HW
2-11

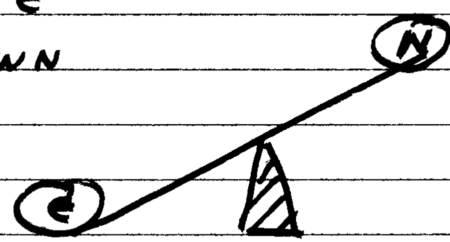
2.3 THE LIMIT OF A SEQUENCE

2.3-2 FALSE THE LIMIT OF A SEQUENCE IS UNIQUE

2.3-3 TRUE. USE OF THIS DEFINITION DOES NOT REQUIRE US TO FIND THE MOST EFFICIENT (SMALLEST) N.

2.3-4 TRUE. YOU CAN THINK OF ε AND N ON A "SEE-SAW"

WHEN ϵ GOES DOWN



N GOES UP

2.3-5 TRUE

$$|x_n - \ell| < \epsilon$$
$$-\epsilon < x_n - \ell < \epsilon$$
$$\ell - \epsilon < x_n < \ell + \epsilon$$

2.3-6 TRUE ONLY THE TERMS FOR VERY LARGE n EFFECT THE VALUE OF THE LIMIT. A FINITE NO. OF TERMS HAS A LAST TERM X_N AND ONLY TERMS AFTER THE N^{TH} TERM EFFECT THE LIMIT VALUE.

HW
2-12

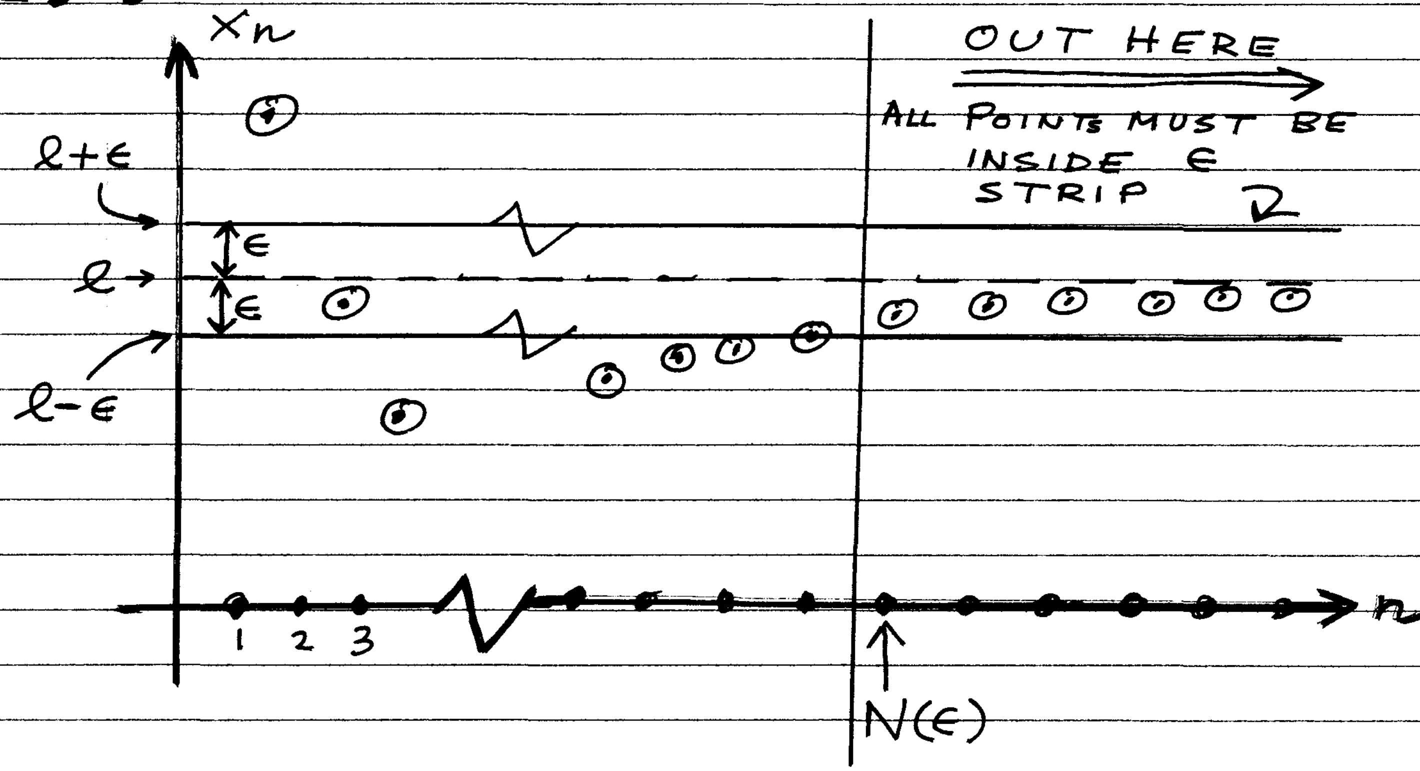
2.3 LIMIT OF SEQ. (CONT.)

