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VARIATIONS ON VIETA'S AND WALLIS' S PRODUCTS FOR PI

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1. Introduction

There are many expressions in the mathematical literature for the number π . The beautiful infinite product of radicals

(1.1) $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ \cdots$$

due to Vieta [5] in 1592, is one of the oldest noniterative analytical expressions for π. The Wallis's product [6] dating from 1655

(1.2) $$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots$$

is also most remarkable. Both are usually included in any list of interesting expressions for π [2]. Historically, these were the first two infinite products found. (For more history see [1] and [3].)

In a recent note in the American Mathematical Monthly [4] a (possibly new) product was given which contained both of the above classical results as special cases.

The VWP:

(1.3) $$\frac{2}{\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}}_{(n\ radicals)}\quad\prod_{n=1}^{\infty}\frac{2^{p+1}n-1}{2^{p+1}n}\cdot\frac{2^{p+1}n+1}{2^{p+1}n}\ .$$

We will call (1.3) the VWP (Vieta-Wallis Product). While (1.1) and (1.2) seem unrelated, the expression (1.3) shows that they are both special cases of the VWP which is a more general "double product". The first product in the VWP consists of the first p factors of Vieta's original infinite product (1.1). The second product in the VWP is a Wallis-like product. We say this because the case where p = 0 gives us the original Wallis's product (1.2), and for other values of p it is the original Wallis's product with factors deleted. Notice also that the Wallis-like product in the VWP provides us with the error factor needed to make the Vieta product (1.1) exact when only a finite number of factors are used . We will return to the VWP in the next section and examine these features in detail.

In this paper we show how many new variations of the VWP can be obtained. One example is

(1.4) $$\frac{3}{\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}}}}}_{\leftarrow---n\ radicals---\rightarrow}\quad\prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+1}n-1}{3\cdot 2^{p+1}n}\cdot\frac{3\cdot 2^{p+1}n+1}{3\cdot 2^{p+1}n}\right).$$

We will show how to derive (1.4) in section 5.

All the mathematical derivations in this paper, with one exception, require only a precalculus background. The exception is the infinite product expansion for the sine function. This expression is motivated in section 3. This material can be presented in courses in precalculus, calculus, and real analysis. In addition, the convergence of these products could be studied on a computer, making the material suitable for courses in numerical analysis and computer programming. Exercises for students have been included that should be useful in the classroom.

2. A close look at the VWP

We now examine in detail how the VWP has the features mentioned in the introduction. The VWP given in (1.3) yields Vieta's product (1.1) as the limiting case as p goes to infinity, and the Wallis's product (1.2) as the case p=0. For each intermediate value of p = 1, 2, 3, ... we obtain "united Vieta-Wallis-like products":

p=0: $$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{11\cdot 13}{12\cdot 12}\cdots \quad \text{(original Wallis's product)}$$

p=1: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{19\cdot 21}{20\cdot 20}\cdots$$

p=2: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{23\cdot 25}{24\cdot 24}\cdot\frac{31\cdot 33}{32\cdot 32}\cdots$$

p=3: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{31\cdot 33}{32\cdot 32}\cdot\frac{47\cdot 49}{48\cdot 48}\cdot\frac{63\cdot 65}{64\cdot 64}\cdots$$

...

$p\to\infty$: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots \quad \text{(Vieta's original product).}$$

An examination of the above special cases of the VWP shows that each time we increase p by one, we increase the Vieta's product by one new radical factor, and remove alternate factors from the Wallis-like product. The first author unexpectedly discovered the VWP while trying to derive Vieta's product (1.1).

Problems:

2.1 Using a calculator, show that the original Wallis product (the case where $p = 0$ above), converges slowly. Note that $2/\pi = 0.636619772\cdots$, while the Wallis product involving 10 factors (denominator $20 \cdot 20$) is $0.651953423\cdots$.

2.2 Using four Wallis-like factors in the VWP to approximate $2/\pi$ for p = 1, 2, 3, and a calculator, show that the VWP gives the following values:

p = 1, VWP = 0.643877899...; p = 2, VWP = 0.63882540...; p = 3, VWP = 0.637170412... ..

3. The infinite product expansion for *sin x*

In this section, we provide intuitive motivation for Euler's infinite product expansion of the sine function:

(3.1) $$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = x\prod_{n=1}^{\infty}\left(\frac{\pi n - x}{\pi n}\cdot\frac{\pi n + x}{\pi n}\right).$$

Suppose we were required to find a polynomial with zeroes at the three points $x = -\pi, 0$, and π. The most general solution is

(3.2) $$p(x) = c(x+\pi)x(x-\pi) = -\pi^2 cx + cx^3,$$

where c is a constant. If in addition, we require that the polynomial have derivative $p'(0) = 1$, then we see that $c = -1/\pi^2$ and we get

(3.3) $$p(z) = x\left(1-\frac{x^2}{\pi^2}\right),$$

which gives the first two factors of the product (3.1). Notice that the function sin(x) is approximated by (3.3) for small x since sin(x) has zeroes at these three points as well as a derivative which is 1 when $x = 0$. But sin(x) also has zeroes at all integer multiples of π, so we are not surprised that the expression (3.1) is a valid representation for sin(x).

