CLASSROOM CAPSULES

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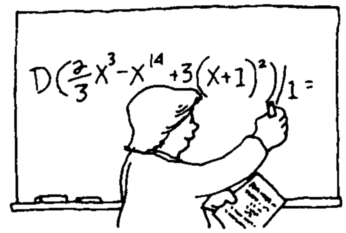

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

A Triple Angle Formula for Tangent

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As is well known,

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}. \tag{1}$$

In this capsule, we derive an alternative formula for $\tan 3\theta$ and show its applications.

Fact. *If* $\tan 3\theta$ *is defined,*

$$\tan^2\theta + \tan^2(60^\circ - \theta) + \tan^2(60^\circ + \theta) = 9\tan^2 3\theta + 6. \tag{2}$$

Proof. Let $x = \tan\theta$, and observe the following:

$$\begin{aligned}
&\tan^2\theta + \tan^2(60^\circ - \theta) + \tan^2(60^\circ + \theta)\\
&\quad = x^2 + \left(\frac{\sqrt{3} - x}{1 + \sqrt{3}x}\right)^2 + \left(\frac{\sqrt{3} + x}{1 - \sqrt{3}x}\right)^2\\
&\quad = x^2 + \frac{(\sqrt{3} - x)^2(1 - \sqrt{3}x)^2 + (\sqrt{3} + x)^2(1 + \sqrt{3}x)^2}{(1 - 3x^2)^2}\\
&\quad = x^2 + \frac{6x^4 + 44x^2 + 6}{(1 - 3x^2)^2}\\
&\quad = \frac{9x^6 + 45x^2 + 6}{(1 - 3x^2)^2}\\
&\quad = \frac{9(x^6 - 6x^4 + 9x^2) + 6(9x^4 - 6x^2 + 1)}{(1 - 3x^2)^2}\\
&\quad = 9\tan^2 3\theta + 6
\end{aligned}$$

by (1). So the assertion is proved.

Examples. Letting $\theta = 10^\circ$ and 20° in (2), we have

$$\tan^2 10^\circ + \tan^2 50^\circ + \tan^2 70^\circ = 9\tan^2 30^\circ + 6 = 9, \tag{3}$$

$$\tan^2 20^\circ + \tan^2 40^\circ + \tan^2 80^\circ = 9\tan^2 60^\circ + 6 = 33. \tag{4}$$

Hence, combining (3) and (4), and the values $\tan 30^\circ$ and $\tan 60^\circ$, we obtain

$$\tan^2 10^\circ + \tan^2 20^\circ + \tan^2 30^\circ + \cdots + \tan^2 80^\circ = \frac{136}{3}. \tag{5}$$

If we let $\theta = 5^\circ,\ 15^\circ$ and 25° in (2), we obtain

$$\tan^2 5^\circ + \tan^2 55^\circ + \tan^2 65^\circ = 9\tan^2 15^\circ + 6 = 69 - 36\sqrt{3}, \tag{6}$$

$$\tan^2 15^\circ + \tan^2 45^\circ + \tan^2 75^\circ = 9\tan^2 45^\circ + 6 = 15, \tag{7}$$

$$\tan^2 25^\circ + \tan^2 35^\circ + \tan^2 85^\circ = 9\tan^2 75^\circ + 6 = 69 + 36\sqrt{3}. \tag{8}$$

Thus, (6), (7), and (8) yield

$$\tan^2 5^\circ + \tan^2 15^\circ + \tan^2 25^\circ + \cdots + \tan^2 85^\circ = 153. \tag{9}$$

Therefore, it follows from (5) and (9) that

$$\tan^2 5^\circ + \tan^2 10^\circ + \tan^2 15^\circ + \cdots + \tan^2 80^\circ + \tan^2 85^\circ = \frac{595}{3}.$$

Remarks.

(i) For $\theta = 7.5^\circ$ and 22.5° the exact value of the left hand side of (2) is easily obtained.

(ii) From (2) we can derive a triple angle formula for cotangent:

$$\cot^2\theta + \cot^2(60^\circ - \theta) + \cot^2(60^\circ + \theta) = 9\cot^2 3\theta + 6.$$

So the formulas of (3)–(9) can be rewritten in terms of cotangent.

