REAL ANALYSIS
DR T J OSLER

LCD 1

SOLUTIONS TO HW ON LIMITS, CONTINUITY & DIFFERENTIABILITY

2 (a) PROVE $\lim_{x \to 2} 3x-2 = 4$

$$|f(x)-\ell| = |(3x-2)-4| = |3x-6| = |3(x-2)|$$

$$= 3|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{2}$$

USE $\delta = \frac{\epsilon}{2}$

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2 (b) PROVE $\lim_{x \to 3} x^2 = 9$

$$|f(x)-\ell| = |x^2-9| = |x+3| \cdot |x-3|$$

SUPPOSE WE RESTRICT x TO $2 < x < 4$, $\Rightarrow |x-3| < 1$

THEN $|x+3| < 7$, SO

$$|f(x)-\ell| < 7|x-3| < \epsilon \Rightarrow |x-3| < \frac{\epsilon}{7}$$

USE $\cancel{\delta = \frac{\epsilon}{7}}$ $\delta = \mathrm{MIN}\left[1, \frac{\epsilon}{7}\right]$

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2 (c) PROVE $\lim_{x \to 2} x^3 = 8$

$$|f(x)-\ell| = |x^3-8| = |(x-2)(x^2+2x+4)|$$

$$= |x^2+2x+4|\,|x-2|$$

RESTRICT x TO $1 < x < 3 \Rightarrow |x-2| < 1$

THEN $|x^2+2x+4| < |3^2+2\cdot 3+4| = 19$

$$|f(x)-\ell| < 19|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{19}$$

USE $\delta(\epsilon) = \mathrm{MIN}\left[1, \frac{\epsilon}{19}\right]$

2 (d) PROVE $\lim\limits_{x \to a} x^2 = a^2$

$|f(x) - \ell| = |x^2 - a^2| = |x+a||x-a|$

RESTRICT x TO $|x-a| < 1 \Rightarrow$

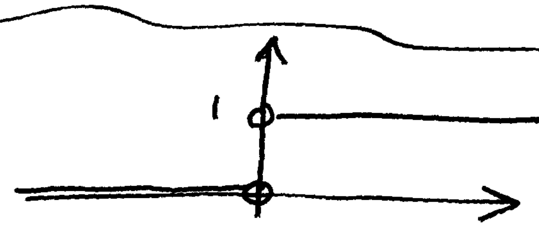
$-1 < x - a < 1 \Rightarrow a - 1 < x < a + 1$

THEN $|x+a| < |2a| + 1$ AND

$|f(x) - \ell| < (|2a| + 1)|x-a| < \epsilon \Rightarrow |x-a| < \dfrac{\epsilon}{2|a|+1}$

USE $\delta(\epsilon) = \text{MIN}\left[1, \dfrac{\epsilon}{2|a|+1}\right]$

3.1 $f(x) = \begin{cases} 1 & \text{FOR } 0 < x \\ 0 & \text{FOR } x < 0 \end{cases}$

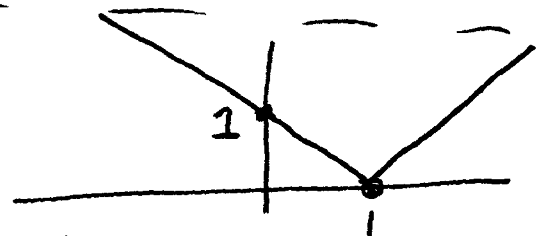


(a) $\lim\limits_{x \to 0} f(x)$ DNE

(b) $f(x)$ IS DISCONTINUOUS AT $x = 0$

(c) " " NOT DIFFERENTIABLE AT $x = 0$

3.2 $f(x) = |x - 1|$

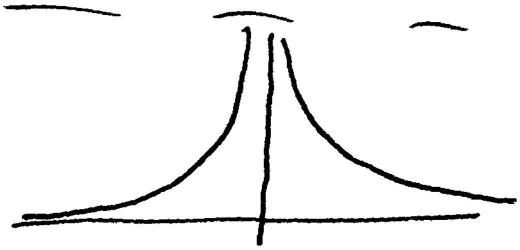


(a) $\lim\limits_{x \to 1} f(x) = 0$

(b) $f(x)$ IS CONT. AT $x = 1$

(c) " " NOT DIFF. AT $x = 1$

3.3 $f(x) = \dfrac{1}{x^2}$

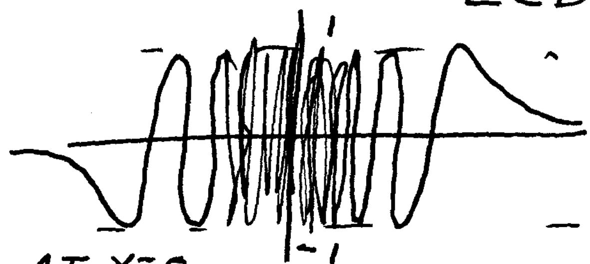


(a) $\lim\limits_{x \to 0} f(x)$ DNE

(b) NOT CONT. AT $x = 0$

(c) " DIFF "