We have given intuitive motivation for this infinite product. This product converges for all values of x. A full discussion of the theory behind our motivation is found in the classic textbook [7]. In the next section we will see how (3.1) generates the Wallis-like products in our VWP and its extensions.

Problems:

3.1 Set $\theta = \pi/2$ in Euler's product expansion (3.1) for $\sin\theta$ and obtain the original Wallis product (1.2).

3.2 In advanced books on analysis [7, p. 32], it is shown that an infinite product of the form $(1-a_1)(1-a_2)(1-a_3)\cdots$, (where all the $a_k > 0$), converges if the corresponding series $a_1 + a_2 + a_3 + \cdots$ converges. Show that our infinite product for $\sin x$, (3.1), converges for all x.

4. The derivation of extended VWP's

To derive the VWP (1.3) and our new extensions we start by applying the double angle formula for the sine function p times to obtain

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}$$

$$= 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}$$

$$\ldots$$

(4.1) $$\sin\theta = 2^p\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\sin\frac{\theta}{2^p}$$

Next we use the infinite product (3.1) for the sine function, (valid for all x), with $x=\theta/2^p$, to replace the last factor in (4.1). We get after dividing by θ

$$\text{(4.2)} \qquad \frac{\sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\prod_{n=1}^{\infty}\left(\frac{2^p\pi n-\theta}{2^p\pi n}\cdot\frac{2^p\pi n+\theta}{2^p\pi n}\right).$$

We evaluate each of the cosine factors in (4.2) in terms of $\cos\theta$ by repeated use of the half-angle formula for the cosine. (Here we will assume $-\pi/2\le\theta\le\pi/2$ so that the cosines are never negative.)

$$\cos\frac{\theta}{2}=\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$$

$$\cos\frac{\theta}{2^2}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}$$

...

$$\text{(4.3)} \qquad \cos\frac{\theta}{2^p}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}$$
$$(p\ \ radicals)$$

Combining (4.3) with (4.2) we obtain

$$\text{(4.4)} \qquad \frac{\sin\theta}{\theta}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}\ \prod_{n=1}^{\infty}\left(\frac{2^p\pi n-\theta}{2^p\pi n}\cdot\frac{2^p\pi n+\theta}{2^p\pi n}\right)$$
$$(n\ radicals)$$

If we set $\theta=\pi/2$ in (4.4) and simplify we obtain the VWP (1.3). The new relation (1.4) was obtained from (4.4) by setting $\theta=\pi/6$. By letting $\theta=r\pi/s$ where r and s are natural numbers with $r<s$, we can obtain new extended VWP relations provided both $\sin(\theta)$ and $\cos(\theta)$ are known exactly in closed form.

5. Further extensions of the VWP

Now we try values of θ other than $\pi/2$ in (4.7) to obtain new VWP expressions. If we set $\theta = \pi/6$ in (4.7) we get

$$(5.1)\qquad \frac{3}{\pi} = \prod_{n=1}^{p} \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}}}}}_{\longleftarrow --- n\ radicals --- \longrightarrow} \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+1}n-1}{3\cdot 2^{p+1}n}\cdot\frac{3\cdot 2^{p+1}n+1}{3\cdot 2^{p+1}n}\right).$$

It is interesting to examine special cases. We begin with $p = 0$ so that no radicals appear. We get

$$(5.2)\qquad \frac{3}{\pi} = \prod_{n=1}^{\infty}\left(\frac{6n-1}{6n}\frac{6n+1}{6n}\right) = \frac{5}{6}\frac{7}{6}\cdot\frac{11}{12}\frac{13}{12}\cdot\frac{17}{18}\frac{19}{18}\cdots$$

This last relation is a generalization of the original Wallis product of rational factors. (1.2). Next we try $p = 1$ and get

$$(5.3)\qquad \frac{3}{\pi} = \sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}\,\frac{11}{12}\frac{13}{12}\cdot\frac{23}{24}\frac{25}{24}\cdot\frac{35}{36}\frac{37}{36}\cdot\frac{47}{48}\frac{49}{48}\cdots\ .$$

When we set $p = 2$ we get

$$(5.4)\qquad \frac{3}{\pi} = \sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}}\,\frac{23}{24}\frac{25}{24}\cdot\frac{47}{48}\frac{49}{48}\cdot\frac{71}{72}\frac{73}{72}\cdot\frac{95}{96}\frac{97}{96}\cdots\ .$$

The following table shows the results of our efforts to expand (4.7) with various values of θ.