(iii) It is not likely to have a *simple* formula

$$\tan^2\theta + \tan^2(\phi - \theta) + \tan^2(\phi + \theta) = f(\tan 3\theta),$$

where ϕ is some specific angle and the right hand side is a function of $\tan 3\theta$ like the right hand side of (2).

The Murder Mystery Method for Determining Whether a Vector Field is Conservative

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We describe here a variation of the usual procedure for determining whether a vector field is conservative and, if it is, for finding a potential function. We have used this

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method, which we call the *murder mystery method*, in our own classes for many years; students love it.

It is helpful to make a diagram of the structure underlying potential functions and conservative vector fields. For functions of two variables, this is shown in the first drawing in Figure 1. The potential function f is shown at the top. Slanted lines represent derivatives of f; derivatives with respect to x go to the left, while derivatives with respect to y go to the right. The second line thus gives the components of $\vec{\nabla} f$. The bottom line shows the mixed second derivatives, which can of course be taken in either order.

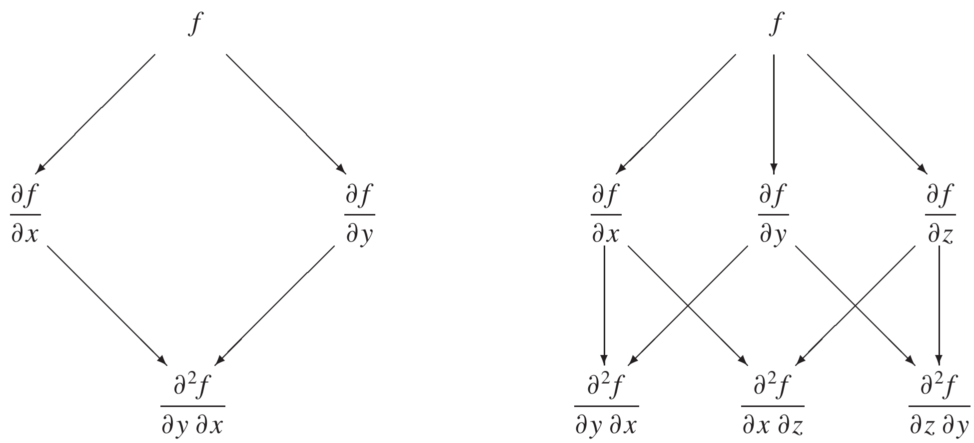

Figure 1. Potential functions made easy.

But we are not given f. Rather, we are given a vector field, such as

$$\vec{\boldsymbol{F}} = y\hat{\boldsymbol{\imath}} + (x + 2y)\hat{\boldsymbol{\jmath}} \tag{1}$$

and need to determine whether it is conservative. That is, we need to determine whether $\vec{\boldsymbol{F}}$ is the gradient of some potential function f. This is the second line of the diagram! We could start by checking the mixed derivatives. However, what we really want is the potential function; we should be moving up the diagram, not down. What happens if we simply integrate both components, as shown in the first drawing in Figure 2? The potential function is clearly contained in the results of these two integrals; it is just a question of combining them correctly.

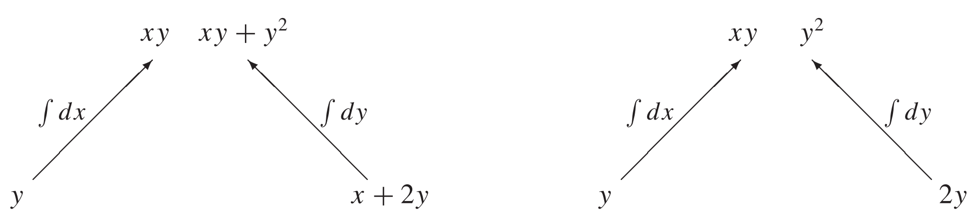

Figure 2. Finding 2-d potential functions by integration.

Furthermore, there is enough information here to determine whether $\vec{\boldsymbol{F}}$ is conservative in the first place; there is no need to check the derivatives. For example, had we been given the vector field

$$\vec{\boldsymbol{H}} = y\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} \tag{2}$$

and attempted the same procedure, we would obtain the second drawing in Figure 2. Simply by noticing that xy, a function of two variables, only occurs once, we see that $\vec{\boldsymbol{H}}$ is not conservative.

