REAL ANALYSIS

SOLUTIONS TO HOMEWORK

PART II

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CH. 2. SEQUENCES

2.1 DEFINITION OF A SEQUENCE

2.1-1 (a) $-1, 1, -1, 1, -1, \ldots$

(b) $1, 3, 5, 7, 9, \ldots$, (c) $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{12}$

(d) $-2, 4, -8, 16, -32, \ldots$

(e) $2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, \frac{7776}{3125}, \ldots$

(f) $1, 0.806852819, 0.734721044,$
$0.697038971, 0.673894542, \ldots$

2.1-2 (a) $(-1)^{n+1}$, (b) $\frac{(-1)^{n+1}}{n}$

(c) $\frac{1}{n^2}$, (d) $1+\frac{1}{n}$, (e) $\frac{(-1)^{n+1}}{n!}$

(f) i^{n-1}, (g) $\frac{i^{n-1}}{2^{n-1}}$, (h) $\frac{1}{2n-1}$

(i) $\frac{(-1)^{n+1}\, 2n}{2n+1}$, (j) $\frac{2^n}{n!}$

2.1-3

$c, \log n, n, n^2, n^3, 2^n, e^n, 3^n,$

$n!, n^n$

2.1 DEF. OF SEQ. (CONT.)

2.1-4

(a) $\lim \frac{2n+1}{n+3} = \lim \frac{2n}{n} = \lim 2 = 2$

(b) $\lim \frac{n^2+2n-1}{3n^2+1} = \lim \frac{n^2}{3n^2} = \lim \frac{1}{3} = \frac{1}{3}$

(c) $\lim \frac{2n^2+1}{3n^2+\log n} = \lim \frac{2n^2}{3n^2} = \lim \frac{2}{3} = \frac{2}{3}$

(d) $\lim \frac{n^5+5}{e^n+2} = \lim \frac{n^5}{e^n} = 0$

(SINCE $e^n \to \infty$ faster than n^5)

(e) $\lim \frac{e^n}{n!} = 0$ ($n! \to \infty$ faster than e^n)

(f) $\lim \frac{e^n+n}{n^n+2} = \lim \frac{e^n}{n^n} = 0$ ($n^n \to \infty$ faster than e^n)

(g) $\lim \left(1+\frac{1}{n}\right)^n = e$ (DEFINITION OF e)

(h) $\lim\limits_{n\to\infty} \left(1+\frac{2}{n}\right)^n$

$\boxed{\text{CALL } \frac{2}{n} = \frac{1}{m} \quad \text{THEN } n = 2m}$

$= \lim\limits_{m\to\infty} \left(1+\frac{1}{m}\right)^{2m}$

$= \lim\limits_{m\to\infty} \left\{\left(1+\frac{1}{m}\right)^m\right\}^2 = e^2$

2.1 DEF. OF SEQ. (CONT.)

2.1-4 (CONTINUED)

(i) $$\lim \frac{n^2+1}{n^2\log n} = \lim \frac{n^2}{n^2\log n}$$

$$= \lim \frac{1}{\log n} = 0$$

(j) $$\lim \frac{n!\,e^n}{n^n\sqrt{n}}$$

USING STIRLING'S FORMULA

$$n! \approx \sqrt{2\pi n}\,\frac{n^n}{e^n}$$ WE GET

(LARGE n)

$$\lim \sqrt{2\pi n}\,\frac{n^n}{e^n} \cdot \frac{e^n}{n^n\sqrt{n}} =$$

$$\lim \sqrt{2\pi} = \sqrt{2\pi}$$

2.2 RECURSION FORMULAS

2.2-1 (a) $X_1 = 1, \quad X_{n+1} = X_n + 3$

1, 4, 7, 10, 13, 16, ...

$X_n = 3n - 2$

2.2 RECURSION FORMULAS

2.2-1 (CONTINUED)

(b) $x_1 = 1, \quad x_{n+1} = 1 - x_n$

$1, 0, 1, 0, 1, 0, \ldots$

$x_n = \frac{1}{2}\left(1 + (-1)^{n+1}\right)$

(c) $x_1 = 1, \quad x_{n+1} = 2x_n$

$1, 2, 4, 8, 16, 32, \ldots$

$x_n = 2^{n-1}$

(d) $x_1 = 1, \quad x_{n+1} = \frac{n}{n+1} x_n$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$\cancel{1, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6}, \frac{2}{7}, \ldots}$

$\cancel{x_n = \frac{2}{n+1}}$ $\quad x_n = \frac{1}{n}$

2.2-2 FIBONACCI SEQ.

$x_1 = x_2 = 1, \quad x_{n+2} = x_n + x_{n+1}$

(a) $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$

2.2 RECURSION FORMULAS (CONT.)

2.2-2 (CONTINUED)

(b) FIND $\lim \frac{x_{n+1}}{x_n} = \ell$

$$x_{n+2} = x_{n+1} + x_n$$

DIVIDE BY x_{n+1}

$$\frac{x_{n+2}}{x_{n+1}} = \frac{x_{n+1}}{x_{n+1}} + \frac{x_n}{x_{n+1}}$$

FOR LARGE n, $\frac{x_{n+2}}{x_{n+1}} \approx \ell$

AND $\frac{x_n}{x_{n+1}} \approx \frac{1}{\ell}$

$$\rightarrow \ell = 1 + \frac{1}{\ell}$$

$$\ell^2 = \ell + 1$$

$$\ell^2 - \ell - 1 = 0$$

$$\ell = \frac{1 \pm \sqrt{1+4}}{2}$$ (USE + SINCE ℓ IS POSITIVE)

$$\boxed{\ell = \frac{1+\sqrt{5}}{2}} \approx 1.618033989 \ldots$$

(GOLDEN SECTION)