θ	**Vieta Wallis Product**
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2\pi} = \prod_{n=1}^{p} \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\left(\frac{1}{2}\right)^2}}}}}_{\longleftarrow --- n\,radicals ---- \longrightarrow} \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p}n-1}{3\cdot 2^{p}n}\cdot\frac{3\cdot 2^{p}n+1}{3\cdot 2^{p}n}\right)$

$\frac{\pi}{4}$	$\frac{2\sqrt{2}}{\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{2^{p+2}n-1}{2^{p+2}n}\cdot\frac{2^{p+2}n+1}{2^{p+2}n}\right)$ $\longleftarrow - -\ n\,radicals - - - - \longrightarrow$
$\frac{\pi}{5}$	$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{8}\left(1+\sqrt{5}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{5\cdot 2^{p}n-1}{5\cdot 2^{p}n}\cdot\frac{5\cdot 2^{p}n+1}{5\cdot 2^{p}n}\right)$ $\longleftarrow - -\ n\,radicals - - - - \longrightarrow$
$\frac{\pi}{6}$	$\frac{3}{\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+1}n-1}{3\cdot 2^{p+1}n}\cdot\frac{3\cdot 2^{p+1}n+1}{3\cdot 2^{p+1}n}\right)$ $\longleftarrow - -\ n\,radicals - - - - \longrightarrow$
$\frac{\pi}{10}$	$\frac{5\left(\sqrt{5}-1\right)}{2\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(5+\sqrt{5}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{5\cdot 2^{p+1}n-1}{5\cdot 2^{p+1}n}\cdot\frac{5\cdot 2^{p+1}n+1}{5\cdot 2^{p+1}n}\right)$ $\longleftarrow - -\ n\,radicals - - - - \longrightarrow$
$\frac{\pi}{12}$	$\frac{3\sqrt{2}\left(\sqrt{3}-1\right)}{\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+2}n-1}{3\cdot 2^{p+2}n}\cdot\frac{3\cdot 2^{p+2}n+1}{3\cdot 2^{p+2}n}\right)$ $\longleftarrow - -\ n\,radicals - - - - \longrightarrow$

Problems:

5.1 Using $\cos(\pi/5) = \dfrac{1+\sqrt{5}}{4}$, and $\sin(\pi/5) = \sqrt{\dfrac{5-\sqrt{5}}{8}}$, derive the entry where $\theta = \pi/5$ in the above table.

5.2 Using the entry where $\theta = \pi/5$ in the above table, derive the following by taking p = 0 (no radicals). $\dfrac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi} = \left(\dfrac{4\cdot 6}{5\cdot 5}\right)\left(\dfrac{9\cdot 11}{10\cdot 10}\right)\left(\dfrac{14\cdot 16}{15\cdot 15}\right)\left(\dfrac{19\cdot 21}{20\cdot 20}\right)\cdots$.

5.3 Using the entry where $\theta = \pi/5$ in the above table, derive similar formulas by taking p =1, 2, 3. Answers:

$p = 1$,

$$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi} = \sqrt{\frac{5+\sqrt{5}}{8}}\left(\frac{9\cdot 11}{10\cdot 10}\right)\left(\frac{19\cdot 21}{20\cdot 20}\right)\left(\frac{29\cdot 31}{30\cdot 30}\right)\left(\frac{39\cdot 41}{40\cdot 40}\right)\cdots .$$

$p = 2$,

$$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{8}}}\left(\frac{19\cdot 21}{20\cdot 20}\right)\left(\frac{39\cdot 41}{40\cdot 40}\right)\left(\frac{59\cdot 61}{60\cdot 60}\right)\cdots .$$

$p = 3$,

$$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{8}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{8}}}}\left(\frac{39\cdot 41}{40\cdot 40}\right)\left(\frac{79\cdot 81}{80\cdot 80}\right)\left(\frac{119\cdot 121}{120\cdot 120}\right)\cdots$$

5.4 Using the entry where $\theta = \pi/5$ in the above table, derive the following Vieta-like product by taking $p \to \infty$.

$$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{8}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{8}}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{8}}}}}\cdots$$

5.5 Using the formulas obtained in problems 5.2 and 5.3, derive the following product:

$$\sqrt{\frac{5+\sqrt{5}}{8}} = \left(\frac{4\cdot 6}{5\cdot 5}\right)\left(\frac{14\cdot 16}{15\cdot 15}\right)\left(\frac{24\cdot 26}{25\cdot 25}\right)\left(\frac{34\cdot 36}{35\cdot 35}\right)\cdots .$$

This completes our investigation of the expanded Vieta-Wallis Product.

References

[1] P. Beckmann, *A History of Pi,* St. Martin's Press, New York, New York, 1971

[2] L. Berggren, J. Borwein and P. Borwein, *Pi, A Source Book,* Springer, New York, 1997, pp. 686-689.

[3] W. Dunham, *Journey Through Genious, The Great Theorems of Mathematics,* Penguin, 1990.

[4] T. J. Osler, *The united Vieta's and Wallis's products for pi,* American Mathematical Monthly, 106 (1999), pp. 774-776.

[5] F. Vieta, Variorum de Rebus Mathematicis Reponsorum Liber VII, (1593) in: *Opera Mathematica, (reprinted)* Georg Olms Verlag, Hildesheim, New York, 1970, pp. 398-400 and 436-446.

[6] J. Wallis, Computation of π by Successive Interpolations, (1655) in: *A Source Book in Mathematics, 1200-1800* (D. J. Struik, Ed.), Harvard University Press, Cambridge, MA, 1969, pp. 244-253.

[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis,* Cambridge University Press, Fourth Ed., 1927, p. 137.