2.3-7 ANY VALUE LARGER THAN $N = \frac{2}{\epsilon}$ IS OK.

(a) NO, (b) OK, (c) OK

(d) NO

2.3-8



2.3-9 TRUE

2.3-10 ANY CONSTANT SEQUENCE

$c, c, c, c, \ldots$

2.3-11 FALSE — AS ϵ GOES UP, N USUALLY GOES DOWN.

HW
2-13

2.4 USING THE DEF. OF LIMIT

2.4-1 THE PRELIMINARY ANALYSIS DID SEVERAL THINGS:

(1) IT DETERMINED A SUITABLE CHOICE FOR N AS A FUNCTION OF ϵ

$$N(\epsilon) = \frac{1}{\epsilon}$$

(2) IT GAVE US SEVERAL STEPS INVOLVING THE ALGEBRA OF INEQUALITIES THAT WE LATER COULD REVERSE TO FORM A "NICER" PROOF.

(3) THIS PRELIMINARY ANALYSIS IS ITSELF A "PROOF". WE CARRY OUT THE "FORMAL PROOF" AFTER IT, ONLY TO MAKE IT READ NICELY. (THE FORMAL PROOF FOLLOWS THE WORDS OF THE DEFINITION OF LIMIT CLOSELY.)

2.4-2

(a) $\lceil 123.4 \rceil = 124$

(b) $\lceil 100 \rceil = 100$

(c) $\lceil -1.5 \rceil = -1$

2.4-3 $N(\epsilon) = \lceil \frac{1}{\epsilon} \rceil + 1$

(a) $N(.1) = \lceil \frac{1}{.1} \rceil + 1 = \lceil 10 \rceil + 1 = 11$

(b) $N(.02) = \lceil \frac{1}{.02} \rceil + 1 = \lceil 50 \rceil + 1 = 51$

(c) $N(.001) = \lceil \frac{1}{.001} \rceil + 1 = \lceil 1000 \rceil + 1 = 1001$

(d) "increase"

2.4-4 THE PRELIMINARY ANALYSIS IS ITSELF A PROOF. THE SECOND "FORMAL" PROOF SIMPLY READS IN THE STYLE OF THE DEFINITION OF LIMIT, MAKING IT "NICER".

2.4-5 ANY LARGER VALUE OF N IS EQUALLY VALID. THUS

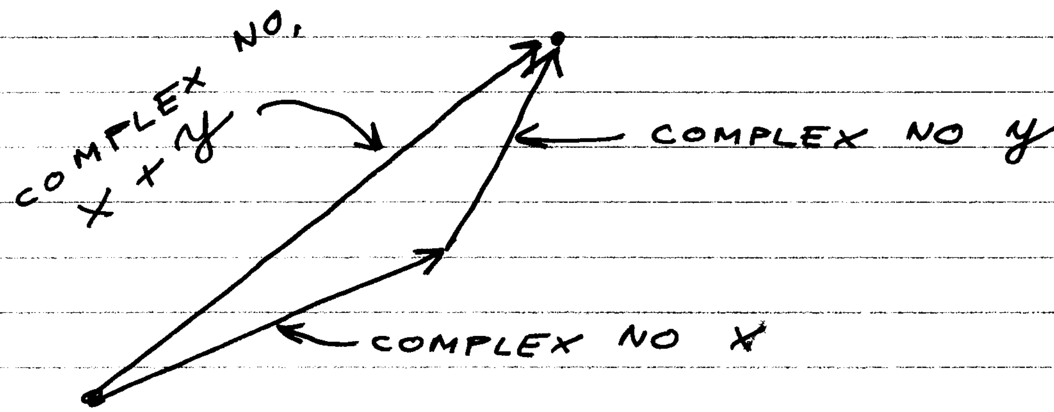
$$N = \left[\frac{1}{\epsilon}\right] + 2 \quad \text{or}$$

