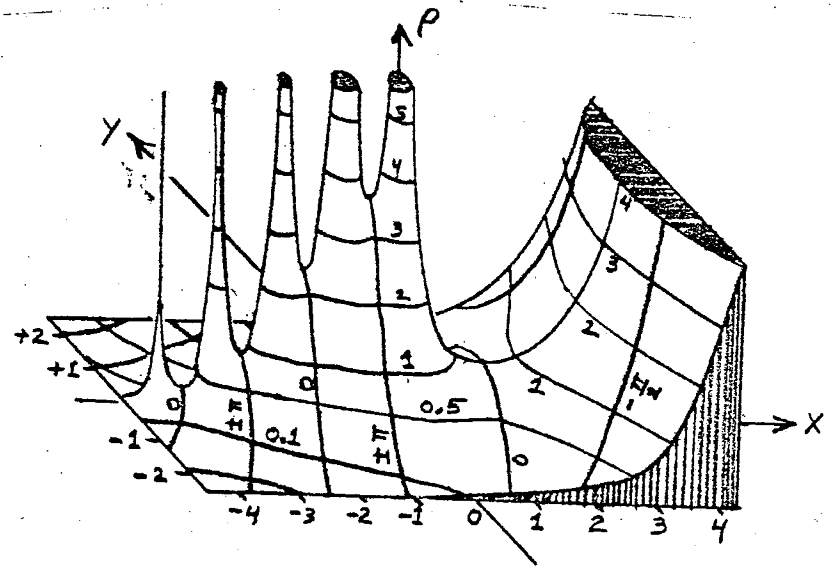
We note that the graph of $\Gamma(x)$ tends to infinity near $x=0, -1, -2, -3, \ldots,$ and thus the factorial of any negative integer is undefined.

Figure 1 The function $u = \Gamma(x)$ for real numbers x.

$$x! = \Gamma(x+1)$$



<u>Figure 2</u> Relief map for the gamma function $\Gamma(z) = \rho e^{i\phi}$. The Modulus ρ is plotted vertically over the complex z-plane.



<u>Figure 3</u> Contour map for the function $\Gamma(z) = \rho e^{i\phi}$. Lines of constant ρ (solid) and lines of constant ϕ (dotted) are plotted over the complex z-plane.

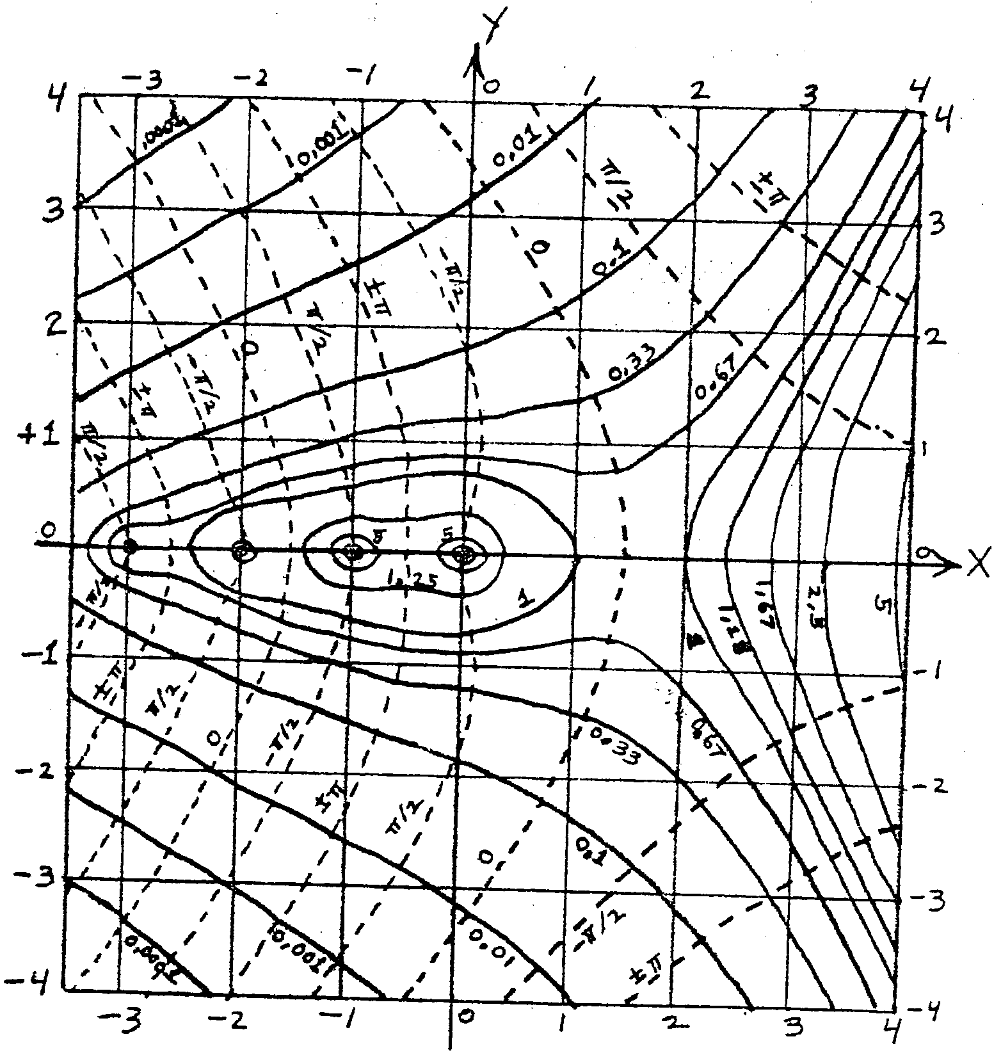


Figure 2 shows a relief map of the gamma function in which the modulus ρ from the relation

$$\Gamma(z) = \rho e^{i\phi}$$

is plotted vertically over the z-plane. Figure 3 is a contour map of the "Mountains" shown in Figure 2. These two figures give us a picture of the nature of the gamma function for complex values of z, whereas Figure 1 is useful only for real z.

A study of Figure 2 reveals that:

1. $\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$. It appears that $\Gamma(z)$ is a meromorphic function.
2. As z approaches infinity in a direction parallel to, and in the direction of, the positive x-axis, the modulus of $|\Gamma(z)|$ tends rapidly to infinity. Therefore

$$\lim_{\substack{x \to +\infty \\ y \text{ constant}}} |\Gamma(x+iy)| = +\infty$$

3. As z approaches infinity in a direction parallel to the y-axis, the modulus of $\Gamma(z)$ tends to zero. Thus

$$\lim_{\substack{y \to \pm\infty \\ x \text{ constant}}} \Gamma(x+iy) = 0$$

4. $\Gamma(z)$ has no zeros.

We can even estimate values of the gamma function from Figure 3.

Example 1

Estimate the value of $\Gamma(2 + 2i)$.

Solution

At the point where $x=2$ and $y=2$ on Figure 3 we have, approximately,

$$\rho = 0.33 \quad \text{and} \quad \phi = \text{about } \cancel{2\pi/3}. \; \frac{\pi}{3}$$

Therefore

$$\begin{aligned} \Gamma(z) &= \rho e^{i\phi} \\ \Gamma(2 + 2i) &\approx 0.33\, e^{i\cancel{2}\pi/3} \\ &\approx 0.33\left(+\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) \\ &\approx \cancel{-}0.17 + 0.3\, i \end{aligned}$$

Problems

16. Estimate $\Gamma(z)$ at points where z equals (a) $-2+3i$, (b) $4 - i$, (c) $-1/2$.

17. Estimate $z!$ at the points where z equals (a) $-3+3i$, (b) $3 - i$, (c) $-3/2$, and (d) $-5/2$.

7.6 Gamma function identities

To gain familiarity with the gamma function and its properties, we will begin by using the relations listed in Table 1. For the moment, we accept these as being true . In the following section we will indicate how many of these relations arise.

Analytic definitions

Any of the three relations (1), (2) , or (3) could be used to build a rigorous theory of the gamma function. Notice that (1) is only valid for z in the half-plane $\mathrm{Re}(z) > 0$, whereas there is no restriction on (2) or (3). (Of course, (2) cannot be used at $z = 0, -1, -2, \ldots$, since these are poles of $\Gamma(z)$.)