3.4 $f(x) = \sin\left(\frac{1}{x}\right)$

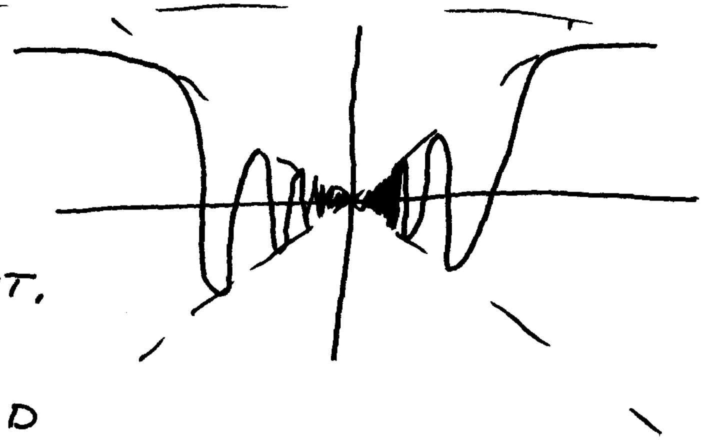


(a) $\lim_{x \to 0} f(x)$ DNE

(b) $f(x)$ IS NOT CONT. AT $x=0$

(c) " " " DIFF. "

3.5 $f(x) = x \sin \frac{1}{x}$

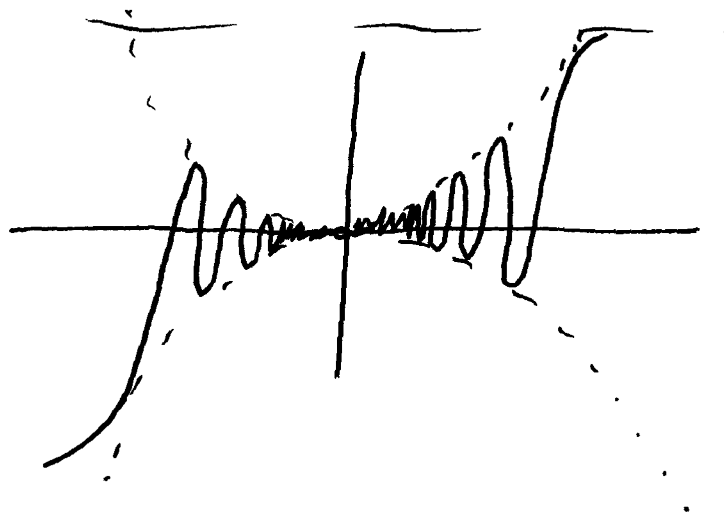


(a) $\lim_{x \to 0} f(x) = 0$

(b) $f(x)$ IS NOT CONT. AT $x=0$ SINCE $f(0)$ IS UNDEFINED

(c) $f(x)$ IS NOT DIFF. AT $x=0$

3.6 $f(x) = x^2 \sin \frac{1}{x}$



(a) $\lim_{x \to 0} f(x) = 0$

(b) $f(x)$ IS NOT CONT. AT $x=0$ SINCE $f(0)$ IS UNDEFINED

(c) $f(x)$ IS NOT DIFF. AT $x=0$

3.7 $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

(a) $\lim_{x \to 0} f(x) = 0$, (b) & (c) $f(x)$ NOT CONT, & NOT DIFF. AT $x=0$

3.8 $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim_{x \to 0} f(x) = 0$

(b) & (c) $f(x)$ IS CONT. AND DIFF, AT $x=0$

3.9 $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim_{x \to 0} f(x) = 0$

(b) $f(x)$ IS CONT. AT $x=0$

(c) " IS NOT DIFF, AT $x=0$

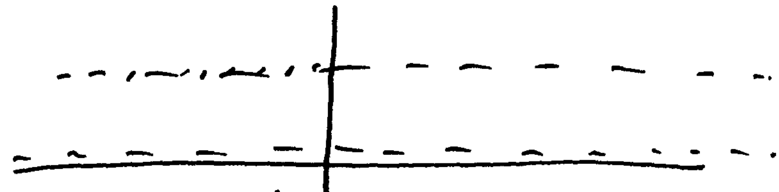
3.10 $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim_{x \to 0} f(x)$ DNE

(b) & (c) $f(x)$ IS NOT CONT. & NOT DIFF, AT $x=0$.