$$N = \left[\frac{1}{\epsilon}\right] + 100 \quad \text{or}$$

$$N = 2\left[\frac{1}{\epsilon}\right] + 1$$

ARE ALL ACCEPTABLE.

2.4-6



$|x+y|$ = LENGTH OF VECTOR $x+y$

$|x|$ = " x

$|y|$ = "

THE INEQUALITY $|x+y| \le |x| + |y|$ IS NOW CLEAR FROM THE "TRIANGLE" ABOVE.

2,4-7

(a) $\lim \frac{1}{n} = 0$

PRELIMINARY ANALYSIS

$$|x_n - \ell| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon$$

THUS $\frac{1}{\epsilon} < n$

FOR ALL $n > \frac{1}{\epsilon}$, WE SEE THAT

$|x_n - \ell| < \epsilon$.

WE WILL NOT BOTHER TO REVERSE THE STEPS FOR A "FORMAL" PROOF. THIS IS LEFT FOR THE STUDENT.

(b) $\lim \frac{5n+2}{n} = 5$

PRELIMINARY ANALYSIS

$$|x_n - \ell| = \left|\frac{5n+5}{n} - 5\right| = \left|\frac{5n+5-5n}{n}\right|$$

$$= \left|\frac{5}{n}\right| = \frac{5}{n} < \epsilon$$

THUS $\frac{5}{\epsilon} < n$

SELECT $N(\epsilon) = \cancel{\left[\frac{5}{\epsilon}\right] + 1}\ \frac{5}{\epsilon}$

HW
2-16

2.~~4~~-7 (CONTINUED)

(c) $\lim \frac{5n+2}{3n+1} = \frac{5}{3}$

PRELIMINARY ANALYSIS:

$$|x_n - \ell| = \left|\frac{5n+2}{3n+1} - \frac{5}{3}\right| = \left|\frac{15n+6-15n-5}{3(3n+1)}\right|$$

$$= \left|\frac{1}{3(3n+1)}\right| = \frac{1}{3(3n+1)}$$

$$< \frac{1}{3(3n)} = \frac{1}{9n} < \epsilon$$

THUS $\frac{1}{9\epsilon} < n$

SELECT $N(\epsilon) = \cancel{\left[\frac{1}{9\epsilon}\right]} \cancel{+1}\ \frac{1}{9\epsilon}$

(d) $\lim \frac{3n^2+n+2}{4n^2+2n+1} = \frac{3}{4}$

PRELIMINARY ANALYSIS:

$$|x_n - \ell| = \left|\frac{3n^2+n+2}{4n^2+2n+1} - \frac{3}{4}\right|$$

$$= \left|\frac{12n^2+4n+8-12n^2-6n-3}{4(4n^2+2n+1)}\right|$$

$$= \left|\frac{-2n+5}{4(4n^2+2n+1)}\right|$$

2.~~3~~4-7 (d) (CONTINUED)

$$|x_n - \ell| < \left|\frac{-2n+5}{4(4n^2)}\right| < \left|\frac{2n+5}{16n^2}\right|$$

$$= \frac{2n+5}{16n^2} < \frac{2n+5n}{16n^2}$$

$$= \frac{7n}{16n^2} = \frac{7}{16n} < \epsilon$$

THUS $\frac{7}{16\epsilon} < n$

SELECT $N(\epsilon) = \cancel{\left\lceil \frac{7}{16\epsilon} \right\rceil + 1}\ \frac{7}{16\epsilon}$