TABLE 1 THE GAMMA FUNCTION

ANALYTIC DEFINITIONS

(1) Euler's integral $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt, \quad \mathrm{Re}(z) > 0.$

(2) Euler's limit $\Gamma(z) = \lim_{N \to \infty} \dfrac{N!\ N^z}{z(z+1)(z+2)\ \ldots\ (z+N)}$

(3) Weierstrass' product $\dfrac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$

where γ is Euler's constant defined by

$$\gamma = \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \log N \right) = 0.57721\ 56649\ 01533\ \ldots$$

SPECIAL VALUES

(4) $\Gamma(n+1) = n!\,, \quad n=0,1,2,\ldots\,.$ (5) $\Gamma(1/2) = \sqrt{\pi}$

FUNCTIONAL EQUATIONS

(6) $\Gamma(z+1) = z\,\Gamma(z)$ (7) $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

DUPLICATION FORMULA

(8) $\Gamma(2z) = \pi^{-1/2}\ 2^{2z-1}\ \Gamma(z)\ \Gamma(z+\tfrac{1}{2})$

BETA FUNCTION

(9) $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \qquad \begin{matrix} \mathrm{Re}(x) > 0 \\ \mathrm{Re}(y) > 0 \end{matrix}$

ANALYTIC BEHAVIOR

(10) $\Gamma(z)$ is a meromorphic function having simple poles at $z = -n$, where $n = 0,1,2,3,\ldots$, with residues $(-1)^n/n!$.

(11) $\Gamma(z)$ has no zeros.

ASYMPTOTIC EXPANSION

(12) $\Gamma(z+1) \sim z^z\, e^{-z} \sqrt{2\pi z}\ \left(1 + \dfrac{1}{12\,z} + \dfrac{1}{288\,z^2} - \dfrac{139}{51840\,z^3} - \ldots \right)$

valid for z having large modulus and away from the negative real axis.

Example 1

Use Euler's integral (1) to evaluate $\Gamma(1)$.

Solution

Set $z=1$ in (1) and get $\Gamma(1) = \int_0^\infty e^{-t}dt = 1.$

Problems

18. Use (1) to show that (a) $\Gamma(2) = 1$, (b) $\Gamma(3) = 2$, (c) $\Gamma(4) = 6$.

19. Use integration by parts on the Euler integral (1) to derive the functional equation (6) .

Functional equations

We now examine the functional equations (6) and (7).

Example 2

Use the functional equation (7) to compute $\Gamma(1/2)$.

Solution

Set $z = 1/2$ in (7) and get $\Gamma(1/2)^2 = \pi/1$. Thus $\Gamma(1/2) = \sqrt{\pi}$. This is $(-1/2)!$.

Problems

20. Using $\Gamma(1/2) = \sqrt{\pi}$ and (6), determine (a) $\Gamma(-1/2)$, (b) $\Gamma(3/2)$, (c) $\Gamma(5/2)$, (d) $\Gamma(7/2)$, (e) $\Gamma(9/2)$.

21. Determine a formula for $\Gamma((2n+1)/2)$, where $n=1,2,3,\ldots$.

Example 3

Use (6) to determine the residue of $\Gamma(z)$ at $z=0$.

Solution

We can write (6) as $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. For z very near zero this relation tends to $\frac{\Gamma(1)}{z}$. Since we know that $\Gamma(1) = 1$, we know that the desired residue is 1 .

Example 4

Determine the residue of $\Gamma(z)$ at $z = -1$.

Solution

Multiply both sides of (6) by $z+1$ and get

$$z(z+1)\Gamma(z) = (z+1)\Gamma(z+1) .$$

Using the functional relation again on the right side of this last expression we get

$$z(z+1)\Gamma(z) = \Gamma(z+2).$$

Solving for $\Gamma(z)$ we have

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)} .$$

As z approaches -1, this last expression approaches

$$\frac{\Gamma(1)}{(-1)(z+1)} = \frac{-1}{z+1} .$$

This last relation reveals that $\Gamma(z)$ has a simple pole at $z = -1$ with residue -1 .

Problem

22. Determine the residue of $\Gamma(z)$ at (a) $z= -2$, (b) $z= -3$, (c) $z= -n$, where $n= 0, 1, 2, 3, \ldots$.

The Beta integral

The beta integral (9) can now be evaluated for selected values of x and y.

Example 5

Evaluate the integral $\int_0^a t^{-1/2} (a - t)^{-1/2}\, dt = I$.

Solution

Set $t = au$ and transform the integral to

$$I = \int_0^1 u^{-1/2}(1 - u)^{-1/2}\, du\ .$$

Take $x = y = 1/2$ in (9) and get

$$I = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \frac{\sqrt{\pi}\,\sqrt{\pi}}{1} = \pi$$

Problems

23. Evaluate $\int_0^a t^2\,(a - t)^{-1/2}\,dt$

24. Evaluate $\int_0^a t^{1/2}(a - t)^{-1/2}\,dt$

25. Evaluate $\int_0^{\pi/2} (\sin\theta)^p\,(\cos\theta)^q\,d\theta$ in terms of gamma functions. (Hint: Set $t = \sin^2\theta$ in (9).)

The asymptotic expansion

The asymptotic expansion (12) is used to calculate $\Gamma(z)$ when the modulus of z is large. However, it cannot be used when z is near the negative real axis. If we use only the first term in this series we have

$$\Gamma(z+1) = z! \approx z^z\, e^{-z}\sqrt{2\pi z}$$

which is known as Stirlings approximation .

If a small electronic calculator is available, we can compute with Stirlings approximation relatively easily. The following table shows that Stirlings approximation is good even when z is as small as 10 (0.83 % error), and gives only a 4% error for $z = 2$.

z	$z!$	$z^z e^{-z} \sqrt{2\pi z}$	% error
2	2	1.919	4.05 %
3	6	5.836	2.73 %
4	24	23.506	2.05 %
10	3,628,800	3,598,696	0.83 %

If in addition to using the first term in (12), we also add the second term, we can improve these approximations to the factorials.

Example 6.

Improve the approximation to 10! in the above table by using the first two terms of. (12) .

Solution

We must multiply Stirlings approximation by $(1 + 1/12z)$. Since $z = 10$ we must then multiply by $1 + 1/120 = 1.008333333$, and get 3,628,685 . The error is now only 0.003 % .

It is interesting to notice that the percentage error in these approximations is always nearly the value of the first term omitted in the series. Thus the percentage error found in the above table when we approximated 10! was about

$$\frac{1}{12\ z} = \frac{1}{120} = 0.0083333 = .83\ \% .$$

Problems

27. Use an electronic calculator to check the entries in the above table.

28. Add the correction term $1/12\ z$ to the first three entries of the table to improve the approximations.

29. Estimate $\Gamma(1 + 4i)$ using Stirlings approximation and compare the result with the value obtained from the contour plot of Figure 3, Section 7.5 .

7.7 Motivation for the gamma function

Suppose we had no knowledge of the gamma function, and we wanted to extend the concept of factorial, $z!$, to values of z other than 0, 1, 2, 3, ... There are many possible, arbitrary relations that could be advanced as definitions for the generalized factorial. But arbitrary definitions are not likely to yield beautiful or useful results. We seek to discover a formula for $z!$ that is,,in some intuitively understood sense, natural . This idea is not new to us, we used it extensively in Chapter 2.

Let us see if we can evolve a formula for $z!$ that seems appropriate. Look at the relation

$$(1)\quad \pi(z) = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\ \cdots}{(z+1)(z+2)(z+3)(z+4)\ \cdots}$$

Set $z = 4$ and observe that

$$\pi(4) = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\ \cdots}{5\cdot6\cdot7\cdot8\ \cdots}$$

$$= 1\cdot2\cdot3\cdot4 = 4!$$

Clearly $\pi(N) = N!$ whenever N is a positive integer. Expression (1) seems natural enough. After all, it is composed of simple arithmetic operations. Can it be used for any arbitrary value of z to give a definition of $z!$? It is a formula into which we can insert any real or complex number z (except -1, -2, -3, ...), but does the formula make sense ? For example, if we set $z = 1/2$

we would get

$$\pi(1/2) \overset{?}{=} (1/2)! \overset{?}{=} \frac{1\cdot2\cdot3\cdot4\ \cdots}{(3/2)(5/2)(7/2)\ \ldots}$$

Clearly both the numerator and the denominator diverge to infinity. This cannot be of use.