2.2 RECURSION FORMULAS (CONT)

3. $x_1 = 1, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

CALL $\lim x_n = x$

THEN FOR LARGE n, $x_n \approx x_{n+1} \approx x$

$$x = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

$$2x = x + \frac{a}{x}$$

$$x = \frac{a}{x}$$

$$x^2 = a$$

$$x = \boxed{\sqrt{a}}$$

$x_1 = 1, \quad x_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = 1.5$

$x_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = 1.41666\ldots$

$x_4 = 1.414215686$

$x_5 = 1.414213562 \leftarrow$ AFTER 5 ITERATIONS WE GET 10 DIGIT ACCURACY ON THE TI-81.

$x_6 =$ SAME

2.2 RECURSION FORMULAS (CONT)

2.2-4 CALL $\lim \frac{x_{n+1}}{x_n} = \ell$

(a) $x_{n+2} = 3x_{n+1} + x_n$

DIVIDE BY x_{n+1}

$$\frac{x_{n+2}}{x_{n+1}} = 3 + \frac{x_n}{x_{n+1}}$$

$$\frac{x_{n+2}}{x_{n+1}} = 3 + \frac{1}{\frac{x_{n+1}}{x_n}}$$

$$\ell = 3 + \frac{1}{\ell}$$

$$\ell^2 = 3\ell + 1$$

$$\ell^2 - 3\ell - 1 = 0$$

$$\ell = \frac{3 \pm \sqrt{9+4}}{2} \quad \left(\begin{array}{l}\text{USE + SINCE}\\ \ell \text{ IS POS.}\end{array}\right)$$

$$\ell = \frac{3+\sqrt{13}}{2} \approx 3.302775638\ldots$$

(b) $x_{n+2} = x_{n+1} - 2x_n$

DIVIDE BY x_{n+1}

$$\frac{x_{n+2}}{x_{n+1}} = 1 - \frac{2x_n}{x_{n+1}}$$

(CONTINUED)

2.2 RECURSION (CONT.)

2.2-4 (b) (CONT.)

$$\frac{x_{n+2}}{x_{n+1}} = 1 - \frac{2}{\frac{x_{n+1}}{x_n}}$$

$$\ell = 1 - \frac{2}{\ell}$$

$$\ell^2 = \ell - 2$$

$$\ell^2 - \ell + 2 = 0$$

$$\ell = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm \sqrt{7}\, i}{2}$$

HOW CAN THIS LIMIT BE A COMPLEX NUMBER WHEN ALL x_n ARE REAL?

LET'S LOOK AT THE SEQ. (x_n)

$1, 1, -1, -3, -1, 5, 7, -3, -17, -11,$
$23, 45, \ldots$

NOW LOOK AT THE SEQ. $\frac{x_{n+1}}{x_n}$

$\frac{1}{1}, \frac{-1}{1}, \frac{-3}{-1}, \frac{-1}{-3}, \frac{5}{-1}, \frac{7}{5}, \frac{-3}{7}, \frac{-17}{-3},$
$\frac{-11}{-17}, \frac{23}{-11}, \frac{45}{23}, \ldots$

$\boxed{\text{THIS SEQ. HAS NO LIMIT!}}$

2.2 RECURSION FORMULAS (CONT.)

2.2-4 (CONTINUED)

(d) $x_{n+2} = 2x_{n+1} + 3x_n$

DIVIDE BY x_{n+1}

$$\frac{x_{n+2}}{x_{n+1}} = 2 + \frac{3x_n}{x_{n+1}}$$

$$\ell = 2 + \frac{3}{\ell}$$

$$\ell^2 = 2\ell + 3$$

$$\ell^2 - 2\ell - 3 = 0$$

$$\ell = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} \quad (\text{USE } +)$$

$$\boxed{\ell = 3}$$

2.2-5 (a) CALL $\lim x_n = x$

$$x_n = \sqrt{2 + x_{n-1}}$$

$$x = \sqrt{2+x}$$

$$x^2 = 2 + x$$

$$x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \quad (\text{USE } +)$$

$$= \boxed{2}$$

2.2 RECURSION FORMULAS (CONT.)

2.2-5 (CONTINUED)

(b) $x_n = \sqrt{3 + x_{n-1}}$

$$x = \sqrt{3+x}$$

$$x^2 = 3 + x$$

$$x^2 - x - 3 = 0$$

$$x = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2} \quad (\text{USE } +)$$

$$x = \frac{1+\sqrt{13}}{2} \approx 2.302775638\ldots$$

ON MY TI-81 I GET

		NO. OF ACCURATE DIGITS
x_1 =	1,732050808	0
x_2 =	2,175327747	1
x_3 =	2,274934669	1
x_4 =	2,296722593	1
x_5 =	2,301460969	3
x_6 =	2,302490167	4
x_7 =	2,302713653	5
x_8 =	2,302762179	5
x_9 =	2,302772715	6
x_{10} =	2,302775003	7
x_{11} =	2,302775500	7
x_{12} =	2,302775608	8
x_{13} =	2,302775631	9
x_{14} =	2,302775636	9
x_{15} =	2,302775637	9
x_{16} =	2,302775638	10
x_{17} =	2,302775638	10

ABOUT EVERY 2 ITERATIONS GIVES ONE MORE ACCURATE DIGIT