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EXTRAS

LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES USED TO GENERALIZE FORMULAS OF WALKER AND CAUCHY

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1. Introduction

There are two extensions of the Leibniz rule for the Nth derivative of the product of two functions

$$D^N(uv) = \sum_{n=0}^{N} \binom{N}{n} D^{N-n} u \; D^n v,$$

one due to J. J. Walker in 1800

$$(1) \qquad D^N[f^N uv] = \sum_{n=0}^{N} W(N, n),$$

where

$$W(\alpha, \omega) = \binom{\alpha}{\omega} D^{\alpha-\omega} [f^{\alpha-\omega} v] \, D^{\omega-1} [f^{\omega} u'],$$

and the other due to A. L. Cauchy in 1826

$$(2) \qquad D^{N-1}[f^N D(uv)] = \sum_{n=0}^{N} C(N, n)$$

where

$$C(\alpha, \omega) = \binom{\alpha}{\omega} D^{\alpha-\omega-1}[f^{\alpha-\omega} v'] \, D^{\omega-1}[f^{\omega} u'].$$

In a recent paper, H. N. Gould [2] discussed the history of (1) and (2), showed several methods of proof, and mentioned their relation to combinatorial identities.

In this paper we extend (1) and (2) to the case where the non-negative integer N is replaced by arbitrary (rational, irrational or complex) α. The resulting derivatives of the form $D_z^{\alpha} g(z)$ are called fractional derivatives and can be defined by

$$(3) \qquad D_z^{\alpha} g(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_0^{(z+)} \frac{g(t)dt}{(t-z)^{\alpha+1}} \, .$$

The definition (3) is explained in detail in [3].

In the next section, we prove that the extension of W a l k e r's fo[rm]ula (1) to fractional derivatives has a series and an integral form

$$\text{(4)} \qquad D_z^{\alpha}[f^{\alpha}\, uv] = \sum_{n=-\infty}^{\infty} W(\alpha,\ an+\gamma)a = \int_{-\infty}^{\infty} W(\alpha,\ \omega+\gamma)d\omega,$$

and that the extension of C a u c h y's formula (2) also assumes a seri[es] and an integral form

$$\text{(5)} \qquad D_z^{\alpha-1}[f^{\alpha}\, D(uv)] = \sum_{n=-\infty}^{\infty} C(\alpha,\ an+\gamma)a = \int_{-\infty}^{\infty} C(\alpha,\ \omega+\gamma)\, d\omega.$$

Here $0 < a \leqslant 1$ and γ is an arbitrary complex number.

In Section **3** we examine special cases of (4) and (5) in which specif[ic] functions are selected for f, u and v.

2. Proofs of the Generalized Formulas

T h e o r e m. (i) *Let R be a simply connected open set in the comple[x] plane having the origin as an interior or boundary point.*

(ii) *Let $u(z) = z^P g(z)$ and $v(z) = z^Q h(z)$ where $g(z)$ and $h(z)$ are analytic o[n] $R \cup \{0\}$.*

(iii) *Let $f(z)$ be analytic and never zero on $R \cup \{0\}$.*

(iv) *Assume that the curves $C(z) = \{t \mid (t-z)/f(t) \mid = \mid -z/f(0) \mid\}$ ar[e] simple and closed for each z such that $C(z) \subset R \cup \{0\}$. Assume also that eac[h] curve defined by $\{t \mid \ |(t-z)/f(t)| = constant\}$ interior to $C(z)$ is simple an[d] closed.*

(v) *Call $S = \{z \mid C(z) \subset R \cup \{0\}\}$.*

Then for $z \in S$, $0 < a \leqslant 1$, and all α and γ such that $\binom{\alpha}{an+\gamma}$ defined, the generalized formula of Walker (4) is true for $Re(P) > 0$, $Re(Q) > -1$, and the generalized formula of Cauchy (5) is true for $Re(P) >$ and $Re(Q) > 0$.

P r o o f. In [3, p. 3, (1.4 a)] the author proved that

$$D_z^{\alpha}\, UV = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}\, [Uq^{an+\gamma}]\, D_z^{an+\gamma-1}\, [V' q^{-an-\gamma}],$$

where $V(0) = 0$. Replacing U by $f^{\alpha}v$, V by u and q by $1/f$ we get the seri[es] form of W a l k e r's formula (4) at once. In [4] the author showed th[at] $\sum_{n=-\infty}^{\infty} \dots a$, could be replaced by $\int_{-\infty}^{\infty} \dots d\omega$ and an by ω in the series for[m] of the Leibniz rule to get a valid integral form. Thus we have shown that t[he] generalized formulas of W a l k e r are simply formulas from [3] and [4] with a change in notation.

