02/29/04

PROOF WITH FEW WORDS
QUADRATIC CONVERGENCE OF THE AGM

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Let $0 < b_0 < a_0$ and define $\varepsilon_0 = a_0 - b_0$, also

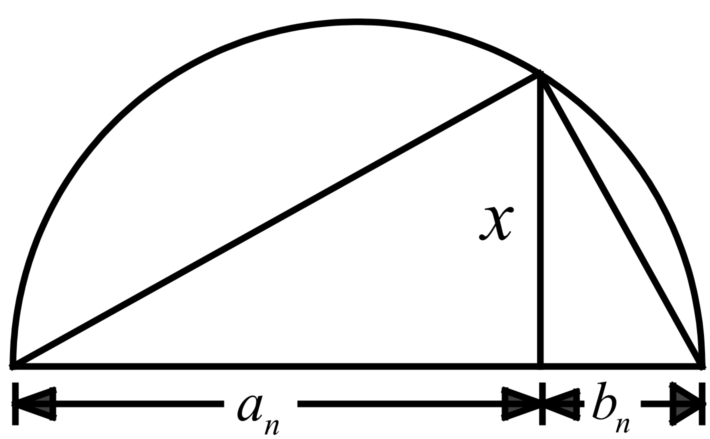
$$a_1 = \frac{a_0 + b_0}{2}, \ b_1 = \sqrt{a_0 b_0}, \ \varepsilon_1 = a_1 - b_1,$$
$$a_2 = \frac{a_1 + b_1}{2}, \ b_2 = \sqrt{a_1 b_1}, \ \varepsilon_2 = a_2 - b_2,$$
$$\dots ,$$

and in general

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \ b_n = \sqrt{a_{n-1} b_{n-1}}, \ \varepsilon_n = a_n - b_n .$$

The arithmetic-geometric mean is defined by $M(a_0, b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. We prove that this limit exists and that the convergence is quadratic ($\varepsilon_{n+1} < c\varepsilon_n^2$).

Below we have a semicircle on the diameter $a_n + b_n$.

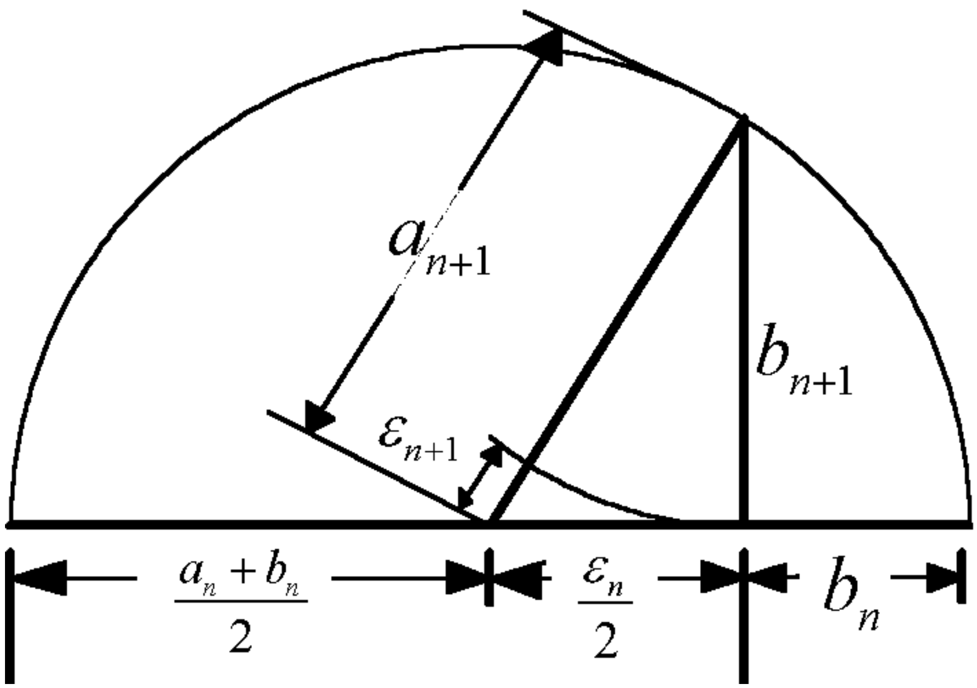


$$\frac{x}{a_n} = \frac{b_n}{x} \quad \Rightarrow \quad x = \sqrt{a_n b_n} = b_{n+1}$$

and the radius $r = \dfrac{a_n + b_n}{2} = a_{n+1}$.

$$\frac{a_n + b_n}{2} - b_n = \frac{a_n - b_n}{2} = \frac{\varepsilon_n}{2}$$

Hence we have the diagram below:



$$b_0 < b_1 < b_2 < \cdots < b_n < \cdots < a_n \cdots < a_2 < a_1 < a_0$$

From the above figure it is clear that

$$\varepsilon_{n+1} < \frac{\varepsilon_n}{2} \quad \text{so} \quad \lim a_n = \lim b_n .$$

$$a_{n+1}^2 - b_{n+1}^2 = \frac{\varepsilon_n^2}{4}$$

$$\left(a_{n+1} + b_{n+1}\right)\left(a_{n+1} - b_{n+1}\right) = \frac{\varepsilon_n^2}{4}$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{4\left(a_{n+1} + b_{n+1}\right)} = \frac{\varepsilon_n^2}{8a_{n+2}} < \frac{\varepsilon_n^2}{8b_0},$$

Hence $M(a_0, b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and the convergence is quadratic.