(e) $\lim \frac{3n^2-2n+1}{5n^2-2n-3} = \frac{3}{5}$

$$|x_n - \ell| = \left|\frac{3n^2-2n+1}{5n^2-2n-3} - \frac{3}{5}\right|$$

$$= \left|\frac{15n^2-10n+5-15n^2+6n+9}{5(5n^2-2n-3)}\right|$$

$$= \left|\frac{-4n+14}{5(5n^2-2n-3)}\right|$$

$$< \left|\frac{4n+14}{5(5n^2-2n-3)}\right|$$

HW
2-18

2.3 - 7 (e) (CONTINUED)

$$|x_n - \ell| < \frac{4n+14n}{|5(5n^2-2n-3)|} = \frac{18n}{5|5n^2-2n-3|}$$

NOW $2n+3 \le n^2$ PROVIDED $n \ge 3$

THUS

$$|5n^2-2n-3| = |5n^2-(2n+3)| \ge |5n^2-n^2|$$

(PROVIDED $n \ge 3$)

Now

$$|x_n - \ell| < \frac{18n}{5|5n^2-n^2|} = \frac{18n}{20n^2} = \frac{18}{20n}$$

$$= \frac{18}{20n} < \epsilon$$

$$\frac{18}{20\epsilon} < n$$

SELECT $N(\epsilon) = \text{MAX}\left\{ \frac{18}{20\epsilon}, 3 \right\}$

(f) $\displaystyle \lim \frac{n^3+1}{n^3-n+3} = 1$

$$|x_n - \ell| = \left| \frac{n^3+1}{n^3-n+3} - 1 \right| =$$

$$\left| \frac{n^3+1-n^3+n-3}{n^3-n+3} \right| =$$

$$\left| \frac{n-2}{n^3-n+3} \right|$$

2.4-7 (f) (CONTINUED)

$$|x_n - \ell| < \left|\frac{n}{n^3 - n + 3}\right| = \left|\frac{n}{n^3 - (n-3)}\right|$$

BUT $|n-3| < \frac{n^3}{2}$ FOR $n \geq 2$

$$|x_n - \ell| < \left|\frac{n}{n^3 - \frac{n^3}{2}}\right| \leq \left|\frac{n}{\frac{n^3}{2}}\right|$$

$$= \frac{2}{n^2} < \epsilon$$

$$\frac{2}{\epsilon} < n^2$$

$$\sqrt{\frac{2}{\epsilon}} < n$$

SELECT $N(\epsilon) = \cancel{\left[\sqrt{\frac{2}{\epsilon}}\right] + 1}, \sqrt{\frac{2}{\epsilon}}$

2.4-7 (g) $\lim \frac{4n}{3n^5+1} = 0$

$$|x_n - \ell| = \left|\frac{4n}{3n^5+1} - 0\right| = \frac{4n}{3n^5+1}$$

$$< \frac{4n}{3n^5} = \frac{4}{3n^4} < \epsilon$$

THUS $\frac{4}{3\epsilon} < n^4 \implies \sqrt[4]{\frac{4}{3\epsilon}} < n$

SELECT $N(\epsilon) = \left[\sqrt[4]{\frac{4}{3\epsilon}}\right]$

2.$\cancel{3}$4-7 (CONTINUED)

(h) $\lim \frac{n^2}{2^n} = 0$

(1) $|x_n - \ell| = \left| \frac{n^2}{2^n} - 0 \right| = \frac{n^2}{2^n}$

FROM THE BINOMIAL THEOREM WE HAVE

$$(1+1)^n = 2^n = 1 + \frac{n}{1} + \frac{n}{1} \cdot \frac{n-1}{2} + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \cdots + 1$$

SINCE ALL THE TERMS (BINOMIAL COEFFICIENTS) IN THE ABOVE SUM ARE POSITIVE,

$$2^n > \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3} = \frac{n(n-1)(n-2)}{6}$$

GOING BACK TO (1) WE CAN SUBSTITUTE THIS SMALLER QUANTITY INTO THE DENOMINATOR TO GET

$$|x_n - \ell| = \frac{n^2}{2^n} < \frac{n^2}{\frac{n(n-1)(n-2)}{6}} = \frac{6n^2}{n(n-1)(n-2)}$$