We describe this to students as a *murder mystery*. A crime has been committed by the unknown murderer f; your job is to find the identity of f by interviewing the witnesses. Who are the witnesses? The components of the vector field. What do they tell you? Well, you have to integrate ("interrogate") them! Now for the fun part.

If two witnesses say they saw someone with red hair, that doesn't mean the suspect has two red hairs! So if you get the same clue more than once, you only count it once.

On the other hand, some clues require corroboration. These witnesses were situated in such a way that each could only look in one direction. Thus, one witness, the x-component, only sees terms involving x, etc. If a clue contains more than one variable, it should have been seen by more than one witness! In fact, functions of n variables should occur precisely n times. In the case of the vector field $\vec{\boldsymbol{H}}$, the clue xy was only seen by one witness, not both; somebody is lying! In short, clues must be consistent.

Here is the Murder Mystery Method in a nutshell:

- *Integrate:* Integrate the x-component with respect to x, etc.
- *Check consistency:* Functions of n variables must occur exactly n times.
- *Combine clues:* Use each clue once to determine the potential function.
  (If the consistency check fails, the vector field is not conservative.)

The power of the murder mystery method is even more apparent in three dimensions. We encourage the reader to try to find a potential function for the vector field $\vec{\boldsymbol{G}}$ defined by

$$\vec{\boldsymbol{G}} = yz\,\hat{\boldsymbol{\imath}} + (xz + z)\,\hat{\boldsymbol{\jmath}} + (xy + y + 2z)\,\hat{\boldsymbol{k}} \tag{3}$$

using this method! The underlying structure is shown in the second drawing in Figure 1, where now y derivatives are shown going straight down, and z derivatives go to the right.

Consistency is traditionally checked by computing the last line of the appropriate diagram in Figure 1. We reiterate that this is not necessary with the Murder Mystery Method. This computation can be introduced afterwards to motivate the algebraic formula for curl in rectangular coordinates. (We save the geometric definition until after we have covered Stokes' Theorem.)

Finally, this method also generalizes to bases adapted to other coordinate systems, such as polar coordinates—especially since the "differentiation" step, namely the calculation of the curl, is omitted. For example, the natural polar basis consists of $\hat{\boldsymbol{r}}$, the unit vector in the radial direction, and $\hat{\boldsymbol{\phi}}$, the unit vector tangent to circles $r =$ constant. Noting that

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{4}$$

leads quickly to a version of the Murder Mystery Method in polar coordinates; the only real change is the need to include a factor of r before integrating with respect to ϕ. The procedure is similar in cylindrical or spherical coordinates.

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Visual Proof of Two Integrals

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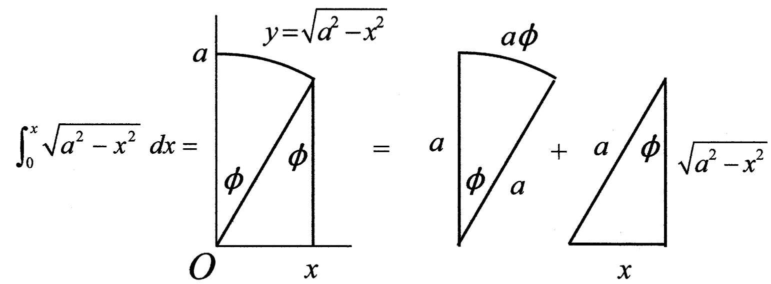
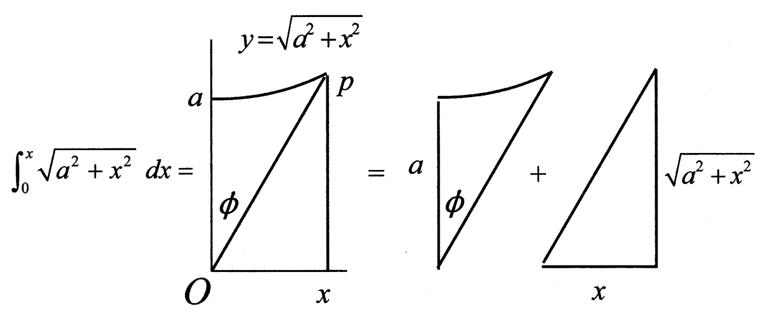


$$= \frac{a(a\,\phi)}{2} + \frac{x\sqrt{a^2-x^2}}{2}$$

$$= \frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2}$$

$$\int \sqrt{a^2-x^2}dx = \frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2} + C.$$