Can we alter (1) so that it becomes a mathematically useful expression, i.e. a convergent limit ? As a first attempt we might look at

$$(2) \qquad (z)! \overset{?}{=} \lim_{N\to\infty} \frac{1\cdot2\cdot3\cdot4\ \cdots\ N}{(z+1)(z+2)(z+3)\ \ldots\ (z+N)}$$

Does (2) generate factorials when z is a positive integer and does it converge ? To answer this question, we note that the right side of (2) is nearly

$$\frac{1\cdot2\cdot3\cdot4\ \cdots\ N}{(z+1)(z+2)\ \ldots\ (z+N)}$$

when N is a very large number compared to z . Select $z = 4$ and $N = 1000$. Now (2) is approximately

$$4! \overset{?}{\approx} \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\ \cdots 1000}{5\cdot6\ \cdots\ 1000\cdot1001\cdot1002\cdot1003\cdot1004}$$

$$(3) \quad 4! \overset{?}{\approx} \frac{1\cdot2\cdot3\cdot4}{1001\cdot1002\cdot1003\cdot1004} \quad .$$

This last expression is certainly not an approximation to $4!$. How can we alter (3) so that it will be approximately $4!$ and simultaneously , useful for arbitrary z ? Try putting into the numerator $1000^4 = N^z$. Expression (2) now becomes altered as

(4) $$z! \overset{?}{=} \lim_{N\to\infty} \frac{1\cdot2\cdot3\cdots N \quad N^z}{(z+1)(z+2)\cdots(z+N)}$$

and (3) becomes

(5) $$4! \overset{?}{\approx} \underline{1\cdot2\cdot3\cdot4} \; \frac{1000^4}{1001\cdot1002\cdot1003\cdot1004}$$

Now the expression (5) does look good because the ratio

$$\frac{1000^4}{1001\cdot1002\cdot1003\cdot1004}$$

is nearly one making the right side approximately $4!$ as desired. It now seems reasonable that (4) should converge and give us a generalization of the factorial as a convergent limit. Since

$$z! = \Gamma(z+1) = z\Gamma(z)$$

we see that

$$\Gamma(z) = \frac{z!}{z}$$

and thus (4) becomes

(6) $$\Gamma(z) = \lim_{N\to\infty} \frac{N! \quad N^z}{z(z+1)(z+2)\cdots(z+N)}$$

which we recognize as Euler's limit from Table 1 of the previous section.

From (6) we can obtain the product of Weierstrass

$$(7) \quad \frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

where $\gamma = \lim\limits_{N} \left(\sum\limits_{k=1}^{N} 1/k - \log N \right)$.

From (6) we have

$$\frac{1}{\Gamma(z)} = \lim_{N \to \infty} z \, \frac{(z+1)}{1} \, \frac{(z+2)}{2} \, \frac{(z+3)}{3} \cdots \frac{(z+N)}{N} \, N^{-z}$$

$$(8) \quad \frac{1}{\Gamma(z)} = \lim_{N \to \infty} z \left(1 + \frac{z}{1}\right) \left(1 + \frac{z}{2}\right) \left(1 + \frac{z}{3}\right) \cdots \left(1 + \frac{z}{N}\right) e^{-z \log N}$$

In Example 2 of Section 7.3 we saw that the product $\prod \left(1 + \frac{z}{n}\right)$ diverges, but with exponential convergence factors $e^{-z/n}$ it converges. Thus we alter (8) by inserting factors $e^{-z/n}$ in appropriate locations and also inserting $e^{z/n}$ to insure that the right hand side does not change its value.

$$\frac{1}{\Gamma(z)} = \lim_{N \to \infty} \left[z \left(1 + \frac{z}{1}\right) e^{-z/1} \left(1 + \frac{z}{2}\right) e^{-z/2} \cdots \left(1 + \frac{z}{N}\right) e^{-z/N} \right]$$

$$\cdot \; e^{z/1} \; e^{z/2} \cdots e^{z/N} \; e^{-z \log N}$$

$$= \lim_{N \to \infty} \left[z \left(1 + \frac{z}{1}\right) e^{-z/1} \cdots \left(1 + \frac{z}{N}\right) e^{-z/N} \right]$$

$$\exp\left(\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N} - \log N \right) z \right)$$

$$= z \; e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

This last relation is the Weierstrass product.

7.8 Derivation of gamma function identities

Most of the identities for the gamma function can be obtained easily from the product of Weierstrass. We will now derive the identity

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,. \tag{1}$$

Using $x\,\Gamma(x) = \Gamma(x+1)$ we have

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\Gamma(z)\;(-z)\Gamma(-z)} = \frac{-1}{z\;\Gamma(z)\Gamma(-z)}$$

$$= -\frac{1}{z}\quad z\,e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

$$\cdot\;(-z)\,e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$$

$$= z\prod_{n=-\infty}^{\infty}{}'\left(1+\frac{z}{n}\right)e^{-z/n}\,.$$

From Section 7.3, Example 2, we recognize this last expression as the infinite product representation for the function $\dfrac{\sin \pi z}{\pi}$. Thus the identity is proved.

Example 1

Derive the duplication formula (Table 1, (8)) for the gamma function from Euler's limit.

Solution

Replace N by 2N and z by 2z in Euler's limit and get

$$\Gamma(2z) = \lim_{N\to\infty} \frac{(2N)! \quad (2N)^{2z}}{(2z)(2z+1)(2z+2)(2z+3)\ \ldots\ (2z+2N)}$$

Factoring 2 from each element of the denominator gives

$$\Gamma(2z) = \lim \frac{(2N)! \quad (2N)^{2z}}{2^{2N+1}\ z\ (z+\frac{1}{2})(z+1)(z+\frac{3}{2})\ \ldots (z+\frac{2N-1}{2})(z+N)}$$

Collecting factors with whole mumbers and those with fractions in the denominator we get

$$\Gamma(2z) = \lim_{N\to\infty} \frac{1\cdot 2\cdot 3\ \cdots\ N\cdot N^{z}}{z(z+1)\ldots(z+N)} \cdot \frac{(N+1)(N+2)\ \cdots\ (2N)\quad N^{z}}{(z+\frac{1}{2})(z+\frac{3}{2})\ \ldots (z+\frac{2N-1}{2})} \cdot 2^{2z-2N-1}$$

Now we see Euler's limit for $\Gamma(z)$ appearing on the right side, and Euler's limit for $\Gamma(z+\frac{1}{2})$ is almost there also. If we multiply and divide by

$$(N-1)! \quad (N-1)^{z+\frac{1}{2}}$$

we get

$$\Gamma(2z) = \lim_{N\to\infty} \frac{1\cdot 2\cdot 3\ \cdots\ N\cdot N^{z}}{z(z+1)\ldots(z+N)} \ \lim_{N\to\infty} \frac{(N-1)! \quad (N-1)^{z+1/2}}{(z+\frac{1}{2})(z+\frac{3}{2})\ldots(z+\frac{1}{2}+N-1)}$$

$$\lim_{N\to\infty} \frac{(N+1)(N+2)\ldots(2N)\cdot 2^{-2N}\, N^{z}}{(N-1)!\cdot(N-1)^{z+1/2}} \cdot 2^{2z-1}$$

$$= 2^{2z-1}\ \Gamma(z)\ \Gamma(z+\tfrac{1}{2}) \lim_{N\to\infty} \frac{(N+1)(N+2)\ldots(2N) \quad 2^{-2N}}{(N-1)! \quad (N-1)^{1/2}} \left[\frac{N}{N-1}\right]^{z}$$

Since $\lim_{N \to \infty} \left[\frac{N}{N - 1}\right]^z$ is 1, we see that this last limit is a constant C. We now have

$$\Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \tfrac{1}{2}) C$$

To find C, set $z = 1$ and get

$$\Gamma(2) = 2 \Gamma(1) \Gamma(3/2) C$$

$$1 = 2 (1) \frac{\sqrt{\pi}}{2} C$$

Thus $C = \pi^{-1/2}$ and we have derived the duplication formula.

Problems

30. Using Euler's limit, show that $\Gamma(z+1) = z\Gamma(z)$.

7.9 Motivation for Stirlings approximation

The very beautiful asymptotic expansion for large $|z|$

(1) $$\Gamma(z+1) = z! \approx z^z e^{-z} \sqrt{2\pi z}\ \left(1 + \frac{1}{12\, z} + \dots\right)$$

is difficult to derive, even when z is a positive integer. Nevertheless, we can give a relatively simple argument to demonstrate That (1) is reasonable. We will show that for large positive integers N,

(2) $$N! \approx C\, N^N e^{-N} \sqrt{N} \qquad \text{(Stirling's approximation)}$$

where C is a constant. We will not show that $C = \sqrt{2\pi}$, but we will get a numerical value for C that is within 3 % of this value.

Before the days of the inexpensive electronic calculators, $N!$ for large N, would be computed using logarithms. Thus it is natural to begin by seeking an approximation for

(3) $$\log N! = \log 1 + \log 2 + \log 3 + \dots + \log N$$

for large N.

The right side of (3) can be visualized as the sum of the ordinates (shown dotted) above the points where $x = 1, 2, \dots, N$, in Figure 1. The right side of (3) is also visualized as the sum of all the areas of the rectangles shown in Figure 2.

