A collection of numbers whose proof of irrationality is like that of the number e

Thomas J. Osler and Nicholas Stugard
Mathematics Department
Rowan University
Glassboro, NJ 08028

Osler@rowan.edu

Introduction

In some elementary courses we show that $\sqrt{2}$ is irrational. We also show that other roots like $\sqrt{3}$, $\sqrt[3]{2}$, etc., are irrational. Much less often, we show that the number *e*, the base of the natural logarithm, is irrational, even though a proof is available that uses only elementary calculus (see [3], p. 425). In this short note we repeat this proof and find a collection of other numbers that can be proved to be irrational in a similar way. While this work is not new (see [2]), the presentation here is simpler than others we have seen. It is appropriate for students in calculus, advanced calculus and real analysis.

We can define *e* by the infinite series $e=\sum_{n=0}^{\infty}\frac{1}{n!}$. To show that *e* is irrational, we give a proof by contradiction. Assume that $e=\frac{N}{D}$, where *N* and *D* are natural numbers with no common factor. Then we have

$$\frac{N}{D}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\frac{1}{D!}+\sum_{n=D+1}^{\infty}\frac{1}{n!}.$$

Multiply both sides by *D*! to get

$$N(D-1)!=D!+\frac{D!}{1!}+\frac{D!}{2!}+\frac{D!}{3!}+\cdots\frac{D!}{D!}+\sum_{n=D+1}^{\infty}\frac{D!}{n!}. \quad (1)$$

The left-hand side of (1) is an integer, and so are all the terms before $\sum_{n=D+1}^{\infty} \frac{D!}{n!}$ on the right-hand side. Thus $\sum_{n=D+1}^{\infty} \frac{D!}{n!}$ must be an integer. But

$$\sum_{n=D+1}^{\infty} \frac{D!}{n!} = \frac{1}{D+1} + \frac{1}{(D+1)(D+2)} + \frac{1}{(D+1)(D+2)(D+3)} + \cdots. \qquad (2)$$

$$< \frac{1}{D+1} + \frac{1}{(D+1)^2} + \frac{1}{(D+1)^3} + \cdots.$$

This last sum is a geometric series of the type $x + x^2 + x^3 + \cdots = \frac{x}{1-x}$, so we get after letting $x = \frac{1}{D+1}$, $\sum_{n=D+1}^{\infty} \frac{D!}{n!} < \frac{\frac{1}{D+1}}{1 - \frac{1}{D+1}} = \frac{1}{D}$. This means that $0 < \sum_{n=D+1}^{\infty} \frac{D!}{n!} < 1$, and thus $\sum_{n=D+1}^{\infty} \frac{D!}{n!}$ cannot be an integer as required. This is the contradiction we sought. We have shown that e must be irrational.

An analysis of the proof

In this section we outline the features of the above proof that will be used in Theorems 1 and 2 to prove that a variety of numbers are irrational. This will make it simpler to prove these new theorems, since several common features will not require repeating.

Suppose we are to prove that the number a is irrational. If the method shown above is to provide a successful proof, we expect the following:

Step A. The number is given as an infinite series $a = \sum_{n=0}^{\infty} \frac{1}{a_n}$.

Step B. Assume that a is rational and express it as the fraction

$$\frac{N}{D} = \sum_{n=0}^{\infty} \frac{1}{a_n}, \tag{3}$$

where *N* and *D* are integers.

Step C. Select a multiplicative factor *FD* and an upper limit *P* for the finite sum. Now multiplying (3) by *FD* gives us

$$FN = \sum_{n=0}^{P} \frac{FD}{a_n} + \sum_{n=P+1}^{\infty} \frac{FD}{a_n}. \tag{4}$$

We see that FN is an integer, and the finite sum $\sum_{n=0}^{P} \frac{FD}{a_n}$ must be shown to be an integer.

Step D. The second sum $\sum_{n=P+1}^{\infty} \frac{FD}{a_n}$ must be shown to satisfy the inequality

$$0 < \left| \sum_{n=P+1}^{\infty} \frac{FD}{a_n} \right| < 1.$$

Step E. Relation (4) now states that an integer equals an integer plus a non zero number whose absolute value is less than one. This is impossible and we have proved that *a* is irrational.

Familiar numbers that are irrational

In the following theorem we list several values of familiar functions that the above method will show to be irrational.