We assume (see [**1**] for example) that the hyperbolic branch $y = \sqrt{a^2+x^2}$ is parametrized by $x = a\sinh\phi$ and $y = a\cosh\phi$, where $\phi = \frac{2A}{a^2}$ and A is the area of the hyperbolic region aOp.



$$= \frac{a(a\,\phi)}{2} + \frac{x\sqrt{a^2+x^2}}{2}$$

$$= \frac{a^2 \sinh^{-1}(x/a)}{2} + \frac{x\sqrt{a^2+x^2}}{2}$$

$$\int \sqrt{a^2+x^2}dx = \frac{a^2 \sinh^{-1}(x/a)}{2} + \frac{x\sqrt{a^2+x^2}}{2}.$$

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Area Relations on the Skewed Chessboard

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To the most casual observer it is obvious that the area of the red (white) squares is equal to the area of the black squares on an ordinary chessboard or checkerboard. But does this still hold on a skewed chessboard such as in Figure 1? The purpose of this article is to answer this and other related questions.

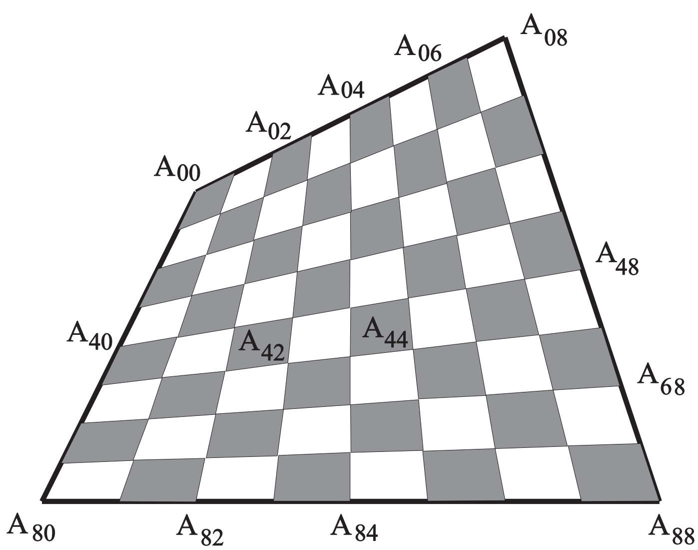

Figure 1.

First we consider more precisely what is meant by a skewed chessboard. Our *skewed chessboard* is any convex quadrilateral in which each side is divided into eight congruent segments whose corresponding endpoints are joined by cross-segments to form sixty-four non-overlapping quadrilaterals.

Since the sides of quadrilateral $A_{00}A_{08}A_{88}A_{80}$ are divided into eight congruent parts, it is natural to wonder whether the cross-segments, such as $A_{40}A_{48}$, are also divided into eight congruent parts. To prove that this is the case, we make repeated use of a result attributed to Pierre Varignon (1654–1722).

Varignon's Theorem. If B, D, F, H are the consecutive midpoints of the sides of quadrilateral *ACEG*, then *BDFH* is a parallelogram.

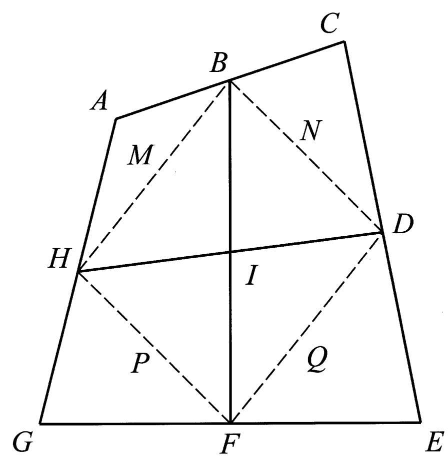

Figure 2.