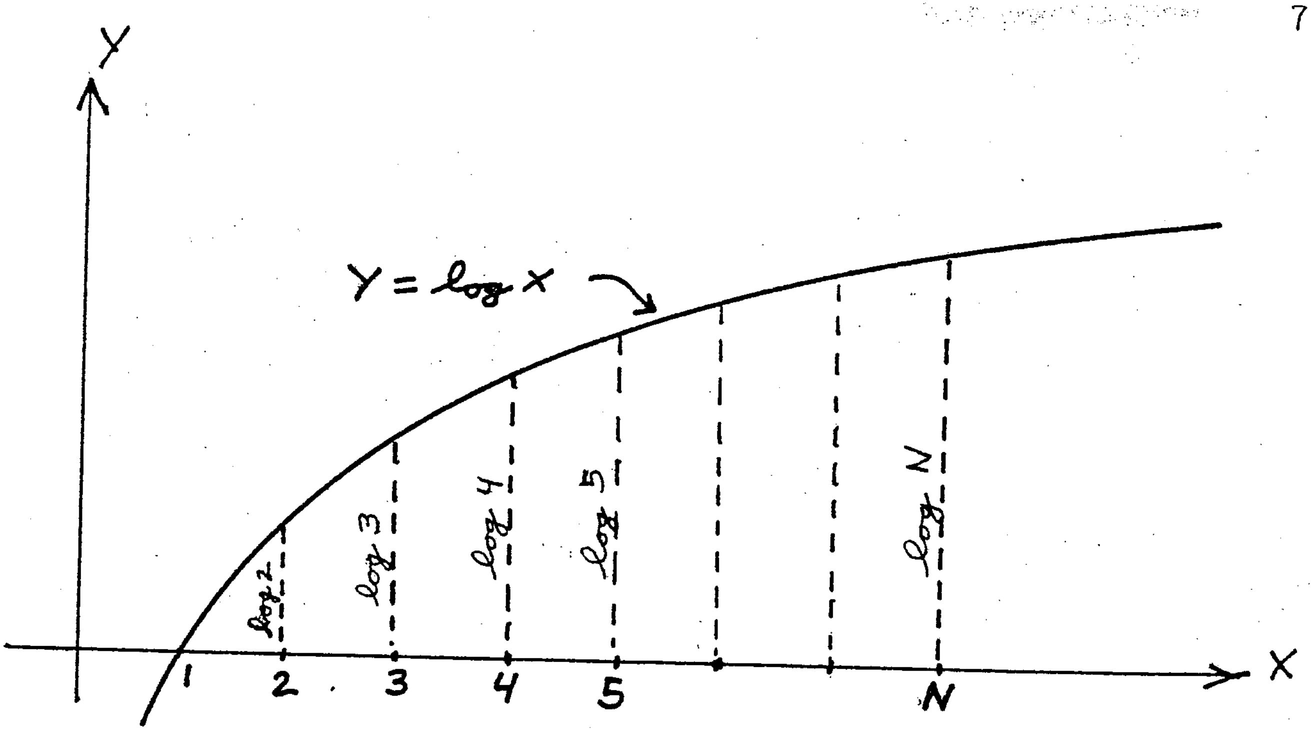


Figure 1

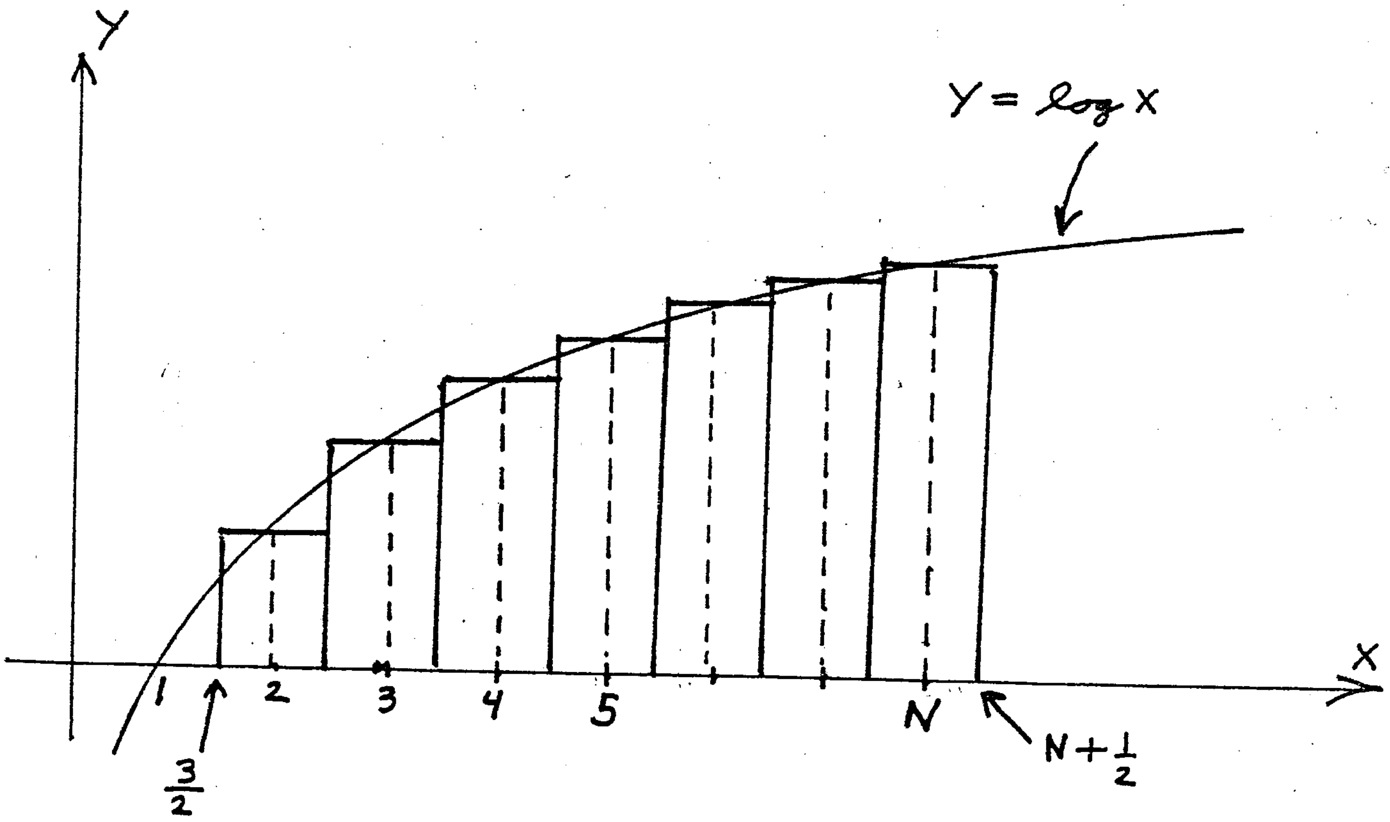


Figure 2

Notice that the area under the rectangles is nearly the area under the curve $y = \log x$ from $x = 3/2$ to $x = N + 1/2$. Thus we have

$$\log N! \approx \int_{3/2}^{N+1/2} \log x \, dx$$

$$\approx x \log x - x \Big|_{3/2}^{N+1/2}$$

$$\approx (N + 1/2)\log(N + 1/2) - (N + 1/2) - \tfrac{3}{2}\log\tfrac{3}{2} + \tfrac{3}{2}$$

(4) $$\approx \log(N + 1/2)^{N + 1/2} + \log(\tfrac{2}{3})^{3/2} - N + 1$$

Taking the exponential of both sides of (4) we get

(5) $$N! \approx (N + \tfrac{1}{2})^{N+\frac{1}{2}} \; e^{-N} \; (\tfrac{2}{3})^{3/2} \; e^{1}$$

Because $(N + \frac{1}{2})^{N+\frac{1}{2}} = N^{N+\frac{1}{2}} (1 + \frac{1}{2N})^{N+\frac{1}{2}}$, (5) becomes

$$N! \approx N^{N+\frac{1}{2}} \; e^{-N} \; (\tfrac{2}{3})^{3/2} \; (1 + \tfrac{1}{2N})^{N+\frac{1}{2}} \; e .$$

Recall that in the calculus we learned that $\lim_{N\to\infty} (1 + \frac{x}{N})^{N} = e^{x}$,

and thus for large N, $(1 + \frac{1}{2N})^{N} \approx e^{1/2}$. Thus we have

$$N! \approx N^{N+\frac{1}{2}} \; e^{-N} \; (\tfrac{2}{3})^{3/2} \; e^{3/2} \; (1 + \tfrac{1}{2N})^{1/2}$$

Now $(\frac{2}{3})^{3/2} e^{3/2} = 2.4395 \ldots$, and thus

$$(6) \quad N! \approx (2.4395)\sqrt{N}\, N^{N} \, e^{-N} .$$

The correct constant in Stirling's approximation is

$$\sqrt{2\pi} = 2.506628275 \ldots$$

and thus our constant in (6) is off by only 2.7 % . Thus we see that Stirling's approximation can be suggested by comparing the rectangles in Figure 2 to the area under the logarithmic curve.