To obtain the generalized formulas of Cauchy we start with (4) and „factor a D_z" from both sides in the form

$$D_z^{\alpha-1} D_z [f^\alpha uv] = \sum_{n=-\infty}^{\infty} \ldots D_z^{\alpha-an-\gamma-1} D_z [f^{\alpha-an-\gamma} v]. \tag{6}$$

(One can say that $D_z^{\alpha-1} D_z G(z) = D_z^\alpha G(z)$ provided $G(0) = 0$. See [3, p. 9]). Now (6) becomes

$$\begin{aligned} D_z^{\alpha-1} [f^\alpha D_z (uv)] + \alpha D^{\alpha-1} [f^{\alpha-1} uvf'] = \sum \ldots D_z^{\alpha-an-\gamma-1} [f^{\alpha-an-\gamma} v'] + \\ + \sum \ldots (\alpha - an - \gamma) D_z^{\alpha-an-\gamma-1} [f^{\alpha-an-\gamma-1} vf']. \end{aligned} \tag{7}$$

If we replace α by $\alpha - 1$ and v by vf' in the generalized Walker's formula (4), we see that the second term on the left hand side of (7) equals the second term on the right hand side of (7). Canceling these terms we have the series form of the generalized Cauchy's formula (5).

The integral form of (5) follows at once from [4]. It is a simple matter to check that the hypothesis of this theorem is sufficient to assure that he above formulas and manipulations are valid by comparison with [3, heorem 4.1 and Corollary 4.1].

3. Examples

In this section we select specific functions for $u(z)$, $v(z)$, $f(z)$ and we examine the resulting generalized formulas of Cauchy (5). We do not give any examples of the generalized formulas of Walker (4), since the above proof demonstrated that (4) is simply a result given previously in [3] and [4] with a change in notation.

We give only the series form of (5). In each example $\sum_{-\infty}^{\infty} \ldots a$ can be replaced by $\int_{-\infty}^{\infty} \ldots d\omega$ and an can be replaced by ω to get the corresponding integral form.

In all the examples given below, we have set $u(z) = z^P$ and $v(z) = z^Q$ with $\mathrm{Re}(P)$ and $\mathrm{Re}(Q)$ positive in (5). The resulting fractional derivatives were found in [1, pp. 186—7].

Example 1. Let $f(z) = e^{Az}$ in (5) and get

$$\frac{\Gamma(P + a + 1)\, {}_1F_1 \left(\begin{matrix} P + Q \\ P + Q - \alpha + 1 \end{matrix} \middle| \alpha A z \right)}{\Gamma(P + Q - \alpha + 1)\, \Gamma(\alpha + 1)\, \Gamma(P + 1)\, \Gamma(Q + 1)} =$$

$$= \sum_{n=-\infty}^{\infty} \frac{{}_1F_1 \left(\begin{matrix} Q \\ Q - \alpha + an + \gamma + 1 \end{matrix} \middle| (\alpha - an - \gamma) A z \right) {}_1F_1 \left(\begin{matrix} P \\ P - an - \gamma + 1 \end{matrix} \middle| (an + \gamma) A z \right) a}{\Gamma(\alpha - an - \gamma + 1,\, \Gamma(Q - \alpha + an + \gamma + 1)\Gamma(an + \gamma + 1)\Gamma(P - an - \gamma + 1)}.$$

To obtain the generalized formulas of Cauchy we start with (4) and „factor a D_z" from both sides in the form

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$$= \sum_{n=-\infty}^{\infty} \frac{{}_1F_1\left(\begin{matrix} Q \\ Q - \alpha + an + \gamma + 1 \end{matrix}\middle| (\alpha - an - \gamma)Az\right) {}_1F_1\left(\begin{matrix} P \\ P - an - \gamma + 1 \end{matrix}\middle| (an + \gamma)Az\right) a}{\Gamma(\alpha - an - \gamma + 1, \Gamma(Q - \alpha + an + \gamma + 1)\Gamma(an + \gamma + 1)\Gamma(P - an - \gamma + 1)}.$$