NOW $(n-1)(n-2) > \left(n - \frac{n}{2}\right)\left(n - \frac{n}{2}\right)$ PROVIDED $n \geq 4$

$$= \frac{n^2}{4}$$

THUS

$$|x_n - \ell| < \frac{6n^2}{n(n-1)(n-2)} < \frac{6n^2}{n \frac{n^2}{4}} \quad \left\{ \begin{array}{c} \text{IF} \\ n \geq 4 \end{array} \right\}$$

2.~~3~~4-7 (h) (CONTINUED)

$$|x_n - l| < \frac{24n^2}{n^3} = \frac{24}{n} < \epsilon$$

THUS $\frac{24}{\epsilon} < n$

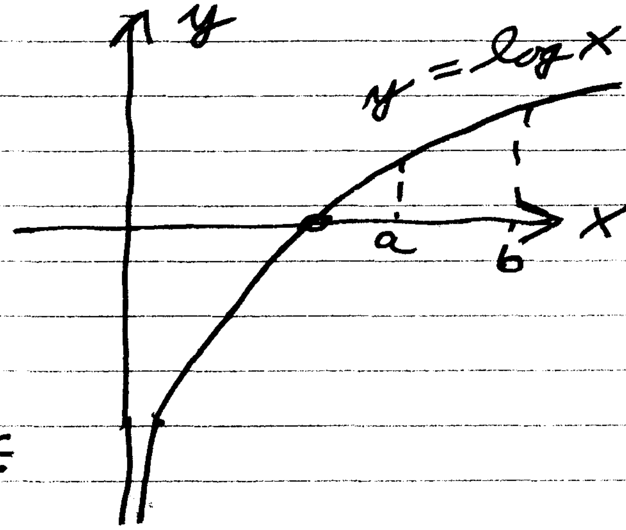
SELECT $N(\epsilon) = \text{MAX}\left\{\left[\frac{24}{\epsilon}\right], 4\right\}$

2.~~3~~4-7 (i) $\lim \frac{2^n}{3^n} = 0$

(1) $|x_n - l| = \left|\frac{2^n}{3^n} - 0\right| = \frac{2^n}{3^n} < \epsilon$

SINCE $y = \log x$ IS AN INCREACING FUNCTION WE HAVE:
IF $a < b$ THEN

$\log a < \log b$



THUS WE CAN WRITE FROM (1)

$\log\left(\frac{2}{3}\right)^n < \log \epsilon$

(2) $n \log\left(\frac{2}{3}\right) < \log \epsilon$

SINCE BOTH $\log\left(\frac{2}{3}\right)$ AND $\log \epsilon$ ARE NEGATIVE (IF $\epsilon < 1$) WE REWRITE (2) SO THAT THE LOGS WILL BE POSITIVE, 2,4-7 (i) (CONTINUED)

MULTIPLYING (2) BY (-1) WE GET

$$-n \log\left(\tfrac{2}{3}\right) > -\log \epsilon$$

$$n \log\left(\tfrac{2}{3}\right)^{-1} > \log \epsilon^{-1}$$

$$n \log \tfrac{3}{2} > \log \tfrac{1}{\epsilon}$$

$$n > \frac{\log \frac{1}{\epsilon}}{\log \frac{3}{2}}$$

SELECT $N(\epsilon) = \left[\dfrac{\log \frac{1}{\epsilon}}{\log \frac{3}{2}} \right] \cancel{+1}$.

2,4-8

(a) PROVE $\lim (-1)^n$ D.N.E.

PROOF

ASSUME BY CONTRADICTION THAT THIS LIMIT DOES EXIST. CALL IT l. THEN GIVEN ANY ϵ

(1) $\quad |(-1)^n - l| < \epsilon$ FOR ALL $n \geq N(\epsilon)$.