Proof. ([**1**, p. 53]). By the midsegment, or midline theorem, the line through the midpoints B and D of $\triangle ACE$ is parallel to and one-half the third side, *AE*. Similarly *HF* is parallel to and one-half *AE* for $\triangle AGE$. Therefore, *BD* and *HF* are both parallel and congruent which implies that *BDFH* is a parallelogram.

As a corollary of Varignon's Theorem, we note that *HD* and *BF* bisect each other. This is the actual result needed.

Theorem 1. *Each cross-segment of an 8 by 8 skewed chessboard is divided into eight congruent segments.*

Proof. By repeated use of the corollary to Varignon's Theorem, we can show that each lattice point A_{ij} (except the original perimeter points of the chessboard) of Figure 1 is the midpoint of some quadrilateral's cross-segments. For example, A_{44} is the midpoint of cross-segments $A_{40}A_{48}$ and $A_{04}A_{84}$ of quadrilateral $A_{00}A_{08}A_{88}A_{80}$. Then A_{42} is the midpoint of cross-segments $A_{40}A_{44}$ and $A_{02}A_{82}$ of quadrilateral $A_{00}A_{04}A_{84}A_{80}$. Likewise, A_{41} is the midpoint of cross-segments $A_{40}A_{42}$ and $A_{01}A_{81}$ of quadrilateral $A_{00}A_{02}A_{82}A_{80}$, and so on. Thus, $A_{41}, A_{42}, A_{43}, \ldots, A_{47}$ divide $A_{40}A_{48}$ into eight congruent segments. This process can be repeated to show that each cross-segment, whether "horizontal" or "vertical," is divided into eight congruent segments.

Next we concentrate on results involving a few blocks (i.e., "squares" of the chessboard) rather than the entire chessboard. For convenience we use "$\triangle ABC$" as both the symbol for the triangle and also the area of the triangle. Additionally we will also use a single capital letter to denote the area of a quadrilateral.

Theorem 2. *If three adjacent blocks of a skewed chessboard adjoin in a single row (or column), then the area of the middle one is the arithmetic mean of the areas of the other two.*

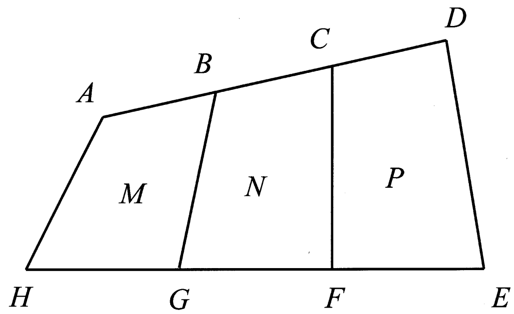

Figure 3.

Proof. Let B, C, F, and G be the respective trisection points of sides AD and EH of quadrilateral $ADEH$ in Figure 3. Additionally, let M, N, and P be the areas of the smaller quadrilaterals formed by the trisection points. We wish to show that $N = (1/2)(M + P)$. It is well known that if two triangles have the same height, then their areas are proportional to their bases. In Figure 3, for example, $\triangle ABG = \triangle CBG$ and $\triangle AHG = (1/3)\triangle AHE$. Therefore,

$$\begin{aligned} M + P &= (\triangle AHG + \triangle ABG) + (\triangle EDC + \triangle EFC) \\ &= \tfrac{1}{3}\triangle AHE + \triangle CBG + \tfrac{1}{3}\triangle EDA + \triangle GFC \\ &= \tfrac{1}{3}(\triangle AHE + \triangle EDA) + (\triangle CBG + \triangle GFC) \\ &= \tfrac{1}{3}(M + N + P) + N = \tfrac{1}{3}(M + P) + \tfrac{4}{3}N \end{aligned}$$

Hence, $M + P = 2N$ which completes the proof.

This result can easily be extended to more than 3 adjacent blocks. For example, if M, N, P, Q are four adjacent blocks in a single row (or column), then $M + Q = N + P$ since $M + P = 2N$ and $N + Q = 2P$ by Theorem 2. Similarly, if five blocks M, N, P, Q, R are in a single row, then $M + R = N + Q = 2P$.