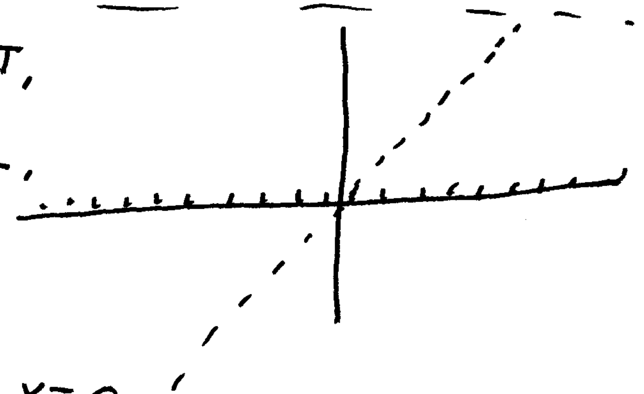
3,11 $f(x) = \begin{cases} 1 & x \text{ RATIONAL} \\ 0 & x \text{ IRRATIONAL} \end{cases}$

(a) $\lim_{x \to 0} f(x)$ DNE



(b) & (c) $f(x)$ IS NOT CONT. & NOT DIFF. AT $x = 0$

3,12 $f(x) = \begin{cases} x & x \text{ RAT.} \\ 0 & x \text{ IRR.} \end{cases}$

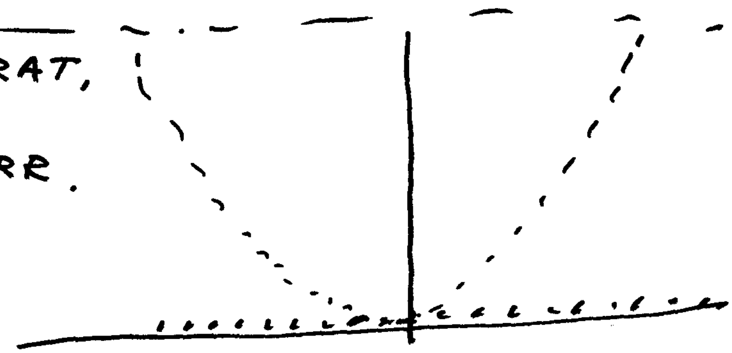


(a) $\lim_{x \to 0} f(x) = 0$

(b) $f(x)$ IS CONT. AT $x = 0$.

(c) " IS <u>NOT</u> DIFF. AT $x = 0$

3,13 $f(x) = \begin{cases} x^2 & x \text{ RAT.} \\ 0 & x \text{ IRR.} \end{cases}$



(a) $\lim_{x \to 0} f(x) = 0$

(b) $f(x)$ IS CONT. AT $x = 0$.

(c) " " DIFF. "

5.1 $f(x) = \begin{cases} 1 & x \text{ RAT} \\ 0 & x \text{ IRR.} \end{cases}$

$f(x)$ IS DISCONT. FOR ALL x.

5.2 $f(x) = \begin{cases} x & x \text{ RAT} \\ 0 & x \text{ IRR,} \end{cases}$

$f(x)$ IS DISCONT. FOR ALL x <u>EXCEPT</u> $x = 0$

5.3 $f(x) = \begin{cases} x^2 & x \text{ RAT} \\ 0 & x \text{ IRR} \end{cases}$

$f(x)$ IS DISCONT. FOR ALL x <u>EXCEPT</u> $x = 0$

5.4 $f(x) = \frac{1}{x}$ IS NOT CONT. AT $x = 0$.

5.5 $f(x) = \frac{\sin x}{x}$ IS NOT CONT. AT $x = 0$ SINCE $f(0)$ IS UNDEFINED.

5.6 $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

HAS <u>NO</u> DISCONTINUITIES

5.7 $f(x) = \begin{cases} \frac{x}{e^x - 1} \\ 1 & x = 0 \end{cases}$ HAS <u>NO</u> DISCONTINUITIES

7.1 $f(x) = \begin{cases} 1 & x \text{ RAT,} \\ 0 & x \text{ IRR,} \end{cases}$

HAS NO DIFF. POINTS.

7.2 $f(x) = \begin{cases} x & x \text{ RAT,} \\ 0 & x \text{ IRR.} \end{cases}$

HAS NO DIFF. POINTS

7.3 $f(x) = \begin{cases} x^2 & x \text{ RAT} \\ 0 & x \text{ IRR} \end{cases}$

IS DIFF. ONLY AT $x = 0$.

7.4 $f(x) = \frac{1}{x}$ IS DIFF AT ALL POINTS EXCEPT $x = 0$.

7.5 $f(x) = \frac{\sin x}{x}$ IS DIFF AT ALL POINTS EXCEPT $x = 0$

7.6 $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

IS DIFF AT ALL POINTS

8.1 F, 8.2 T, 8.3 F

8.4 F, 8.5 T

10 IN PROBLEM 2(D) TAKE

$\delta(\epsilon) = \text{MIN}\left[1, \frac{\epsilon}{3}\right]$