2.4 - 8 (a) (CONTINUED)

FOR $n \geq N(\epsilon)$ THAT ARE EVEN, (1) BECOMES

$$|1-\ell| < \epsilon \quad \text{or}$$

(2) $$|\ell - 1| < \epsilon$$

AND FOR n ODD WE GET

$$|-1-\ell| < \epsilon \quad \text{or}$$

(3) $$|1+\ell| < \epsilon$$

WRITING (2) AND (3) AS

(4) $$-\epsilon < \ell - 1 < \epsilon$$

(5) $$-\epsilon < \ell + 1 < \epsilon$$

AND ADDING 1 TO (4) AND (-1) TO (5) WE GET

$$1-\epsilon < \ell < 1+\epsilon$$

$$-1-\epsilon < \ell < -1+\epsilon$$

FOR $\epsilon = \frac{1}{4}$ THESE READ

(6) $$\frac{3}{4} < \ell < \frac{5}{4} \quad \text{AND}$$

(7) $$-\frac{5}{4} < \ell < -\frac{3}{4}$$

NO SUCH NUMBER ℓ EXISTS THAT SATISFIES BOTH (6) & (7)

2,4-8 (b)

PROVE THAT $\lim n$ DOES NOT EXIST.

PROOF

BY CONTRADICTION, ASSUME THE LIMIT EXISTS. CALL IT ℓ. GIVEN ϵ THERE EXISTS $N(\epsilon)$ SUCH THAT

$|n-\ell| < \epsilon$ FOR ALL n SUCH THAT $N(\epsilon) < n$

THIS CAN BE WRITTEN AS

$$-\epsilon < n-\ell < \epsilon$$

$$\ell-\epsilon < n < \ell+\epsilon$$

FOR <u>ALL</u> n GREATER THAN $N(\epsilon)$.

THIS IS IMPOSSIBLE SINCE THIS LAST EXPRESSION DOES NOT ALLOW n TO EXCEED $\ell+\epsilon$.

2,4-8 (c)

PROVE THAT $\lim \sin\left(\frac{n\pi}{3}\right)$

DOES NOT EXIST.

THIS IS THE SEQUENCE

$$\sin\frac{\pi}{3},\ \sin\frac{2\pi}{3},\ \sin\frac{3\pi}{3},\ \sin\frac{4\pi}{3},\ \sin\frac{5\pi}{3},$$

$$= \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},$$

HW
2-25

2.5 - 8 (c)

THUS WHEN n IS A MULTIPLE OF 3,

(1) $\sin \frac{n\pi}{3} = 0$ $\quad (n = 3k, \ k \in \mathbb{Z})$

AND

(2) $\sin \frac{n\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad$ (FOR $n = 6k+1$, $k \in \mathbb{Z}$)

PROOF

BY CONTRADICTION, ASSUME THE LIMIT EXISTS AND CALL IT ℓ.

(3) $\left|\sin \frac{n\pi}{3} - \ell\right| = |0 - \ell| = |\ell|$

FOR $n = 3k$ FROM (1) ABOVE

AND

(4) $\left|\sin \frac{n\pi}{3} - \ell\right| = \left|\frac{\sqrt{3}}{2} - \ell\right| = \left|\ell - \frac{\sqrt{3}}{2}\right|$

FOR $n = 6k+1$ FROM (2),

GIVEN $\epsilon > 0$, THERE EXISTS $N(\epsilon)$ SUCH THAT

(5) $\left|\sin \frac{n\pi}{3} - \ell\right| < \epsilon$ $\quad$ FOR ALL $n \geq N(\epsilon)$

2.5-8 (d) (CONTINUED)

RELATION (5) READS

(6) $|\ell| < \epsilon$ FROM (3) FOR CERTAIN n AND

(7) $\left|\ell - \frac{\sqrt{3}}{2}\right| < \epsilon$ FROM (4) FOR CERTAIN OTHER n.

WRITE (6) AND (7) AS

(8) $-\epsilon < \ell < \epsilon$

(9) $-\epsilon < \ell - \frac{\sqrt{3}}{2} < \epsilon$

ADD $\frac{\sqrt{3}}{2}$ TO (9) TO GET

(10) $\frac{\sqrt{3}}{2} - \epsilon < \ell < \frac{\sqrt{3}}{2} + \epsilon$

SELECT $\epsilon = \frac{\sqrt{3}}{8}$, THEN (8) & (10) BECOME

(11) $-\frac{\sqrt{3}}{8} < \ell < \frac{\sqrt{3}}{8}$ AND

(12) $\frac{3\sqrt{3}}{8} < \ell < \frac{5\sqrt{3}}{8}$

NO NUMBER ℓ EXISTS SATISFYING (11) & (12).