Review Problems for Chapter 7

1. (a) Find an infinite product representation for $\cosh \pi z$.

   (b) Evaluate the product $\prod_{n=0}^{\infty} (1 + 8(2n+1)^{-2})$

2. Express the most general entire function having zeros at the points $z = i\, n^{3/4}$, where $n = 0, 1, 2, \ldots,$ as an infinite product.

3. Expand $\tanh \pi z$ in partial fractions.

4. Evaluate the integral $\int_0^{\infty} e^{-st}\, t^{3/2}\, dt$.

5. Evaluate the integral $\int_0^1 t^{a-1} (1-t)^{3-a}\, dt$, $0 < a < 4$.

6. Test the integral $\int_1^{\infty} \Gamma(x+1)\, x^{-x}\, dx$ for convergence .

7. Evaluate $\oint_{|z| = 3/2} \Gamma(z)\, dz$

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 7

1 (a) Replace z by $2z$ in (9) and get

$$\boxed{\sin 2z = 2z \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2\pi^2}\right)}$$

(b) Replace z by $3z$ in (9) $\boxed{\sin 3z = 3z \prod_{n=1}^{\infty}\left(1 - \frac{9z^2}{n^2\pi^2}\right)}$

(c) $\sinh z = -i \sin iz$. Replace z by iz in (9) and get

$$\boxed{\sinh z = z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)}$$

(d) Replace z by $(2z)$ in problem 1(c) and get

$$\boxed{\sinh 2z = 2z \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{n^2\pi^2}\right)}$$

(e) Replace z by πz in problem 1(c) and get

$$\boxed{\sinh \pi z = \pi z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2}\right)}$$

(f) $e^{2z}-1 = e^{z}(e^{z}-e^{-z}) = 2e^{z}\left[\frac{e^{z}-e^{-z}}{2}\right] = 2e^{z}\sinh z$

$$= \boxed{2e^{z} z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)}$$

(g) $\frac{e^{z}-1}{z} = \frac{2e^{\frac{z}{2}}}{z}\left(\frac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{2}\right) = \frac{2e^{\frac{z}{2}}}{z}\sinh\frac{z}{2}$

$$= \boxed{e^{\frac{z}{2}} \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4n^2\pi^2}\right)}$$

1/ (Continued)

(k) $$e^{az} - e^{-bz} = e^{\frac{a-b}{2}z}\left(e^{\frac{a+b}{2}z} - e^{-\frac{a+b}{2}z}\right)$$

$$= 2e^{\frac{a-b}{2}z} \sinh \frac{a+b}{2} z$$

$$= 2e^{\frac{a-b}{2}z}\left(\frac{a+b}{2}z\right)\prod_{n=1}^{\infty}\left(1 + \frac{(a+b)^2 z^2}{4\pi^2 n^2}\right)$$

$$= \boxed{(a+b)e^{\frac{a-b}{2}z}\, z \prod_{n=1}^{\infty}\left(1 + \frac{(a+b)^2 z^2}{4\pi^2 n^2}\right)}$$

2/ (a) $\cos z$ has zeros at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$

i.e, $\pm\frac{(2n+1)\pi}{2}$, where $n = 0, 1, 2, \ldots$,

Then we expect

$$\cos z = g(z)\prod_{n=0}^{\infty}\left(1 - \frac{z^2}{\left[\frac{(2n+1)\pi}{2}\right]^2}\right)$$

$$= g(z)\prod_{n=0}^{\infty}\left(1 - \frac{2^2 z^2}{(2n+1)^2\pi^2}\right)$$

When z is very small, $\cos z \approx 1$ and

$1 - \frac{4z^2}{\pi^2(2n+1)^2} \approx 1$, Therefore $1 \approx g(0)\cdot 1$

Thus $g(0) = 1$, and as before we guess $g(z) \equiv 1$,

We have

$$\boxed{\cos z = \prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{\pi^2(2n+1)^2}\right)}$$

2/ (Continued)

(b) Since $\cosh z = \cos iz$ we replace z by iz in problem 2(a) to get

$$\boxed{\cosh z = \prod_{n=0}^{\infty}\left(1 + \frac{4z^2}{\pi^2(2n+1)^2}\right)}$$

(c) $$e^{i2\pi z} + 1 = 2e^{i\pi z}\left(\frac{e^{i\pi z} + e^{-i\pi z}}{2}\right)$$

$$= 2e^{i\pi z}\cos \pi z$$

$$= \boxed{2e^{i\pi z}\prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{(2n+1)^2}\right)}$$

(d) $$\boxed{z\prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{(2n+1)^2}\right)}$$

(e) $$e^{az} + e^{-bz} = e^{\frac{a-b}{2}z}\left(e^{\frac{a+b}{2}z} + e^{-\frac{a+b}{2}z}\right)$$

$$= 2e^{\frac{a-b}{2}z}\cosh\left(\frac{a+b}{2}z\right)$$

Now use problem 2(b) and get

$$e^{az} + e^{-bz} = \boxed{2e^{\frac{a-b}{2}z}\prod_{n=0}^{\infty}\left(1 + \frac{(a+b)^2 z^2}{\pi^2(2n+1)^2}\right)}$$

3/ (a) Set $z = 1$ in problem 1(e) and get $\boxed{\frac{\sinh \pi}{\pi}}$.

(b) Set $z = \frac{\pi}{2}$ in problem 2(b) and get $\boxed{\cosh \frac{\pi}{2}}$,

(c) Set $z = \pi$ in problem 2(a) and get $\cos \pi = \boxed{-1}$, 4/ (a) Since $\lim_{n\to\infty} n^2 \neq 0$, the product diverges,

(b) Since $\sum |n^{-3}| = \sum n^{-3}$ converges, the product converges,

(c) Since $\sum |e^{-n}| = \sum e^{-n}$ converges, the product converges,

(d) Since $\lim_{n\to\infty} e^n \neq 0$, the product diverges,

(e) Since $\sum \frac{1}{\log n}$ diverges, the product diverges,

(f) Since $\sum \left|\frac{1}{n^n}\right|$ converges, the product converges,

5/ All these products converge for all z since the series $\sum |u_n|$ in each case behaves like $\sum n^{-2}$ which converges.

6/ Since

$$\boxed{e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)^2}$$

converges for all z $\left(\sum \left|\frac{z}{n^2}\right| = |z| \sum n^{-2} \text{ converges}\right)$, no exponential convergence factors are required. Here $h(z)$ is an arbitrary entire function.

7/ $e^{h(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{in}\right)^3$ does not converge because $\sum \frac{z}{in}$ diverges. The factor $e^{\frac{z}{in}} \approx 1 + \frac{z}{in}$ will remove the $\frac{1}{n}$ type term since $\left(1 - \frac{z}{in}\right) e^{\frac{z}{in}} \approx \left(1 - \frac{z}{in}\right)\left(1 + \frac{z}{in}\right) = \left(1 + \frac{z^2}{n^2}\right)$. Thus

$$\boxed{e^{h(z)} z \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{in}\right) e^{\frac{z}{in}}\right]^3}$$

converges.

8/ The product $\prod_{n=1}^{\infty}\left(1-\frac{z}{n^{2/3}}\right)^2$ diverges since $\sum \frac{z}{n^{2/3}}$ diverges. We try a convergence factor of the type $e^{\frac{z}{n^{2/3}}} = 1 + \frac{z}{n^{2/3}} + \frac{z^2}{2n^{4/3}} + \cdots$

Thus $\left(1-\frac{z}{n^{2/3}}\right)e^{\frac{z}{n^{2/3}}}$ behaves like

$$\begin{array}{l} 1 + \frac{z}{n^{2/3}} + \frac{z^2}{2n^{4/3}} \\ 1 - \frac{z}{n^{2/3}} \\ \hline 1 + \frac{z}{n^{2/3}} + \frac{z^2}{2n^{4/3}} \\ \quad - \frac{z}{n^{2/3}} - \frac{z^2}{n^{4/3}} - \cdots \\ \hline 1 \quad - \frac{z^2}{2n^{4/3}} \end{array}$$

for large n, Since $\sum \frac{1}{n^{4/3}}$ converges, we have

$$\boxed{e^{h(z)}\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{n^{2/3}}\right)e^{\frac{z}{n^{2/3}}}\right]^2}$$

where $h(z)$ is an arbitrary entire function,

9/ The product $\prod_{n=1}^{\infty}\left(1-\frac{z}{n^{1/3}}\right)$ diverges since $\sum \frac{z}{n^{1/3}}$ diverges. Try the convergence factor

$$e^{\frac{z}{n^{1/3}}} = 1 + \frac{z}{n^{1/3}} + \frac{z^2}{2n^{2/3}} + \cdots$$