Theorem 1: All the numbers

$$e^{1/M} = \sum_{n=0}^{\infty} \frac{1}{n!M^n}, \ \sin(1/M) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!M^{2n+1}}, \ \cos(1/M) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!M^{2n}},$$

$$\sinh(1/M) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!M^{2n+1}}, \text{ and } \cosh(1/M) = \sum_{n=0}^{\infty} \frac{1}{(2n)!M^{2n}},$$

with $M = 1, 2, 3, \cdots$ are irrational.

Proof: We refer to the steps in the *analysis of the proof* given in the previous section. Steps A and B are the same for all our numbers. In step C, for the number $e^{1/M}$ use the upper limit $P = D$ and the multiplicative factor $FD = D!M^D$. Equation (4) now reads

$$N(D-1)!M^D = D!M^D + \frac{D!M^{D-1}}{1!} + \frac{D!M^{D-2}}{2!} + \frac{D!M^{D-3}}{3!} + \cdots + \frac{D!}{D!} + \sum_{n=D+1}^{\infty} \frac{D!}{n!M^{n-D}}.$$

It is clear that all terms left of the infinite sum are integers. In step D we see that

$$\begin{aligned}\sum_{n=D+1}^{\infty} \frac{D!}{n!M^{n-D}} &= \frac{1}{(D+1)M} + \frac{1}{(D+1)\left(D+2\right)M^2} + \frac{1}{(D+1)\left(D+2\right)(D+3)M^3} + \cdots \\ &< \frac{1}{(D+1)M} + \frac{1}{(D+1)^2M^2} + \frac{1}{(D+1)^3M^3} + \cdots \\ &= \frac{1}{(D+1)M-1}.\end{aligned}$$

It is clear that this last expression is less than 1. Step E now follows and the theorem is proved for the number $e^{1/M}$.

For the numbers $\sin(1/M)$ and $\sinh(1/M)$, in step C use the upper limit

$$P = \begin{cases} (D-1)/2 & \text{for odd } D \\ D/2 & \text{for } D \text{ even} \end{cases}$$

and the multiplicative factor $FD = (2P+1)!M^{2P+1}$. For $\cos(1/M)$ and $\cosh(1/M)$ use the upper limit

$$P = \begin{cases} (D+1)/2 & \text{for odd } D \\ D/2 & \text{for } D \text{ even} \end{cases}$$

and the multiplicative factor $FD = (2P)!M^{2P}$. We leave step D to the reader for these last numbers. Theorem 1 is now proved.

Generalized hypergeometric functions

Beyond the trigonometric, exponential and hyperbolic functions there is a large world of *special functions* which include the Bessel functions, the gamma function, the error function, etc. For our purpose, we will use a class of functions known as *generalized hypergeometric functions* (see [1], page 204). To understand these functions, we first introduce the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

This is a generalization of the factorial since $(1)_n = n!$. The generalized hypergeometric function can now be defined as

$${}_pF_q\left(\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

Many elementary functions can be expressed as generalized hypergeometric functions.

For example, ${}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; x\right) = {}_1F_1\left(\begin{matrix} a \\ a \end{matrix}; x\right) = e^x$, and ${}_1F_0\left(\begin{matrix} 1 \\ - \end{matrix}; x\right) = {}_2F_1\left(\begin{matrix} 1 & a \\ & a \end{matrix}; x\right) = \frac{1}{1-x}$.

We can now prove the following theorem:

Theorem 2: Let $b_1, b_2, b_3, \cdots, b_q$ be positive integers. The numbers

$${}_1F_q\left(\begin{matrix} & & 1 & \\ b_1 & b_2 & \cdots & b_q \end{matrix}; \frac{1}{M}\right) = \sum_{n=0}^{\infty} \frac{1}{(b_1)_n (b_2)_n \cdots (b_q)_n M^n}$$

with $M = 1, 2, 3, \cdots$ are irrational.

Proof: Again we use the steps in the *analysis of the proof* given in a previous section to abbreviate the discussion. In step C use the upper limit $P = D$ and the multiplicative factor $FD = (b_1)_D (b_2)_D \cdots (b_q)_D M^D$. The remaining steps are easy and left to the reader.

References

[1] Graham, Ronald L., Knuth, Donald L., Patashnik, Oren, *Concrete Mathematics*, Addison-Wesley Publishing, Massachusetts, 1989.

[2] Lord, N. J. and Sandor J., *On some irrational series*, Mathematics Magazine, 65(1992), pp. 53-55.

[3] Spivak, Michael, *Calculus*, (Third Edition), Publish or Perish, Inc., Houston, 1994.