Theorem 3. *If four adjacent blocks of a skewed chessboard adjoin so that all share a common vertex, then the sum of the areas of two "diagonal" blocks is equal to the sum of the areas of the other two blocks.*

Proof. In Figure 2, the vertex I is shared by the four blocks of quadrilateral $ACEG$. We must show that $M + Q = P + N$. Since the diagonals of parallelogram $BDFH$ bisect each other,

$$\triangle HIB = \triangle BID = \triangle DIF = \triangle FIH.$$

Since B and H are the midpoints of two sides of $\triangle ACG$, and D and F are the midpoints of two sides of $\triangle CEG$,

$$\triangle ABH + \triangle DEF = \left(\tfrac{1}{4}\right)\triangle ACG + \left(\tfrac{1}{4}\right)\triangle CEG = \left(\tfrac{1}{4}\right)(\text{quad } ACEG).$$

In the same manner, $\triangle HGF + \triangle BCD = (1/4)(\text{quad } ACEG)$. By combining these equalities we obtain

$$\begin{aligned} M + Q &= (\triangle ABH + \triangle BIH) + (\triangle DEF + \triangle FID) \\ &= (\triangle ABH + \triangle DEF) + (\triangle BIH + \triangle FID) \\ &= \left(\tfrac{1}{4}\right)(\text{quad } ACEG) + (\triangle BID + \triangle FIH) \\ &= (\triangle HGF + \triangle BCD) + (\triangle BID + \triangle FIH) \\ &= (\triangle HGF + \triangle FIH) + (\triangle BID + \triangle BCD) \\ &= P + N. \end{aligned}$$

We note that Theorem 3 is a known result and appears in [**2**].

Combining Theorems 1, 2, and 3, we can prove additional results by adding and simplifying several equations. Thus (Figure 4), if we have a 3 by 3 skewed quadrilateral where each side is trisected, we obtain:

(1) $A + F = D + C$
(2) $A + I = 2E = G + C$
(3) $B + H = D + F$
(4) $B + F + G = D + H + C$, and
(5) $E = \frac{1}{9}(A + B + C + D + E + F + G + H + I)$.

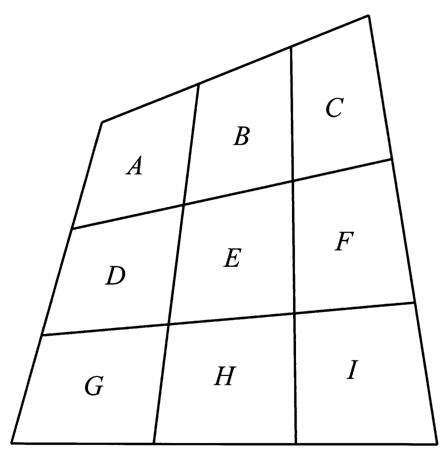

Figure 4.

In [**3**], Greenberg gave an arduous, but clever, proof of (5). Since this note was inspired by (5), we now prove it as follows.

$$\begin{aligned} &A + B + C + D + E + F + G + H + I \\ &\quad = (A + G) + (C + I) + (B + H) + (D + F) + E \\ &\quad = 2D + 2F + 2E + 2E + E \\ &\quad = 2(D + F) + 5E \\ &\quad = 2(2E) + 5E \\ &\quad = 9E. \end{aligned}$$

Therefore $E = \frac{1}{9}$ times the area of the quadrilateral.

Using Theorems 1, 2, and 3, the reader can obtain many additional area relations on n by n skewed chessboards. Finally, we return to the original question. The cross-segments in bold print of Figure 1 divide the skewed chessboard $A_{00}A_{08}A_{88}A_{80}$ into sixteen 2 by 2 blocks. By Theorem 3, the sum of the areas of the black blocks is equal to the sum of the areas of the white blocks for each of the sixteen 2 by 2 blocks. Therefore, the total area of the white blocks equals the total area of the black blocks. Do comparable relationships hold for cubes in a skewed 3-dimensional chessboard?