Thus the product $\left(1-\frac{z}{n^{1/3}}\right)e^{\frac{z}{n^{1/3}}}$, for large n, behaves like

9/ (continued)

$$\begin{array}{llllll}
1 + & \frac{z}{n^{1/3}} & + \frac{z^2}{2n^{2/3}} & + \frac{z^3}{6n} & + \cdots & \\
1 - & \frac{z}{n^{1/3}} & & & & \\
\hline
1 + & \frac{z}{n^{1/3}} & + \frac{z^2}{2n^{2/3}} & + \frac{z^3}{6n} & + \cdots & \\
 - & \frac{z}{n^{1/3}} & - \frac{z^2}{n^{2/3}} & - \frac{z^3}{2n} & - \cdots & \\
\hline
1 & & - \frac{z^2}{2n^{2/3}} & - \frac{z^3}{3n} & - \cdots &
\end{array}$$

Since $\sum \frac{z^2}{2n^{2/3}}$ diverges also, the product still diverges, Now try the additional convergence factor

$$e^{\frac{z^2}{2n^{2/3}}} = 1 + \frac{z^2}{2n^{2/3}} + \frac{z^4}{8n^{4/3}} + \cdots$$

Thus the factor $\left(1 - \frac{z}{n^{1/3}}\right) e^{\frac{z}{n^{1/3}}} e^{\frac{z^2}{2n^{2/3}}}$ behaves like

$$\begin{array}{llllll}
1 - & \frac{z^2}{2n^{2/3}} & - \frac{z^3}{3n} & - & & \\
1 + & \frac{z^2}{2n^{2/3}} & & + \frac{z^4}{8n^{4/3}} & & \\
\hline
1 - & \frac{z^2}{2n^{2/3}} & - \frac{z^3}{3n} & & & \\
 + & \frac{z^2}{2n^{2/3}} & & - \frac{z^4}{4n^{4/3}} & - \frac{z^5}{6n^{5/3}} & \\
 & & & + \frac{z^4}{8n^{4/3}} & . & - \frac{z^6}{16n^2} \\
\hline
1 & & - \frac{z^3}{3n} & - \frac{z^4}{8n^{4/3}} & - \cdots &
\end{array}$$

9/ (continued)

This last factor behaves like $1 - \frac{z^3}{3n}$ for large n which is still too slow because $\sum \frac{z^3}{3n}$ diverges. One more convergence factor $e^{\frac{z^3}{3n}}$ will remove the $\frac{1}{n}$ type term and replace it with a term of the type $\frac{1}{n^{4/3}}$ which is sufficient to insure convergence of the infinite product. Thus we have

$$\boxed{e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^{1/3}}\right) e^{\frac{z}{n^{1/3}} + \frac{z^2}{2\,n^{2/3}} + \frac{z^3}{3\,n}}}$$

10/ Looking at the solution to problem 2(a), we anticipate

$$\cos z = \left[\prod_{n=0}^{\infty}\left(1 - \frac{2z}{\pi(2n+1)}\right)\right]\left[\prod_{n=0}^{\infty}\left(1 + \frac{2z}{\pi(2n+1)}\right)\right].$$

Neither of these products converge because $\sum \frac{2z}{\pi(2n+1)}$ diverges at the same rate as $\sum \frac{1}{n}$. Convergence factors $e^{\pm\frac{2z}{\pi(2n+1)}}$ are required,

$$\boxed{\cos z = \left[\prod_{n=0}^{\infty}\left(1 - \frac{2z}{\pi(2n+1)}\right)e^{\frac{2z}{\pi(2n+1)}}\right] \cdot \left[\prod_{n=0}^{\infty}\left(1 + \frac{2z}{\pi(2n+1)}\right)e^{-\frac{2z}{\pi(2n+1)}}\right]}$$

11/(a)
$$\frac{\sinh \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right)$$

$$\frac{\pi \cosh \pi z}{\sinh \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\frac{2z}{n^2}}{1 + \frac{z^2}{n^2}}$$

$$\boxed{\pi \coth \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}}$$

(b)
$$\cos z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{\pi^2 (2n+1)^2}\right)$$

$$-\frac{\sin z}{\cos z} = \sum_{n=0}^{\infty} \frac{\frac{-8z}{\pi^2(2n+1)^2}}{1 - \frac{4z^2}{\pi^2(2n+1)^2}}$$

$$\boxed{-\tan z = 8z \sum_{n=0}^{\infty} \frac{1}{4z^2 - \pi^2(2n+1)^2}}$$

(c)
$$\cosh z = \prod_{n=0}^{\infty} \left(1 + \frac{4z^2}{\pi^2 (2n+1)^2}\right)$$

$$\frac{\sinh z}{\cosh z} = \sum_{n=0}^{\infty} \frac{\frac{8z}{\pi^2(2n+1)^2}}{1 + \frac{4z^2}{\pi^2(2n+1)^2}}$$

$$\boxed{\tanh z = 8z \sum_{n=0}^{\infty} \frac{1}{4z^2 + \pi^2(2n+1)^2}}$$

(d)
$$\cos z = \left[\prod_{n=0}^{\infty} \left(1 - \frac{2z}{\pi(2n+1)}\right) e^{\frac{2z}{\pi(2n+1)}}\right] \left[\prod_{n=0}^{\infty} \left(1 + \frac{2z}{\pi(2n+1)}\right) e^{-\frac{2z}{\pi(2n+1)}}\right]$$

$$\frac{-\sin z}{\cos z} = \sum_{n=0}^{\infty} \left[\frac{\frac{-2}{\pi(2n+1)}}{1 - \frac{2z}{\pi(2n+1)}} + \frac{\frac{2}{\pi(2n+1)} e^{\frac{2z}{\pi(2n+1)}}}{e^{\frac{2z}{\pi(2n+1)}}}\right]$$

$$+ \sum_{n=0}^{\infty} \left[\frac{\frac{2}{\pi(2n+1)}}{1 + \frac{2z}{\pi(2n+1)}} + \frac{\frac{-2}{\pi(2n+1)} e^{-\frac{2z}{\pi(2n+1)}}}{e^{-\frac{2z}{\pi(2n+1)}}}\right]$$

11/ (d) (continued)

$$\boxed{\begin{aligned} -\tan z = {} & 2\sum_{n=0}^{\infty}\left[\frac{1}{2z-\pi(2n+1)}+\frac{1}{\pi(2n+1)}\right] \\ & +2\sum_{n=0}^{\infty}\left[\frac{1}{2z+\pi(2n+1)}-\frac{1}{\pi(2n+1)}\right] \end{aligned}}$$

12/ $$\boxed{f(z) = g(z)+\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}}$$

No alteration is needed since $\sum \frac{1}{n^2}$ converges.

13/ The function $f(z)=g(z)+\sum\limits_{n=-\infty}^{\infty}\frac{(-1)^n}{z-n}$

requires alteration since $\sum \frac{1}{n}$ diverges,

$$\frac{1}{z-n} = -\frac{1}{n}\left(\frac{1}{1-\frac{z}{n}}\right) = -\frac{1}{n}\left(1+\frac{z}{n}+\cdots\right) = -\frac{1}{n}-\frac{z}{n^2}-\cdots$$

Thus since $\frac{1}{z-n}+\frac{1}{n} \approx \frac{z}{n^2}$ we have

$$\boxed{\begin{aligned} & f(z) = g(z)+\frac{1}{z}+\sum_{n=-\infty}^{\infty}{}'\left[\frac{(-1)^n}{z-n}+\frac{(-1)^n}{n}\right] \\ & \text{where } g(z) \text{ is entire,} \end{aligned}}$$

14/ $\csc \pi z = \frac{1}{\sin \pi z}$ has simple poles at $z=n$

where n is any integer with residue $\frac{(-1)^n}{\pi}$,

From problem 13 we have

$$\csc \pi z = g(z)+\frac{1}{\pi z}+\frac{1}{\pi}\sum_{n=-\infty}^{\infty}{}'\left[\frac{(-1)^n}{z-n}+\frac{(-1)^n}{n}\right].$$

The function $g(z)\equiv 0$, although we will not demonstrate this,

$$\boxed{\pi \csc \pi z = \frac{1}{z}+\sum_{n=-\infty}^{\infty}{}'(-1)^n\left[\frac{1}{z-n}+\frac{1}{n}\right]}$$

15/ $f(z) = g(z) + \sum_{n=1}^{\infty} \frac{1}{z - n^{1/3}}$ does not converge since it behaves like $\sum n^{-1/3}$,

$$\frac{1}{z-n^{1/3}} = -\frac{1}{n^{1/3}}\left(\frac{1}{1-\frac{z}{n^{1/3}}}\right) =$$

$$= -\frac{1}{n^{1/3}}\left(1 + \frac{z}{n^{1/3}} + \frac{z^2}{n^{2/3}} + \frac{z^3}{n} + \cdots\right)$$

$$= -\frac{1}{n^{1/3}} - \frac{z}{n^{2/3}} - \frac{z^2}{n} - \frac{z^3}{n^{4/3}} - \cdots$$

Thus for large n we have

$$\frac{1}{z-n^{1/3}} + \frac{1}{n^{1/3}} + \frac{z}{n^{2/3}} + \frac{z^2}{n} \approx -\frac{z^3}{n^{4/3}}$$

Since $\sum n^{-4/3}$ converges we have

$$\boxed{f(z) = g(z) + \sum_{n=1}^{\infty}\left[\frac{1}{z-n^{1/3}} + \frac{1}{n^{1/3}} + \frac{z}{n^{2/3}} + \frac{z^2}{n}\right]}$$

where $g(z)$ is an arbitrary entire function,

16/(a) At $z = -2+3i$ on FIG 3, we have $\rho = 0,001$ and $\phi = \frac{\pi}{2}$. Thus $\Gamma(-2+3i) \approx 0,001\,\rho^{i\pi/2} \approx \boxed{0,001\,i}$

(b) At $z = 4-i$ on FIG 3, we have $\rho = 5$ and $\phi = -\frac{\pi}{2}$, Thus $\Gamma(4-i) \approx 5\,e^{-i\pi/2} \approx \boxed{-5i}$

(c) Using Fig 1, we have $\Gamma(-\frac{1}{2}) \approx \boxed{-3,6}$

17/ Since $z! = \Gamma(z+1)$, the answers to (a), (b) and (c) are the same as those of the previous problem.

(d) Using Figure 1 we have $(-\frac{5}{2})! = \Gamma(-\frac{5}{2}+1) = \Gamma(-\frac{3}{2})$ $\approx \boxed{2.4}$

18/(a) Set $z=2$ in (1) and get $\Gamma(2) = \int_0^\infty e^{-t}\,t\,dt$.

Integrating by parts we have

$$\Gamma(2) = \int_0^\infty t\,d(-e^{-t}) = -te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}dt = 0+1.$$

(b) Set $z=3$ in (1) and get $\Gamma(3) = \int_0^\infty e^{-t}t^2\,dt$

$$\Gamma(3) = \int_0^\infty t^2 d(-e^{-t})$$
$$= -t^2e^{-t}\Big|_0^\infty + 2\int_0^\infty t\,e^{-t}dt$$
$$= 0 \quad + 2\,\Gamma(2) \quad = \quad 2$$

(c) Set $z=4$ in (1) and get $\Gamma(4) = \int_0^\infty e^{-t}t^3\,dt$

$$= \int_0^\infty t^3\,d(-e^{-t}) = -t^3e^{-t} + 3\int_0^\infty e^{-t}t^2\,dt$$
$$= \quad 0 + 3\;\Gamma(3) \quad = \quad 3\cdot 2 = 6.$$

19/ Replace z by $z+1$ in (1) to get

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt = \int_0^\infty t^z d(-e^{-t}) =$$
$$-t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}dt = 0 + z\,\Gamma(z),$$

20/(a) $\Gamma(-\frac{1}{2}) = \dfrac{(-\frac{1}{2})\Gamma(-\frac{1}{2})}{(-\frac{1}{2})} = \dfrac{\Gamma(1-\frac{1}{2})}{(-\frac{1}{2})} = -2\,\Gamma(\frac{1}{2}) = \boxed{-2\sqrt{\pi}}$

(b) $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \boxed{\frac{\sqrt{\pi}}{2}}$

(c) $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\,\frac{\sqrt{\pi}}{2} = \boxed{\frac{3\sqrt{\pi}}{2^2}}$

(d) $\Gamma(\frac{7}{2}) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\cdot\frac{3\sqrt{\pi}}{2^2} = \boxed{\frac{5\cdot 3\sqrt{\pi}}{2^3}}$

(e) $\Gamma(\frac{9}{2}) = \frac{7}{2}\Gamma(\frac{7}{2}) = \frac{7}{2}\cdot\frac{5\cdot 3\sqrt{\pi}}{2^3} = \boxed{\frac{7\cdot 5\cdot 3\sqrt{\pi}}{2^4}}$

21/ AN EXAMINATION OF THE RESULTS OF THE PREVIOUS PROBLEMS SHOWS THAT FOR (b), $n=1$; (c) $n=2$; (d) $n=3$; (e) $n=4$;

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-2)(2n-1)\sqrt{\pi}}{2\cdot 4\cdot 6\cdots(2n-2)\quad 2^n}$$

$$= \frac{(2n-1)!\,\sqrt{\pi}}{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdots(2\cdot(n-1))\quad 2^n}$$

$$= \frac{(2n-1)!\,\sqrt{\pi}}{2^{n-1}(1\cdot 2\cdot 3\cdots(n-1))\quad 2^n}$$

$$= \boxed{\frac{(2n-1)!\,\sqrt{\pi}}{2^{2n-1}\,(n-1)!}}$$

22/ (a) $$\Gamma(z) = \frac{\Gamma(z)\,z\,(z+1)(z+2)}{z(z+1)(z+2)} = \frac{\Gamma(z+1)(z+1)(z+2)}{z(z+1)(z+2)}$$

$$= \frac{\Gamma(z+2)(z+2)}{z(z+1)(z+2)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)}$$

as z nears -2 this last expression becomes

$$\Gamma(z)\Big|_{z \text{ near } -2} \approx \frac{\Gamma(1)}{-2(-1)(z+2)} = \frac{1}{2(z+2)}$$

Residue at -2 = $\boxed{\frac{1}{2}}$

(b) $$\Gamma(z) = \frac{\Gamma(z+3)}{z(z+1)(z+2)}\,\frac{(z+3)}{(z+3)} = \frac{\Gamma(z+4)}{z(z+1)(z+2)(z+3)}$$

as z nears -3 we have

$$\Gamma(z)\Big|_{z \text{ near } -3} \approx \frac{\Gamma(1)}{(-3)(-2)(-1)(z+3)} = \frac{(-1)^3}{3!\,(z+3)}$$

Residue at -3 = $\boxed{\frac{(-1)^3}{3!}}$

22/ (continued)

(c) $$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)}$$

$$\Gamma(z)\Big|_{\substack{\text{near}\\ z=-n}} \approx \frac{\Gamma(1)}{(-n)(1-n)(2-n)\cdots(-1)\ (z+n)} = \frac{(-1)^n}{n!\ (z+n)}$$

Thus residue at $z=-n$ $= \boxed{\dfrac{(-1)^n}{n!}}$

23/ Set $t = ua$ and get $I = \int_0^1 a^2u^2(a-au)^{-\frac{1}{2}}a\,du$

$$I = a^2a^{-\frac{1}{2}}a\int_0^1 u^2(1-u)^{-\frac{1}{2}}du = a^{5/2}\,B(3,\tfrac{1}{2})$$

$$= a^{5/2}\,\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} = a^{5/2}\,\frac{2(\sqrt{\pi})}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}(\sqrt{\pi})} = \boxed{\frac{16\,a^{5/2}}{15}}$$

24/ Set $t = ua$ and get $I = \int_0^1 a^{1/2}u^{1/2}(a-au)^{-1/2}a\,du$

$$I = a\int_0^1 u^{1/2}(1-u)^{-1/2}du = a\,B(\tfrac{3}{2},\tfrac{1}{2})$$

$$= a\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = a\,\frac{(\frac{\sqrt{\pi}}{2})(\sqrt{\pi})}{1} = \boxed{\frac{\pi a}{2}}$$

25/ $t = \sin^2\theta$, $dt = 2\sin\theta\cos\theta\,d\theta$, $1-t = 1-\sin^2\theta = \cos^2\theta$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^{\pi/2}(\sin\theta)^{2x-2}(\cos\theta)^{2y-2}(2\sin\theta\cos\theta\,d\theta)$$

$$= 2\int_0^{\pi/2}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta$$

Set $2x-1 = p$ and $2y-1 = q \Rightarrow x = \frac{p+1}{2}$, $y = \frac{q+1}{2}$

Therefore

$$\int_0^{\pi/2}(\sin\theta)^p(\cos\theta)^q d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \boxed{\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\,\Gamma(\frac{p+q+2}{2})}}$$