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On the Monotonicity of $\left(1+\frac{1}{n}\right)^n$ and $\left(1+\frac{1}{n}\right)^{n+1}$

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Since the function $f(t) = 1/t$ is decreasing on $(0, +\infty)$, for $0 < a < b$ we have

$$f(b)[b-a] \le \int_a^b f(t)\,dt \le f(a)[b-a].$$

For $a = n$ and $b = n+1$, this reduces to

$$\frac{1}{n+1} \le \log\left(1+\frac{1}{n}\right) \le \frac{1}{n}. \tag{1}$$

The inequalities (1) imply (upon multiplication by n) that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$. Although $\{(1+\frac{1}{n})^n\}$ is an increasing sequence and $\{(1+\frac{1}{n})^{n+1}\}$ is a decreasing sequence, this cannot be proved by (1) alone; one must use, for example, the Mean Value Theorem, or the Binomial Theorem, or the Arithmetic-Geometric Mean Inequality [**4, 3, 2**]. Below we refine inequalities (1) to prove these two results, and we get a little bit more.

For any convex function $F(t)$,

$$F\left(\frac{a+b}{2}\right)[b-a] \le \int_a^b F(t)\,dt \le \frac{F(a)+F(b)}{2}[b-a]. \tag{2}$$

(The right-hand side is the area of the trapezoid circumscribed at the endpoints and the left-hand side is the area of the trapezoid inscribed at the midpoint. This is known as Hadamard's Inequality [**1**].) For $F(t) = 1/t$, with $a = n$ and $b = n+1$, the inequalities (2) become

$$\frac{2}{2n+1} < \log\left(1+\frac{1}{n}\right) < \frac{2n+1}{2n(n+1)}. \tag{3}$$

If we multiply (3) by n, we get

$$\frac{2n}{2n+1} < \log\left(1+\frac{1}{n}\right)^n < \frac{2n+1}{2n+2}.$$

Now the left-hand side here, with $n+1$ instead of n, is greater than the right-hand side. Therefore $\{(1+\frac{1}{n})^n\}$ is an increasing sequence.

Similarly, multiplying (3) by $n+1$ we get

$$\frac{2n+2}{2n+1} < \log\left(1+\frac{1}{n}\right)^{n+1} < \frac{2n+1}{2n}.$$

Since the right-hand side, with $n+1$ instead of n, is less than the left-hand side, the sequence $\{(1+\frac{1}{n})^{n+1}\}$ is decreasing.

Inequalities (3) can be written

$$1 < \log\left(1+\frac{1}{n}\right)^{\frac{2n+1}{2}} \quad \text{and} \quad \log\left(1+\frac{1}{n}\right)^{\frac{2n(n+1)}{2n+1}} < 1,$$

which imply

$$\left(1+\frac{1}{n}\right)^{n(1+\frac{1}{2n+1})} < e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

These inequalities refine the inequalities $(1+\frac{1}{n})^n < e < (1+\frac{1}{n})^{n+1}$ furnished by (1).

The left-hand side of (3) is the midpoint approximation M to $\int_n^{n+1}(1/t)\,dt$, and the right-hand side is the trapezoid approximation T. Using Simpson's rule,

$$S = \frac{2}{3}M + \frac{1}{3}T \approx \int_n^{n+1}\frac{1}{t}\,dt = \log\left(1+\frac{1}{n}\right),$$

we have

$$\log\left(1+\frac{1}{n}\right)^{\frac{1}{S}} \approx 1 \quad \text{and} \quad \left(1+\frac{1}{n}\right)^{\frac{1}{S}} \approx e.$$

Since

$$S = \frac{2}{3}\frac{2}{2n+1} + \frac{1}{3}\frac{2n+1}{2n(n+1)} = \frac{12n^2+12n+1}{6n(n+1)(2n+1)},$$

we obtain

$$\frac{1}{S} = n\left(1+\frac{6n+5}{12n^2+12n+1}\right).$$

Thus,

$$e \approx \left(1+\frac{1}{n}\right)^{\frac{1}{S}} = \left(1+\frac{1}{n}\right)^{n(1+\frac{6n+5}{12n^2+12n+1})}.$$

For example, $n = 100$ gives e correct to nine decimal places.

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