28/ For $z=2$, we multiply $1{,}919$ by $\left(1+\frac{1}{12z}\right)$ to get

$$2! \approx (1{,}919)\left(1+\frac{1}{24}\right) = \boxed{1{,}99896}$$

For $z=3$, we multiply $5{,}836$ by $\left(1+\frac{1}{12z}\right)$ to get

$$3! \approx (5{,}836)\left(1+\frac{1}{36}\right) = \boxed{5{,}9981}$$

For $z=4$, we multiply $23{,}506$ by $\left(1+\frac{1}{12z}\right)$ to get

$$4! \approx (23{,}506)\left(1+\frac{1}{48}\right) = \boxed{23{,}9957}$$

29/

$$\Gamma(1+z) \approx z^z e^{-z}\sqrt{2\pi z} = e^{z\log z} e^{-z}\sqrt{2\pi z}$$

$$= e^{4i\left[\log 4 + \frac{\pi}{2}i\right]} e^{-4i}\sqrt{2\pi i 4}$$

$$= 2e^{-2\pi} e^{i(4\log 4 - 4)}\sqrt{2\pi}\, e^{i\frac{\pi}{4}}$$

$$= 2\sqrt{2\pi}\, e^{-2\pi} e^{i\left(4\log 4 - 4 + \frac{\pi}{4}\right)}$$

$$= \boxed{0.00936\, e^{i(2{,}33)}} \leftarrow \text{check this}$$

Looking at Fig. 3, of Section 7,5 we see that

$$\Gamma(1+4i) \approx 0.01\, e^{i\frac{3\pi}{4}} = \boxed{0.01\, e^{i(2{,}36)}}.$$

30/

$$\Gamma(z+1) = \lim_{N\to\infty} \frac{N^{z+1}\, N!}{(z+1)(z+2)\cdots(z+1+N)}$$

$$= z \lim_{N\to\infty} \frac{N!\, N^z}{z(z+1)(z+2)\cdots(z+N)} \cdot \frac{N}{(z+N+1)}$$

$$= z\,\Gamma(z) \cdot \lim_{N\to\infty} \frac{N}{(N+z+1)}$$

$$= z\,\Gamma(z)$$

Solutions to Review Problems from Chapter 7

1 (a) Since $\cos z = \prod_{n=0}^{\infty}\left(1-\frac{4z^2}{\pi^2(2n+1)^2}\right)$, (see Problem 2(a)),

and since $\cosh z = \cos iz$ we replace z by $i\pi z$ this product to get

$$\boxed{\cosh\pi z = \prod_{n=0}^{\infty}\left(1+\frac{4z^2}{(2n+1)^2}\right)},$$

(b) Set $z=\sqrt{2}$ and get $\boxed{\cosh\sqrt{2}}$.

2/ We first try $g(z)z\prod_{n=1}^{\infty}\left(1-\frac{z}{i\,n^{3/4}}\right)$,

but this product diverges since $\sum n^{-3/4}$ diverges. Try the exponential convergence factor

$$e^{\frac{z}{i n^{3/4}}} \approx 1+\frac{z}{i\,n^{3/4}}-\frac{z^2}{2\,n^{3/2}}$$

$$\begin{array}{rllll}
 & 1 & -\frac{z}{i\,n^{3/4}} & & \\
\hline
 & 1 & +\frac{z}{i\,n^{3/4}} & -\frac{z^2}{2\,n^{3/2}} & +\cdots \\
 & & -\frac{z}{i\,n^{3/4}} & +\frac{z^2}{n^{3/2}} & +\cdots \\
\hline
\left(1-\frac{z}{i\,n^{3/4}}\right)e^{\frac{z}{i n^{3/4}}} \approx & 1 & & +\frac{z^2}{n^{3/2}} &
\end{array}$$

Since $\sum n^{-3/2}$ converges, the desired product is

$$\boxed{g(z)z\prod_{n=0}^{\infty}\left(1-\frac{z}{i\,n^{3/4}}\right)e^{\frac{z}{i n^{3/4}}}},$$

where $g(z)=e^{h(z)}$, $h(z)$ any entire function.

3/ Using the product for $\cosh \pi z$ in review problem 1(a), we have

$$\frac{\frac{d}{dz}(\cosh \pi z)}{\cosh \pi z} = \sum_{n=0}^{\infty} \frac{\frac{d}{dz}\left(1 + \frac{4z^2}{(2n+1)^2}\right)}{1 + \frac{4z^2}{(2n+1)^2}}$$

$$\frac{\pi \sinh \pi z}{\cosh \pi z} = \sum_{n=0}^{\infty} \frac{\frac{8z}{(2n+1)^2}}{1 + \frac{4z^2}{(2n+1)^2}}$$

$$\boxed{\tanh \pi z = \frac{8z}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + 4z^2}}$$

4/ Set $st = u$, $du = s\,dt$ and get

$$I = \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{3/2} \frac{du}{s} = \frac{1}{s^{5/2}} \Gamma\left(\frac{5}{2}\right)$$

$$= s^{-5/2} \left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right) = s^{-5/2} \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)$$

$$= \boxed{\frac{3\sqrt{\pi}}{4} s^{-5/2}}$$

5/ $$\int_0^1 t^{a-1} (1-t)^{4-a-1} dt = \frac{\Gamma(a)\Gamma(4-a)}{\Gamma(a+4-a)}$$

$$= \frac{\Gamma(a)\Gamma(3-a)(3-a)}{\Gamma(4)} = \frac{\Gamma(a)\Gamma(2-a)(2-a)(3-a)}{3!}$$

$$= \frac{\Gamma(a)\Gamma(1-a)(1-a)(2-a)(3-a)}{6} = \boxed{\frac{(1-a)(2-a)(3-a)}{6} \frac{\pi}{\sin \pi a}}$$

6/ For large x, $\Gamma(x+1) \approx x^x e^{-x}\sqrt{2\pi x}$, thus $\Gamma(x+1)x^{-x} \approx e^{-x}\sqrt{2\pi x}$ for large x. Since e^{-x} dominates $\sqrt{x}$ for large x, the integral converges.

7/
$$\oint_{|z|=3/2} \Gamma(z)\,dz = 2\pi i\left[\operatorname{Res}(\Gamma(z),0) + \operatorname{Res}(\Gamma(z),-1)\right]$$

$$= 2\pi i\left[1 + \frac{(-1)^1}{1!}\right] = \boxed{0}.$$

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 7

1\. The infinite product itself $\prod_{n=1}^{\infty} (1 - \frac{z}{r_n})^{m_n}$ gives no hint of having any singularities. In fact, if this product converges, it represents an entire function. The function $g(z)$ is therefore, iself an entire function.

2\. The most general form for an entire function without zeros, $g(z)$, is $e^{h(z)}$, where $h(z)$ is any entire function. Thus we can replace $g(z)$ in (5) by $e^{h(z)}$ and get

$$f(z) = e^{h(z)} \prod_{n=0}^{\infty} (1 - \frac{z}{r_n})^{m_n} \tag{1}$$

where $h(z)$ is an arbitrary entire function. This formula (1) is the most general form for an entire function $f(z)$ having zeros at r_n of multiplicity m_n, $n=1, 2, \ldots$, provided the r_n tend to infinity fast enough to make (1) converge. In Conjecture 4 we will modify (1) so that it always converges.

3\. The n^{th} term test for infinite products

A necessary (but not sufficient) condition for the convergence of the infinite product

$$\prod_{n=0}^{\infty} (1 + u_n)$$

is that

$$\lim_{n \to \infty} u_n = 0 .$$

4. Weierstrass' factor theorem

Suppose $f(z)$ is a given entire function having zeros at r_n of multiplicity m_n, where $n = 1, 2, 3, \ldots$. Then $f(z)$ can be expressed in the form of the convergent product

$$f(z) = e^{h(z)} z^{m_o} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{r_n}\right) \exp\left(\frac{z}{r_m} + \frac{1}{2}\left(\frac{z}{r_n} \right)^2 + \ldots \frac{1}{k_n}\left(\frac{z}{r_n} \right)^{k_n}\right)\right]^{m_n}$$

where $h(z)$ is some entire function. The constants k_n, $n = 1, 2, 3, \ldots$, must be determined so as to make the infinite product converge. The factor z^{m_o} reveals a zero of multiplicity m_o at the origin and is to be suppressed in case the origin is not a zero of the given function $f(z)$.

5. Mittag - Leffler's partial fractions theorem

Suppose $f(z)$ is a given meromorphic function having poles at the points $z = r_n$ for positive integral n, and suppose that the principal part of $f(z)$ at r_n is given as

$$P_n(z) = \sum_{k=1}^{m_n} a_{n,k}(z - r_n)^{-k} .$$

Then $f(z)$ can be expressed in the form

$$f(z) = g(z) + \sum_{n=1}^{\infty} [P_n(z) + h_n(z)]$$

where $g(z)$ is some entire function and the functions $h_n(z)$ are polynomials selected so as to make the infinite series converge for all $z \neq r_n$.